ECE 6340
Intermediate EM Waves

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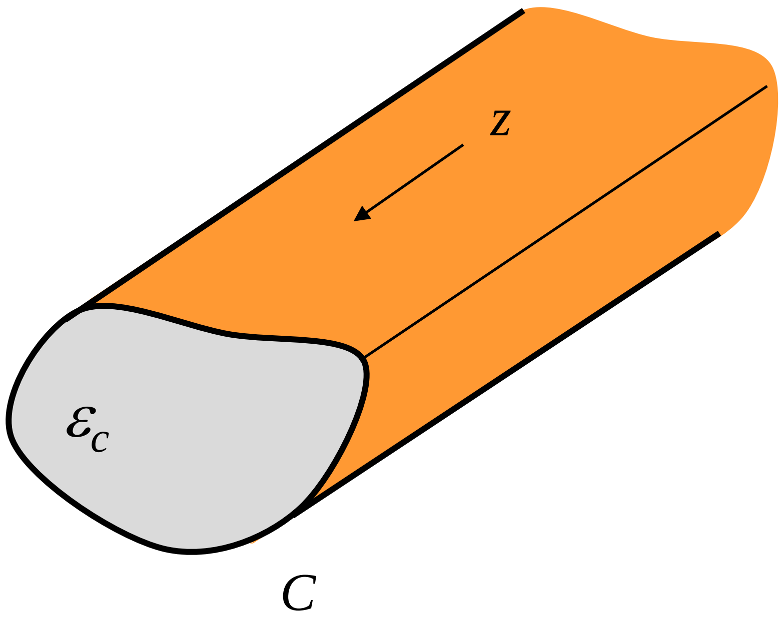
Notes 11

Waveguides

Waveguide geometry:

A hollow pipe waveguide is assumed here.

$$k = \omega\sqrt{\mu\varepsilon_c}$$



Assumptions:

1) The waveguide boundary is PEC
2) The material inside is <u>homogeneous</u> (it may be lossy)
3) A traveling wave exists in the z direction: $\exp(-jk_z z)$.

Comparison of Systems

Transmission lines

- Can propagate a signal at any frequency
- Have minimal dispersion (only due to loss for TEM mode) – good for signal propagation
- May or may not be shielded (e.g., coax vs. twin lead)
- Become very lossy at high frequencies
- Cannot handle high power (unless the cable are very big)

Waveguides

- Can handle high power
- Have low loss (lower than transmission lines)
- Are completely shielded
- Can only be used above the cutoff frequency (not suitable for low frequencies)
- Have large amounts of dispersion (not the best for signal propagation)

Fiber Optic Cables

- Have very low loss (lower than transmission lines or waveguides)
- Have very low dispersion (lower than transmission lines)
- Cannot handle high power
- Require electro-optic interfaces with electronic and RF/microwave equipment

Comparison of Systems (cont.)

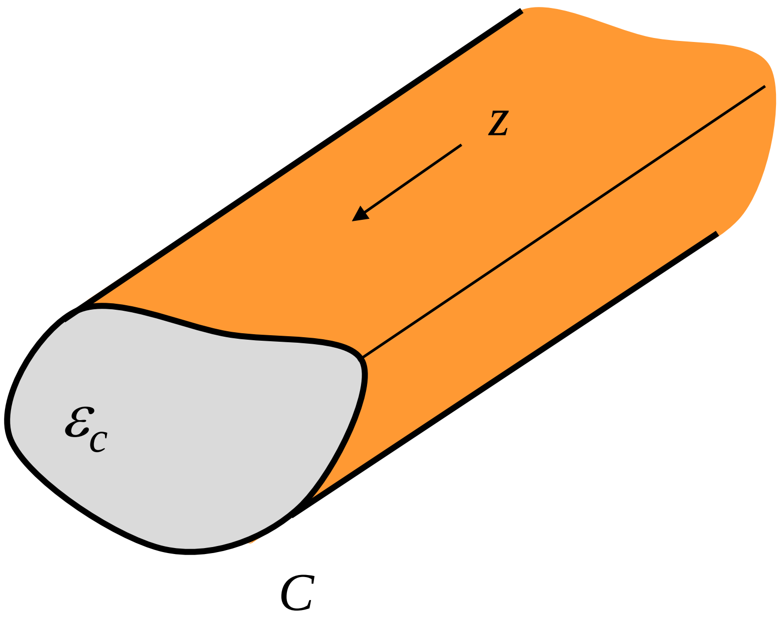
Free Space Propagation (Wireless System with Antennas)

