ECE 6340
Intermediate EM Waves

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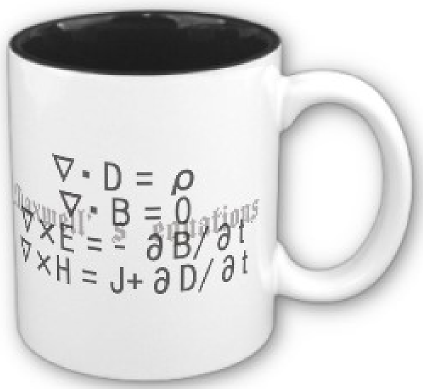


Notes 12

Wave Impedance

Assume TM_z wave traveling in the $+z$ direction

From previous notes:

$$\underline{E}_t^{+} = -Z^{TM}\left(\underline{\hat{z}} \times \underline{H}_t^{+}\right)$$

so $$\underline{\hat{z}} \times \underline{E}_t^{+} = -Z^{TM}\,\underline{\hat{z}} \times \left(\underline{\hat{z}} \times \underline{H}_t^{+}\right)$$

or $$\underline{\hat{z}} \times \underline{E}_t^{+} = -Z^{TM}\left(-\underline{H}_t^{+}\right)$$

$$\underline{H}_t^{+} = \frac{1}{Z^{TM}}\left(\underline{\hat{z}} \times \underline{E}_t^{+}\right)$$

$$Z^{TM} = \frac{k_z}{\omega \varepsilon_c} = \eta \frac{k_z}{k}$$

Wave Impedance (cont.)

For -z wave: $e^{-jk_z z} \rightarrow e^{+jk_z z} = e^{-j(-k_z)z}$ so we can simply substitute

$$k_z \rightarrow -k_z$$

$$\underline{H}_t^- = -\frac{1}{Z^{TM}}\left(\underline{\hat{z}} \times \underline{E}_t^-\right)$$

In this formula the wavenumber and the wave impedance are taken to be the same for negative and positive traveling waves.

Summary:

$$\underline{H}_t^{\pm} = \pm\frac{1}{Z^{TM}}\left(\underline{\hat{z}} \times \underline{E}_t^{\pm}\right)$$

+ sign: +z wave

- sign: - z wave

Wave Impedance (cont.)

TE$_z$ wave:

$$\underline{H}_t^{\pm} = \pm \frac{1}{Z^{TE}} \left(\underline{\hat{z}} \times \underline{E}_t^{\pm} \right)$$

$$Z^{TE} = \frac{\omega \mu}{k_z} = \eta \frac{k}{k_z}$$

Note: $Z^{TE} Z^{TM} = \eta^2$

Transverse Equivalent Network (TEN)

Denote

$$\underline{E}_t(x,y,z) = \underline{E}_{t0}(x,y)\,V(z)$$

where

$$V(z) = V^+ e^{-jk_z z} + V^- e^{+jk_z z}$$

Note: $V(z)$ behaves as a <u>voltage</u> function.

Also, write

$$\underline{E}_{t0}(x,y) = \underline{\hat{e}}_{TM}(x,y)\,\psi_t(x,y)$$

(assume TM_z wave)

Note: We can assume that both the unit vector and the transverse amplitude function are both real (because of the corollary to the "real" theorem).

Hence we have

$$\underline{E}_t(x,y,z) = \underline{\hat{e}}_{TM}(x,y)\,\psi_t(x,y)\,V(z)$$

Note: There is still flexibility here, since there can be any real scaling constant in the definition of ψ_t. We have not uniquely determined ψ_t yet.

Transverse Equivalent Network (cont.)

