ECE 6340
Intermediate EM Waves

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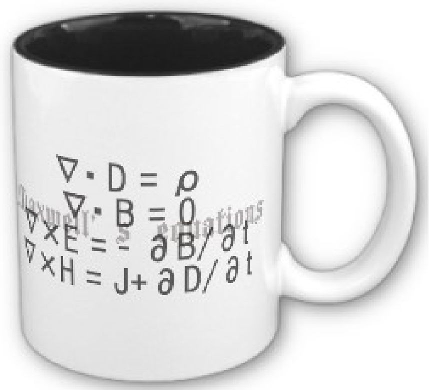

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Wave Equation

Assume $\underline{J}^{i} = \underline{0}\,, \quad \underline{M}^{i} = \underline{0}$

The region is source free and homogeneous.

so $$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$

$$\nabla \times \underline{H} = j\omega\varepsilon_c \underline{E}$$

Then we have $$\nabla \times \left(\nabla \times \underline{E}\right) = -j\omega\mu\left(\nabla \times \underline{H}\right)$$

$$= -j\omega\mu\left(j\omega\varepsilon_c \underline{E}\right)$$

Use $k^{2} = \omega^{2}\mu\,\varepsilon_c$

Hence $$\nabla \times \left(\nabla \times \underline{E}\right) - k^{2}\underline{E} = \underline{0}$$

Vector Wave Equation

Wave Equation (cont.)

Similarly, $$\nabla \times \left(\nabla \times \underline{H} \right) - k^2 \underline{H} = \underline{0}$$

Taking the divergence,

$$\nabla \times \left(\nabla \times \underline{E} \right) - k^2 \underline{E} = \underline{0}$$

$$\Downarrow$$

$$\nabla \cdot \left[\nabla \times \left(\nabla \times \underline{E} \right) \right] - k^2 \nabla \cdot \underline{E} = 0$$

Hence, $$\nabla \cdot \underline{E} = 0$$

Hence $\rho_v = 0$ in the time-harmonic steady state for a homogeneous material, provided $\omega \neq 0$.

The zero-divergence condition is built in to the vector wave equation.

Vector Laplacian

Recall $$\nabla^2 \underline{V} \equiv \nabla\left(\nabla \cdot \underline{V}\right) - \nabla \times \left(\nabla \times \underline{V}\right)$$

In rectangular coordinates,

$$\nabla^2 \underline{V} = \underline{\hat{x}}\,\nabla^2 V_x + \underline{\hat{y}}\,\nabla^2 V_y + \underline{\hat{z}}\,\nabla^2 V_z$$

The vector wave equation can thus be written as

$$\underbrace{\nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2 \underline{E}}_{\nabla\times\nabla\times\underline{E}} - k^2 \underline{E} = 0$$

or

$$\nabla^2 \underline{E} + k^2 \underline{E} = \underline{0}$$

"Vector Helmholtz Equation"

Vector Laplacian (cont.)

Take the rectangular components:

$$\nabla^2 E_x + k^2 E_x = 0$$

$$\nabla^2 E_y + k^2 E_y = 0$$

$$\nabla^2 E_z + k^2 E_z = 0$$

Note: $\nabla \cdot \underline{E} = 0$ is **not implied** from the vector Helmholtz Equation.

(see the example later)

Waveguide Fields

From the "guided-wave theorem," the fields of a guided wave varying as $\exp(-jk_z z)$ may be written in terms of the longitudinal (z) components as

$$E_x = \frac{-j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial y}$$

$$H_x = \frac{j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_z}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_z}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_z}{\partial y}$$

Waveguide Fields (cont.)

These may be written more compactly as

$$\underline{E}_t = \frac{j\omega\mu}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t H_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t E_z\right)$$

$$\underline{H}_t = \frac{-j\omega\varepsilon_c}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t E_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t H_z\right)$$

Waveguide Equations (cont.)

For the special case of TE$_z$ and TM$_z$ fields, we can combine these as follows:

TE$_z$

$$\underline{E}_t = \frac{j\omega\mu}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t H_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t E_z\right)$$

$E_z = 0$

Combine

$$\underline{H}_t = \frac{-j\omega\varepsilon_c}{k^2 - k_z^2}\left(\underline{\hat{z}} \times \nabla_t E_z\right) - \frac{jk_z}{k^2 - k_z^2}\left(\nabla_t H_z\right)$$

$E_z = 0$

Waveguide Equations (cont.)

$$\underline{E}_t = \frac{j\omega\mu}{k^2 - k_z^2}\,\hat{\underline{z}} \times \left(-\left(\frac{k^2 - k_z^2}{jk_z} \right) \underline{H}_t \right)$$

$$= -\left(\frac{\omega\mu}{k_z} \right) \left(\hat{\underline{z}} \times \underline{H}_t \right)$$

Define the wave impedance: $Z^{TE} = \left(\frac{\omega\mu}{k_z} \right)$

Then we have $\underline{E}_t = -Z^{TE} \left(\hat{\underline{z}} \times \underline{H}_t \right)$

Waveguide Equations (cont.)

Summary for TE_z and TM_z cases:

$$\underline{E}_t = -Z^{TE}\left(\underline{\hat{z}}\times\underline{H}_t\right), \quad \mathrm{TE}_z \text{ Fields}$$

$$\underline{E}_t = -Z^{TM}\left(\underline{\hat{z}}\times\underline{H}_t\right), \quad \mathrm{TM}_z \text{ Fields}$$

$$Z^{TE} = \left(\frac{\omega\mu}{k_z}\right) \qquad Z^{TM} = \left(\frac{k_z}{\omega\varepsilon_c}\right)$$

Note: For a wave going in the $-z$ direction, we replace the minus sign with a plus sign.