- No need for transmitter and receiver to be physically connected
- Free space can propagate a signal at any frequency
- Free space has no dispersion
- Requires antennas, which have a limited bandwidth and may introduce dispersion
- Has $1/r^2$ propagation loss (worse then guiding systems for short distances, better than guiding systems for longer distances $(kr >> 1)$
- Usually not used for high power transfer due to propagation loss and interference

Guiding system: $\text{power} \propto e^{-2\alpha z}$

Free-space system: $\text{power} \propto A / r^2$

No TEM Mode



Theorem:

There is no TEM$_z$ mode

Proof:

Assume TEM mode ($k_z = k$):

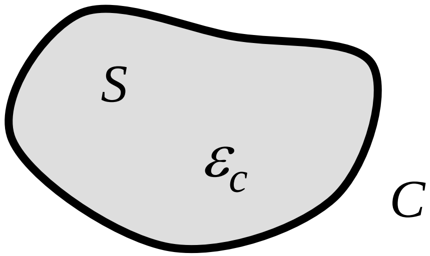
$$\underline{E}(x,y,z) = \underline{E}_{t0}(x,y)\,e^{-jkz}$$

$$\underline{E}_{t0}(x,y) = -\nabla\Phi(x,y)$$

$$\nabla^2\Phi = 0$$

No TEM Mode (cont.)

$$\begin{cases} \nabla^2 \Phi = 0 & \underline{r} \in S \\ \Phi = \text{constant} & \underline{r} \in C \end{cases}$$



Uniqueness theorem of electrostatics:

$$\Phi(x, y) = \text{constant}$$

Hence $\underline{E}_{t0} = -\nabla \Phi = \underline{0}$

$\Rightarrow$ $\underline{E}(x, y, z) = \underline{0}$

$\Rightarrow$ $\underline{H}(x, y, z) = \underline{0}$

$\Rightarrow$ trivial field

Helmholtz Equation for WG

Assume TM_z: $\nabla^2 E_z + k^2 E_z = 0$ $\quad E_z(x,y,z) = E_{z0}(x,y)\,e^{-jk_z z}$

$$\nabla^2 = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)} + \frac{\partial^2}{\partial z^2}$$

$$\nabla_t^2 = \nabla_t \cdot \nabla_t$$

$$\nabla_t = \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y}$$

so $\quad \nabla_t^2 E_z + \left(k^2 - k_z^2\right) E_z = 0$

Helmholtz Equation for WG (cont.)

$$\nabla_t^2 E_z + \left(k^2 - k_z^2\right) E_z = 0$$

Define $k_c^2 \equiv k^2 - k_z^2$ "cutoff wavenumber"

Then $\nabla_t^2 E_z + k_c^2 E_z = 0$

Note:
For a homogeneously-filled WG, k_c will be number, not a function of position.

Factoring out the z dependence,

$$\nabla_t^2 E_{z0} + k_c^2 E_{z0} = 0$$

(2-D Helmholtz equation)

(similarly for H_{z0})

Helmholtz Equation for WG (cont.)

The transverse field components of the 2D fields are given by:

$$E_{x0} = \frac{-j\omega\mu}{k^2 - k_z^2}\frac{\partial H_{z0}}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial x}$$

$$E_{y0} = \frac{j\omega\mu}{k^2 - k_z^2}\frac{\partial H_{z0}}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial y}$$

$$H_{x0} = \frac{j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial y} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_{z0}}{\partial x}$$

$$H_{y0} = \frac{-j\omega\varepsilon_c}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial x} - \frac{jk_z}{k^2 - k_z^2}\frac{\partial H_{z0}}{\partial y}$$

Property of k_c

Theorem

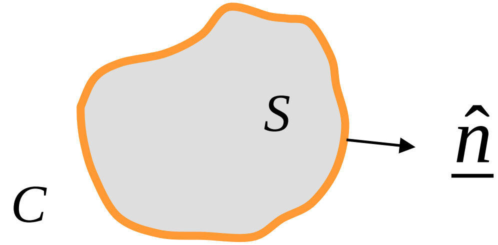
k_c = real number (even if ε_c is complex)

Proof

Let $\underline{V}(x,y) \equiv E_{z0}^{*} \nabla_t E_{z0}$

2-D divergence theorem:

$$\int_S \left(\nabla_t \cdot \underline{V}\right) dS = \oint_C \underline{V} \cdot \underline{\hat{n}}\, dl$$



Property of k_c (cont.)