For the magnetic field we have

$$\underline{H}_t^{\pm} = \pm \frac{1}{Z^{TM}} \left(\underline{\hat{z}} \times \underline{E}_t^{\pm} \right) \quad \Rightarrow \quad \underline{H}_t = \frac{1}{Z^{TM}} \left(\underline{\hat{z}} \times \underline{E}_t^{+} - \underline{\hat{z}} \times \underline{E}_t^{-} \right)$$

Now use $\underline{E}_t^{\pm}(x,y,z) = \underline{\hat{e}}_{TM}(x,y)\psi_t(x,y)V^{\pm}(z)$

$$V^{\pm}(z) = V^{\pm} e^{\mp j k_z z}$$

The result for the transverse magnetic field is then

$$\begin{cases} \underline{H}_t = \dfrac{1}{Z^{TM}} \left(\underline{\hat{z}} \times \underline{\hat{e}}_{TM} \right) \psi_t(x,y)\, V^{+} e^{-j k_z z} \\ \qquad - \dfrac{1}{Z^{TM}} \left(\underline{\hat{z}} \times \underline{\hat{e}}_{TM} \right) \psi_t(x,y)\, V^{-} e^{+j k_z z} \end{cases}$$

Transverse Equivalent Network (cont.)

Define

$$\underline{\hat{h}}_{TM}(x,y) \equiv \underline{\hat{z}} \times \underline{\hat{e}}_{TM}(x,y)$$

and

$$I(z) = \frac{V^{+}e^{-jk_z z}}{Z^{TM}} - \frac{V^{-}e^{+jk_z z}}{Z^{TM}}$$

Note: $I(z)$ behaves as a transmission-line current function.

We then have

$$\underline{H}_t(x,y,z) = \underline{\hat{h}}_{TM}(x,y)\,\psi_t(x,y)\,I(z)$$

Note: We could also introduce a different characteristic impedance ($Z_0 \neq Z^{TM}$), and then have a different ψ_t function for the magnetic field.

Transverse Equivalent Network (cont.)

Summary

$$\underline{E}_t(x,y,z) = \underline{\hat{e}}_{TM}(x,y)\,\psi_t(x,y)\,V(z)$$

$$\underline{H}_t(x,y,z) = \underline{\hat{h}}_{TM}(x,y)\,\psi_t(x,y)\,I(z)$$

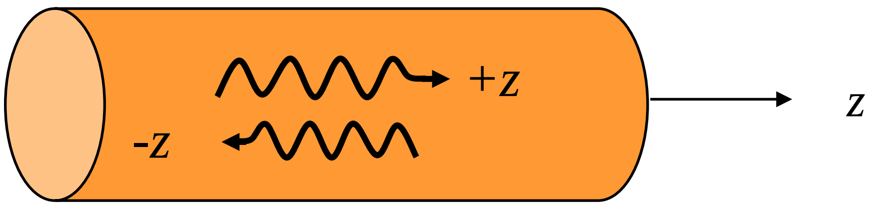
$$V(z) = V^+ e^{-jk_z z} + V^- e^{+jk_z z}$$

$$I(z) = \frac{V^+ e^{-jk_z z}}{Z^{TM}} - \frac{V^- e^{+jk_z z}}{Z^{TM}}$$

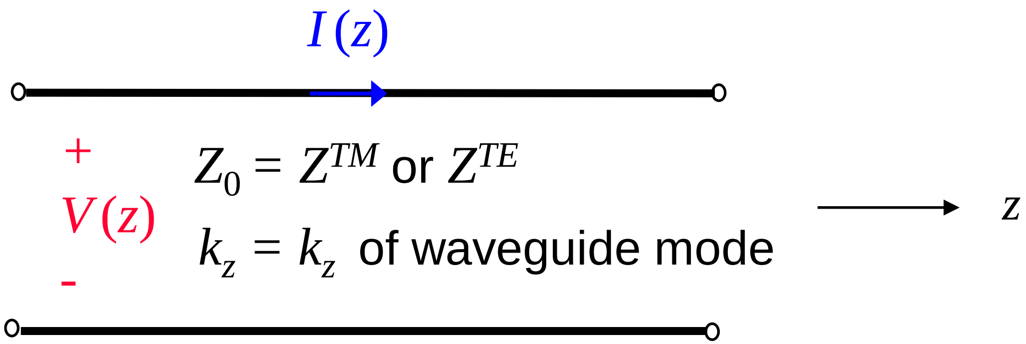
$$\underline{\hat{h}}_{TM}(x,y) \equiv \underline{\hat{z}} \times \underline{\hat{e}}_{TM}(x,y)$$

Transverse Equivalent Network (cont.)