Waveguide Equations (cont.)

Taking the cross product of both sides with $\underline{\hat{z}}$ gives us

$$\underline{H}_t = +\frac{1}{Z^{TE}}\left(\underline{\hat{z}} \times \underline{E}_t\right), \quad \mathrm{TE}_z \text{ Fields}$$

$$\underline{H}_t = +\frac{1}{Z^{TM}}\left(\underline{\hat{z}} \times \underline{E}_t\right), \quad \mathrm{TM}_z \text{ Fields}$$

Note: For any transverse vector we have

$$\underline{\hat{z}} \times \left(\underline{\hat{z}} \times \underline{V}_t\right) = -\underline{V}_t$$

Note: For a wave going in the $-z$ direction, we replace the plus sign with a minus sign.

Waveguide Equations (cont.)

For the special case of a TEM wave:

$$k_z = k$$

$$Z^{TE} = \frac{\omega\mu}{k_z} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon_c}} = \eta$$

$$Z^{TM} = \frac{k_z}{\omega\varepsilon_c} = \frac{k}{\omega\varepsilon_c} = \frac{\omega\sqrt{\mu\varepsilon_c}}{\omega\varepsilon_c} = \eta$$

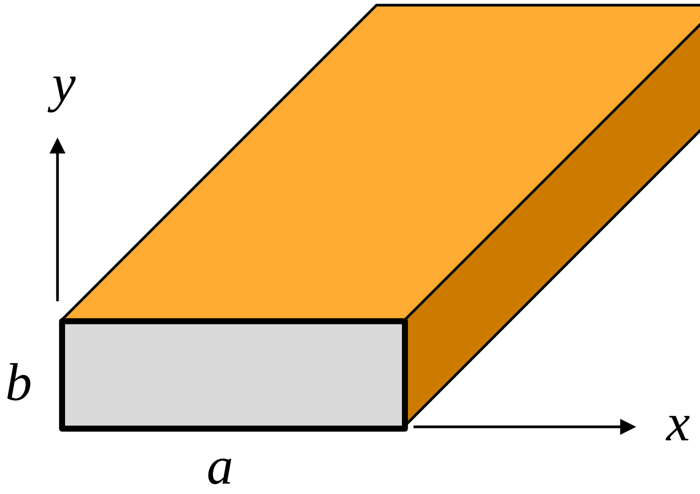
$$Z^{TE} = Z^{TM} = \eta$$ (TEM wave)

Example: Spurious TE_{00} Mode

Rectangular Waveguide

$$\nabla^2 E_z + k^2 E_z = 0$$

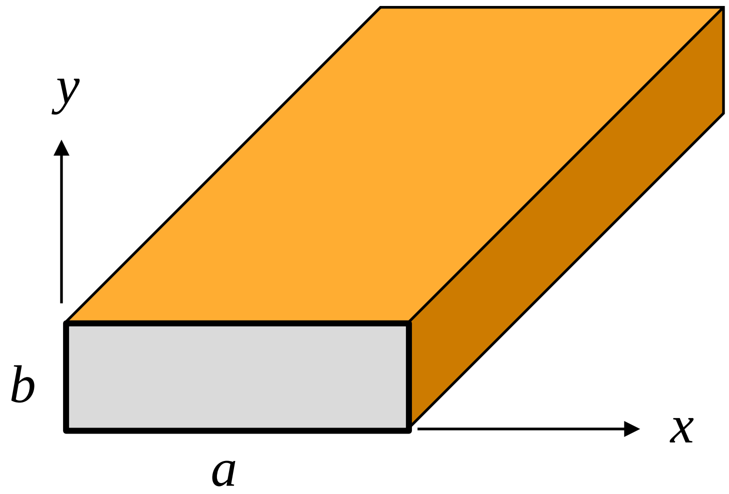
$$\nabla^2 H_z + k^2 H_z = 0$$



In this example we show that the TE_{00} mode of the rectangular waveguide is non-physical, since it violates the divergence condition.

Example (cont.)

Solution from separation of variables (given later):



$$k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

TM$_{mn}$: $$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

TE$_{mn}$: $$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

Example (cont.)

Question: What is the lowest-order mode (smallest *m* and *n*) for a non-trivial field?

TM$_{mn}$: $$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

TE$_{mn}$: $$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

TM$_{mn}$: $m = 1, n = 1$

TE$_{mn}$: $m = 0, n = 0$?

Example (cont.)

TE$_{mn}$ mode:

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z} \qquad k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

TE$_{00}$ mode:

$$H_z = H_0 e^{-jkz}$$

$$H_x = H_y = 0$$

The last equation holds since the normal component of the magnetic field is zero on the PEC, and there is no (x, y) variation inside the waveguide.

$$\underline{H} \cdot \underline{\hat{n}} = 0$$

Example (cont.)

Check the divergence:

$$\nabla \cdot \underline{H} = \frac{\partial \cancel{H_x}}{\partial x} + \frac{\partial \cancel{H_y}}{\partial y} + \frac{\partial H_z}{\partial z}$$

$$= -jk\,H_0\,e^{-jkz}$$

The TE_{00} mode is a <u>non-physical</u> mode.

Hence $\nabla \cdot \underline{H} \neq 0$ (invalid mode)

The TE_{10} mode is actually the mode with the lowest cut-off frequency.

Whenever we solve the vector Helmholtz equation, we should check to make sure that the divergence condition is satisfied!

It turns out that <u>all</u> of the <u>other</u> modes of the rectangular waveguide are physical.