On C: $$\underline{V} \cdot \underline{\hat{n}} = \cancel{E_{z0}^{*}} \left(\nabla_t E_{z0} \cdot \underline{\hat{n}} \right)$$
$$= 0$$

Hence
$$\int_S \left(\nabla_t \cdot \underline{V} \right) dS = 0$$

Therefore
$$\int_S \nabla_t \cdot \left(E_{z0}^{*} \nabla_t E_{z0} \right) dS = 0$$

Use
$$\nabla \cdot \left(\psi \, \underline{A} \right) = \nabla \psi \cdot \underline{A} + \psi \, \nabla \cdot \underline{A}$$

(This also holds for $\nabla_t \cdot$.)

or
$$\int_S \left[\left(\nabla_t E_{z0}^{*} \right) \cdot \left(\nabla_t E_{z0} \right) + E_{z0}^{*} \, \nabla_t^2 E_{z0} \right] dS = 0$$

Property of k_c (cont.)

Hence

$$\int_S \left|\nabla_t E_{z0}\right|^2 dS - \int_S E_{z0}^* k_c^2 E_{z0}\, dS = 0$$

or

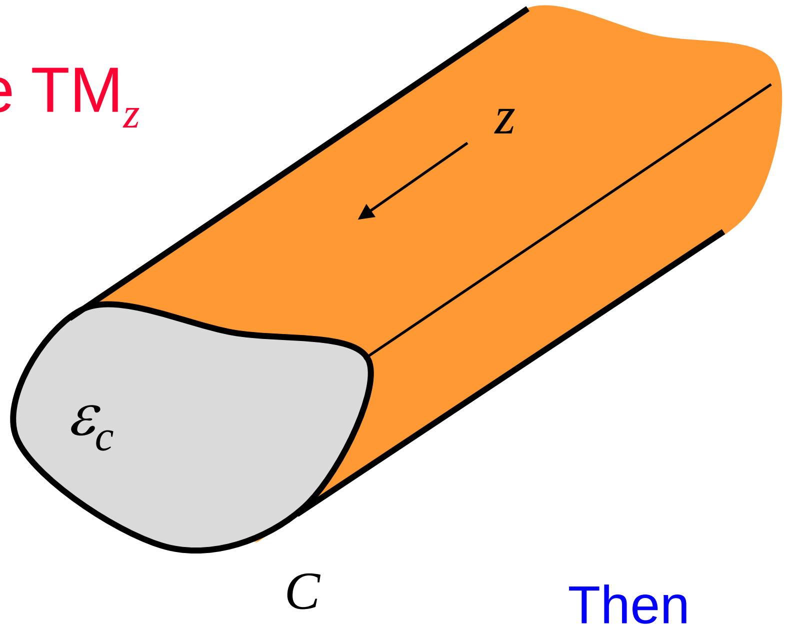
$$\boxed{k_c^2 = \frac{\displaystyle\int_S \left|\nabla_t E_{z0}\right|^2 dS}{\displaystyle\int_S \left|E_{z0}\right|^2 dS}}$$

Hence k_c^2 = real positive number

Note: This is also a <u>variational</u> equation for k_c^2 (proof omitted).

WG Eigenvalue Problem

Assume TM_z



$$\nabla_t^2 E_{z0} + k_c^2 E_{z0} = 0$$

Denote

$$\psi(x,y) = E_{z0}(x,y)$$

$$\lambda = -k_c^2 \text{ (eigenvalue)}$$

Then

$$\nabla_t^2 \psi(x,y) = \lambda \psi(x,y)$$
$$\psi(x,y) = 0\big|_C$$

Note: λ is independent of frequency and permittivity (from the form of the eigenvalue problem).

WG Eigenvalue Problem (cont.)