Waveguide



TEN



$Z_0 = Z^{TM}$ or Z^{TE}

$k_z = k_z$ of waveguide mode

$$V(z) \leftrightarrow \underline{E}_t$$

$$I(z) \leftrightarrow \underline{H}_t$$

Transverse Equivalent Network (cont.)

Power flowing in waveguide

TEN: $$P^{TEN} = \frac{1}{2} V I^{*}$$

WG: $$S_z^{WG} = \frac{1}{2}\left(\underline{E} \times \underline{H}^{*}\right)\cdot \underline{\hat{z}} = \frac{1}{2}\left(\underline{E}_t \times \underline{H}_t^{*}\right)\cdot \underline{\hat{z}}$$

$$= \frac{1}{2}\left[\underline{\hat{e}}_{TM}(x,y)\psi_t(x,y)V(z)\right]\times\left[\underline{\hat{h}}_{TM}(x,y)\psi_t(x,y)I^{*}(z)\right]\cdot \underline{\hat{z}}$$

$$= \frac{1}{2} V I^{*} \left|\psi_t(x,y)\right|^2 \left(\underline{\hat{e}}_{TM} \times \underline{\hat{h}}_{TM}\right)\cdot \underline{\hat{z}}$$

$$\left[\underline{\hat{e}}_{TM} \times \left(\underline{\hat{h}}_{TM}\right)\right]\cdot \underline{\hat{z}} = \left[\underline{\hat{e}}_{TM} \times \left(\underline{\hat{z}} \times \underline{\hat{e}}_{TM}\right)\right]\cdot \underline{\hat{z}} = \left[\underline{\hat{z}}\right]\cdot \underline{\hat{z}} = 1$$

Note : $\underline{A}\times\left(\underline{B}\times\underline{C}\right) = \underline{B}\left(\underline{A}\cdot\underline{C}\right) - \underline{C}\left(\underline{A}\cdot\underline{B}\right)$

Hence $$S_z^{WG} = \frac{1}{2} V I^{*} \left|\psi_t(x,y)\right|^2$$

Transverse Equivalent Network (cont.)

The total complex power flowing down the waveguide is then

$$P^{WG} = \int_S S_z^{WG} dS = \frac{1}{2} V I^* \int_S \left|\psi_t(x,y)\right|^2 dS$$

or

$$P^{WG} = P^{TEN} \int_S \left|\psi_t(x,y)\right|^2 dS$$

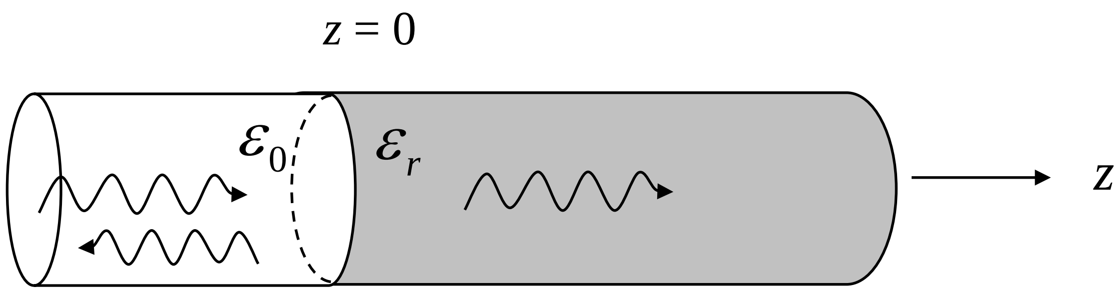
If we <u>choose</u> $\int_S \left|\psi_t(x,y)\right|^2 dS = 1$

Note: This uniquely defines ψ_t, and hence voltage and current.