Assume TE$_z$

Let $$\psi\left(x,y\right)=H_{z0}(x,y)$$

$$\lambda=-k_c^2$$

Then

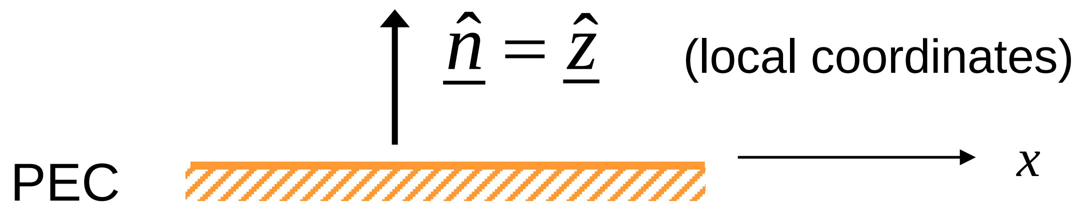
$$\nabla_t^{\;2}\psi\left(x,y\right)=\lambda\psi\left(x,y\right)$$

$$\frac{\partial\psi\left(x,y\right)}{\partial n}=0\Big|_C$$

Note:
The B.C. follows from Ampere's law (proof on next slide).

Once again: λ is independent of frequency and permittivity (from the form of the eigenvalue problem).

WG Eigenvalue Problem (cont.)



$$\boxed{E_x = E_y = 0}$$

$$\nabla \times \underline{H} = j\omega\varepsilon_c \underline{E}$$

Hence

$$\left.\begin{aligned} \frac{\partial \cancel{H_z}}{\partial y} - \frac{\partial H_y}{\partial z} &= 0 \\ \frac{\partial \cancel{H_z}}{\partial x} - \frac{\partial H_x}{\partial z} &= 0 \end{aligned}\right\} \quad \boxed{\frac{\partial \underline{H}_t}{\partial n} = \underline{0}}$$

Note: We also have $H_n = 0$ on PEC.

Summary of Properties for k_c

- The value k_c is always real, even if the waveguide is filled with a lossy material.
- The value of k_c does not depend on the frequency or the material inside the waveguide.
- The value of k_c only depends on the geometrical shape of the waveguide cross section.

"Real" Theorem for Field

Assume TM_z (A similar result holds for H_{z0} of a TE_z mode.)

$$E_{z0}(x,y) = c_1 \, R_1(x,y)$$

(This is valid even if the medium is lossy.)

where $R_1(x,y)$ is a real function, and c_1 is a complex constant.

We assume that the waveguide shape is such that it does not support degenerate modes (more than one mode with the same value of k_c, and hence the same value of wavenumber k_z).

⇓

The shape of the waveguide field corresponding to a particular k_c is unique.

"Real" Theorem for Field

Proof

$$\nabla_t^2 E_{z0} + k_c^2 E_{z0} = 0$$ Take the real part of both sides.

Note that $$\mathrm{Re}(\nabla_t^2 E_{z0}) = \nabla_t^2 \,\mathrm{Re}\, E_{z0}$$

Hence

$$\nabla_t^2 \,\mathrm{Re}\, E_z + k_c^2 \,\mathrm{Re}\, E_z = 0$$

and

$$\mathrm{Re}\, E_z = 0 \;\text{ on } C$$

"Real" Theorem for Field (cont.)

$$\nabla_t^2 \operatorname{Re} E_{z0} + k_c^2 \operatorname{Re} E_{z0} = 0$$

$$\operatorname{Re} E_{z0} = 0 \text{ on } C$$

$$\nabla_t^2 \psi(x,y) = \lambda \psi(x,y)$$

$$\psi(x,y) = 0\big|_C$$

(same eigenvalue problem)

Similarly,

$$\nabla_t^2 \operatorname{Im} E_{z0} + k_c^2 \operatorname{Im} E_{z0} = 0$$

$$\operatorname{Im} E_{z0} = 0 \text{ on } C$$

The solution of the eigenvalue problem for a given eigenvalue is unique to within a multiplicative constant (we assumed nondegenerate modes).

Hence $\operatorname{Im} E_{z0}(x,y) = A \operatorname{Re} E_{z0}(x,y)$

A = real constant

“Real” Theorem for Field (cont.)

Therefore

$$E_{z0}(x,y) = \operatorname{Re} E_{z0}(x,y)[1 + jA]$$
$$= R(x,y)\, c_1$$

(proof complete)

Corollary

$$\underline{E}_{t0}(x,y) = c_2\,\underline{R}_2(x,y)$$

where $R_2\,(x,y)$ is a real vector function, and c_2 is a complex constant.

Note: The transverse field may be out of phase from the longitudinal field (they cannot both be assumed to be real at the same time, in general).

Proof

Assume a TM$_z$ mode (similar proof for TE$_z$ mode).

From the guided-wave equations:

$$E_{x0} = -\frac{jk_z}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial x}$$

$$E_{y0} = -\frac{jk_z}{k^2 - k_z^2}\frac{\partial E_{z0}}{\partial y}$$

Corollary (cont.)

Hence

$$\underline{E}_{t0} = \left(\frac{-jk_z}{k^2 - k_z^2}\right)\nabla_t E_{z0}$$

$$= \underbrace{\left(\frac{-jk_z}{k^2 - k_z^2}\right)}_{c_3}\underbrace{\nabla_t\left(c_1 R(x,y)\right)}_{c_1 \underline{R}_2}$$

$$\underline{R}_2 \equiv \nabla_t R(x,y)$$

Therefore

$$\underline{E}_{t0} = c_2 \underline{R}_2(x,y)$$ with $$c_2 = c_1 c_3$$

Wavenumber

$$k_c^2 \equiv k^2 - k_z^2 = \text{real number}$$

(The value of k_c is independent of frequency and material.)

so
$$k_z = \left(k^2 - k_c^2\right)^{1/2}$$

$$k^2 = \omega^2 \mu \varepsilon_c = k_0^2 \mu_r \varepsilon_{rc}$$

The wavenumber is written as

$$k_z = \beta - j\alpha$$

Attenuation constant

Phase constant

Wavenumber: Lossless Case

$$\varepsilon_c = \varepsilon - j\left(\frac{\sigma}{\omega}\right) = \varepsilon_c' - \cancel{j\varepsilon_c''}$$

$$\varepsilon_c = \varepsilon = \varepsilon' = \text{real}$$

No conductivity

No polarization loss

$$\mu = \mu' = \text{real}$$

$$k_z = \left(\omega^2 \mu \varepsilon - k_c^2\right)^{1/2}$$

real

Conclusion: k_z = real or imaginary

Lossless Case: Cutoff Frequency

$$k_z = \left(\omega^2 \mu \varepsilon - k_c^2\right)^{1/2}$$

Cut-off frequency ω_c: $k_z = 0$ $\omega_c^2 \mu \varepsilon = k_c^2$

Hence

$$\boxed{f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}} k_c}$$

Note: $\boxed{k_c = k\big|_{\omega=\omega_c}}$ (This is a physical interpretation of k_c)

Lossless Case: Wavenumber

Above cutoff $f > f_c$

$$k_z = \sqrt{k^2 - k_c^2}$$
$$= \sqrt{\omega^2 \mu \varepsilon - \omega_c^{\,2} \mu \varepsilon}$$
$$= \omega \sqrt{\mu \varepsilon} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$
$$= k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\beta = k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$
$$\alpha = 0$$

Lossless Case: Wavenumber (cont.)