then $P^{WG} = P^{TEN}$

TEN: Junction

Junction:



same cross sectional shape

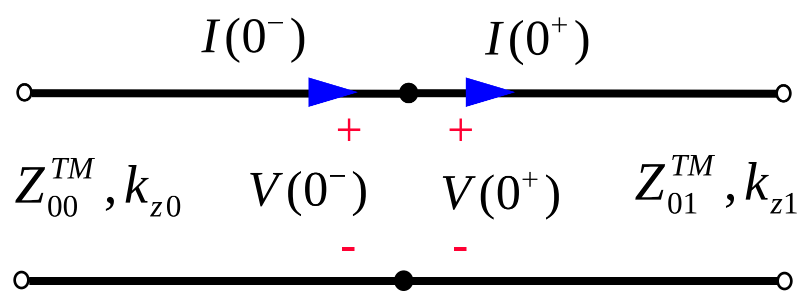
At $z = 0$:

$$\underline{E}_t\left(0^+\right) = \underline{E}_t\left(0^-\right) \qquad \underline{H}_t\left(0^+\right) = \underline{H}_t\left(0^-\right)$$

(EM boundary conditions)

Hence $\quad V(0^+) = V(0^-) \qquad I(0^+) = I(0^-)$

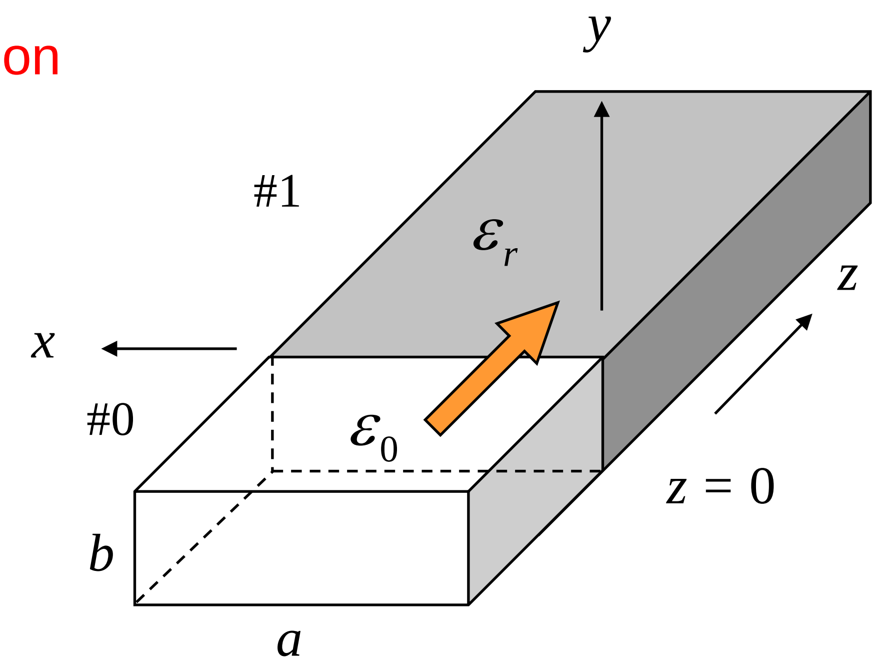
These conditions are also true at a TL junction.



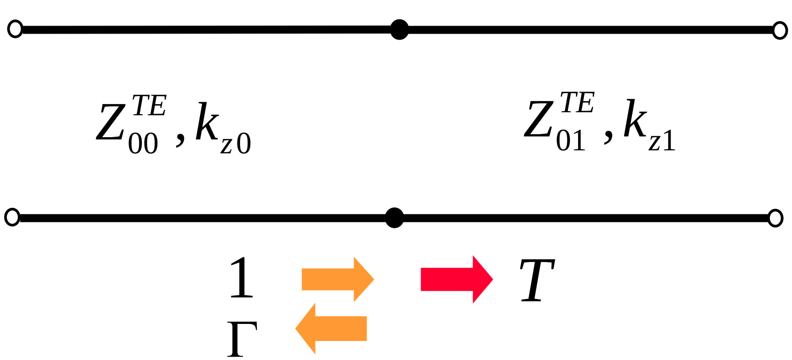
Example: Rectangular Waveguide

TE$_{10}$ mode incident at a junction

The TEN method gives us an exact solution since the two WGs have the same cross sectional shape (the same ψ_t function).