Below cutoff $f < f_c$

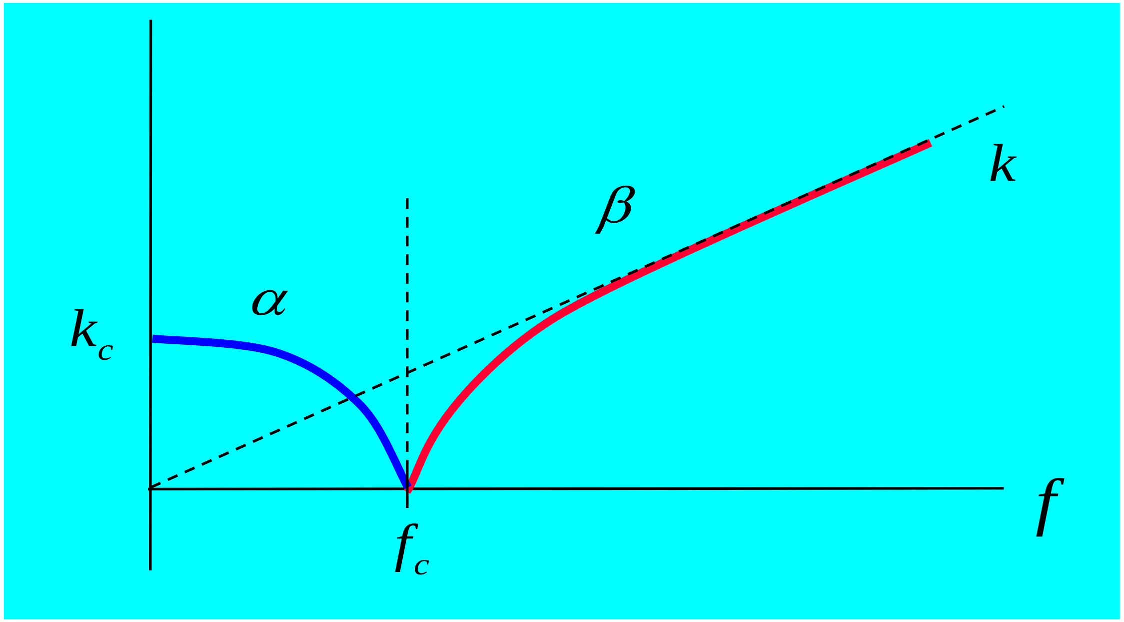
$$
\begin{aligned}
k_z &= \left(k^2 - k_c^2\right)^{1/2} \\
&= -j\sqrt{k_c^2 - k^2} \\
&= -jk_c\sqrt{1-\left(\frac{k}{k_c}\right)^2} \\
&= -jk_c\sqrt{1-\left(\frac{f}{f_c}\right)^2}
\end{aligned}
$$

$$\beta = 0$$

$$\alpha = k_c\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$

Lossless Case: Cutoff Frequency

Lossless (k = real)



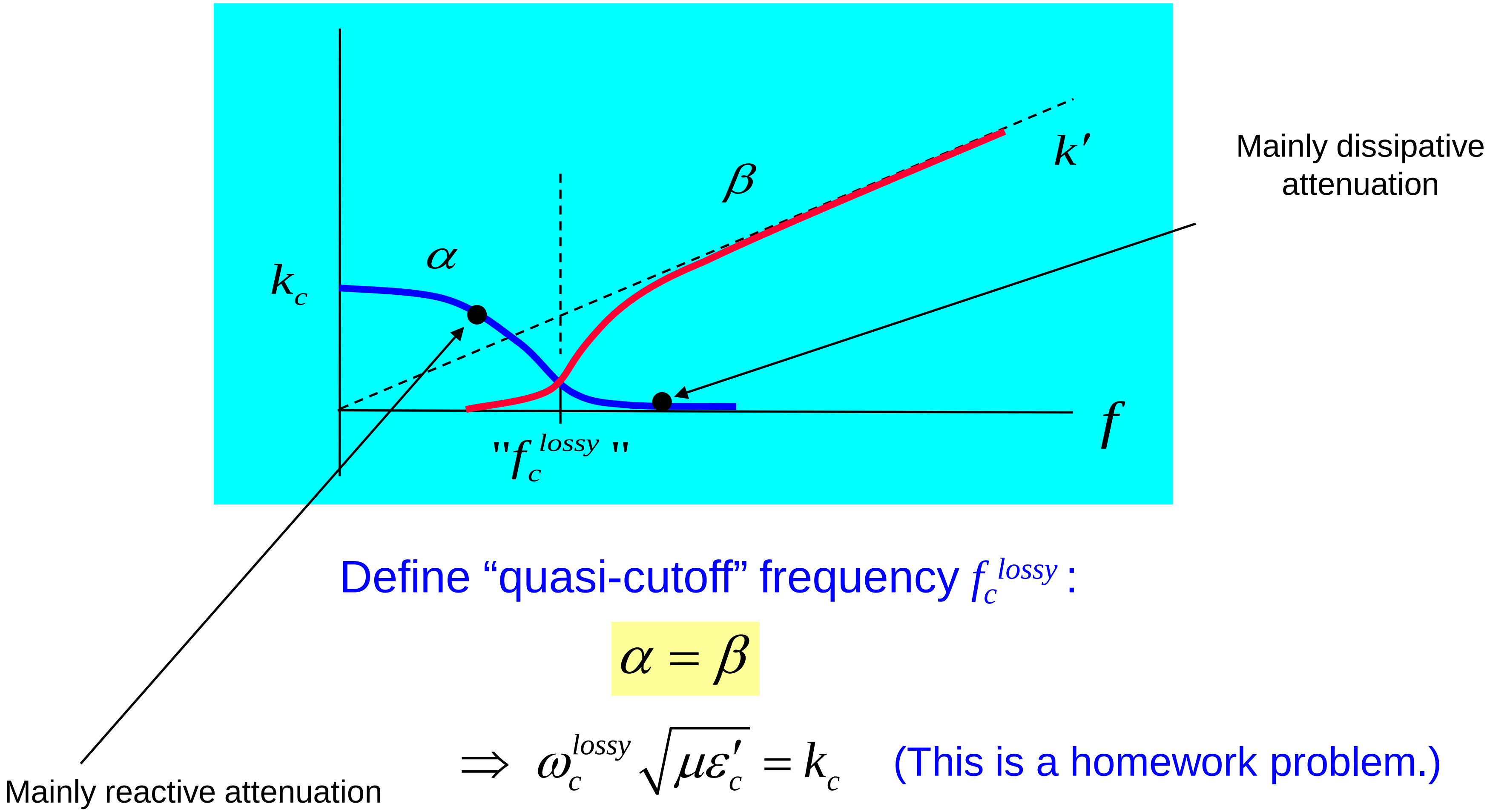
Cutoff frequency f_c:

$$k_z = 0$$

$$(\beta = 0,\ \alpha = 0)$$

Lossy Case: Cutoff Frequency

Lossy $(k = k' - jk'')$



Define "quasi-cutoff" frequency f_c^{lossy}:

$$\alpha = \beta$$

$$\Rightarrow \omega_c^{lossy}\sqrt{\mu\varepsilon_c'} = k_c$$ (This is a homework problem.)

Also $\Rightarrow$ watts = VARS (This is a homework problem.)

Lossless Case: Phase Velocity

Above cutoff: $f > f_c$

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\left[k\sqrt{1-\left(\frac{f_c}{f}\right)^2} \right]}$$

$$= \frac{\omega}{\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

so

$$v_p = \frac{c_d}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

where

$$c_d = \frac{1}{\sqrt{\mu\varepsilon}}$$

Lossless Case: Group Velocity

Above cutoff: $f > f_c$

$$\beta^2 = k^2 - k_c^2 = \omega^2 \mu \varepsilon - k_c^2$$

$$2\beta d\beta = 2\omega d\omega \mu\varepsilon$$

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\mu\varepsilon}\frac{\beta}{\omega} = c_d^2 \frac{1}{v_p}$$

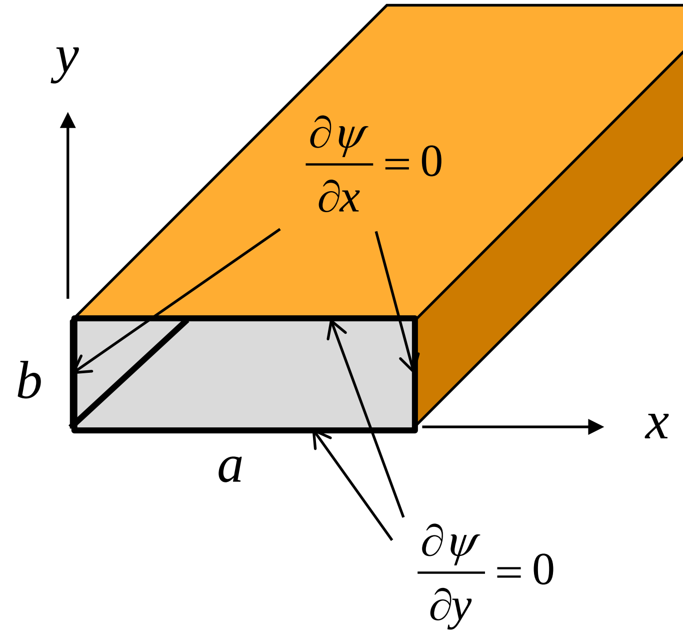
so

$$\boxed{v_p v_g = c_d^2}$$

$$\boxed{v_g = c_d \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Example: Rectangular Waveguide

Assume TE_z mode



The waveguide may be filled with a lossy material.

Eigenvalue Problem:

$$\nabla_t^{\,2} \psi = \lambda \psi$$

$$\frac{\partial \psi}{\partial n} = 0 \Big|_c$$

where

$$\psi(x, y) = H_{z0}(x, y)$$

$$\lambda = -k_c^2$$

Rectangular Waveguide (cont.)

Separation of variables:

Assume: $$\boxed{\psi(x,y) = X(x)Y(y)}$$

so $$X''Y + XY'' = \lambda XY$$

or $$\frac{X''}{X} + \frac{Y''}{Y} = \lambda$$ or $$\frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

Hence $$\frac{X''}{X} = \text{constant} = -k_x^2$$

Rectangular Waveguide (cont.)

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$X'(x) = Ak_x\cos(k_x x) - Bk_x\sin(k_x x)$$

B.C.'s:

$$X'(0) = 0 \Rightarrow A = 0$$ (see note below)

$$X'(a) = 0 \Rightarrow \sin(k_x a) = 0$$

$$\Rightarrow k_x a = m\pi$$

$$\Rightarrow k_x = \frac{m\pi}{a}$$

Hence

$$X(x) = \cos\left(\frac{m\pi x}{a}\right)$$ (setting $B = 1$)

Note: We could also have $k_x = 0$, but this is included as a special case where $A = 0$ and $m = 0$)

Rectangular Waveguide (cont.)

Similarly, $$\frac{Y''}{Y} = -k_y^2 \qquad k_y = \frac{n\pi}{b}$$

$$Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

Hence $$\psi(x, y) = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda$$ so $$\lambda = -k_x^2 - k_y^2 = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Rectangular Waveguide (cont.)

Summary

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$k = k_0\sqrt{\varepsilon_{rc}}$$

Rectangular Waveguide (cont.)

TE$_{10}$ Mode

$$k_c = \frac{\pi}{a} \qquad k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$\underline{E}_t(x,y,z) = -\underline{\hat{y}}\, H_0 \left(\frac{j\omega\mu}{k^2 - k_z^2}\right)\left(\frac{\pi}{a}\right) \sin\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$\underline{H}_t(x,y,z) = -\underline{\hat{x}} \left(\frac{1}{Z_{TE}}\right) H_0 \left(\frac{j\omega\mu}{k^2 - k_z^2}\right)\left(\frac{\pi}{a}\right) \sin\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$Z_{TE} = \frac{\omega\mu}{k_z} = \frac{\eta}{\sqrt{1 - \left(\frac{\pi}{ka}\right)^2}}$$

Rectangular Waveguide (cont.)

TE$_{10}$ Mode

(after simplifying)

$$k_c = \frac{\pi}{a}$$

$$k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$\underline{E}_t(x,y,z) = -j\,\underline{\hat{y}}\,H_0\left(\frac{\omega\mu a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

$$\underline{H}_t(x,y,z) = -j\,\underline{\hat{x}}\left(\frac{1}{Z_{TE}}\right)H_0\left(\frac{\omega\mu a}{\pi}\right)\sin\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

Note that the transverse fields are both 90° out of phase from the longitudinal field,
If the material is lossless and we are above cutoff (Z_{TE} is real).

Rectangular Waveguide (cont.)

TE$_{10}$ Mode

$$H_{z0}(x,y) = H_0 \cos\left(\frac{\pi x}{a}\right) = c_1 R(x,y)$$

$$\underline{E}_{t0}(x,y) = \left(-jH_0\mu\right)\left(\frac{\omega a}{\pi}\right)\left(\underline{\hat{y}}\sin\left(\frac{\pi x}{a}\right)\right) = c_2\underline{R}_1(x,y)$$

$$\underline{H}_{t0}(x,y) = \left(\frac{-j H_0\mu}{Z_{TE}}\right)\left(\frac{\omega a}{\pi}\right)\left(\underline{\hat{x}}\sin\left(\frac{\pi x}{a}\right)\right) = c_3\underline{R}_2(x,y)$$

We see that the results are consistent with the “Real” theorem and its corollary.