TEN:



$$\Gamma = \frac{Z_{01}^{TE} - Z_{00}^{TE}}{Z_{01}^{TE} + Z_{00}^{TE}}$$

$$T = 1 + \Gamma$$

Example: Rectangular Waveguide (cont.)

$$Z_{00}^{TE} = \frac{\omega\mu_0}{k_{z0}} = \frac{\omega\mu_0}{\sqrt{k_0^2 - \left(\frac{\pi}{a}\right)^2}}$$

$$= \frac{\omega\mu_0 / k_0}{\sqrt{1 - \left(\frac{\pi}{k_0 a}\right)^2}}$$

$$= \frac{\eta_0}{\sqrt{1 - \left(\frac{\pi}{k_0 a}\right)^2}}$$

Similarly,

$$Z_{01}^{TE} = \frac{\eta_1}{\sqrt{1 - \left(\frac{\pi}{k_1 a}\right)^2}}$$

$$\eta_0 = 376.730313461771 \; [\Omega]$$

Example (Numerical Results)

X-band waveguide

$f = 10$ GHz

$$a = 2.2225 \quad [\text{cm}]$$

$$b = 1.0319 \quad [\text{cm}]$$

Choose $\varepsilon_r = 2.2$ (teflon)

$$f_c = 6.749 \quad [\text{GHz}] \quad \text{(air-filled guide)}$$

$$Z_{00}^{TE} = 510.25 \quad [\Omega] \qquad Z_{01}^{TE} = 285.18 \quad [\Omega]$$

$$k_{z0} = 154.74 \quad [\text{rad/m}] \qquad k_{z1} = 276.87 \quad [\text{rad/m}]$$

The results are:

$$\Gamma = -0.28295$$

$$T = 0.71705$$

Example (cont.)

$$P_r^{\%} = 100\left(\left|\Gamma\right|\right)^2 = 100\left(\left|-0.28295\right|\right)^2 = 8.00$$

$$P_t^{\%} = 100 - 8.00 = 92.0$$ (conservation of energy and orthogonality)

$$P_r^{\%} = 8.00$$

$$P_t^{\%} = 92.0$$

$$P_z\left(0^-\right) = P_z\left(0^+\right)$$ conservation of energy

$$P_z\left(0^-\right) = P_z^{inc} + P_z^{ref}$$ othogonality (in Notes 13)

$$\Rightarrow P_z^{inc} + P_z^{ref} = P_z^{trans}$$

$$\Rightarrow P_z^{inc} - P_z^{inc}\left|\Gamma\right|^2 = P_z^{trans}$$

Note:

$$P_t^{\%} \neq 100\left|T\right|^2$$

(This is because the impedances of the two guides are different.)

Example (cont.)

Alternative calculation:

$$P^{WG} = P^{TEN} \int_S \left|\psi_t(x,y)\right|^2 dS$$

$$P_t^{\%} = 100\left(\frac{\dfrac{\left|V^t\right|^2}{2Z_{01}^{TE}} \displaystyle\int_S \left|\psi_t(x,y)\right|^2 dS}{\dfrac{\left|V^+\right|^2}{2Z_{00}^{TE}} \displaystyle\int_S \left|\psi_t(x,y)\right|^2 dS} \right) = 100\left(\frac{\dfrac{\left|T\right|^2}{2Z_{01}^{TE}}}{\dfrac{\left|1\right|^2}{2Z_{00}^{TE}}} \right)$$

$$P_t^{\%} = 100\left(\frac{Z_{00}^{TE}}{Z_{01}^{TE}} \right)\left|T\right|^2$$

$$P_t^{\%} = 92.0$$

Example (cont.)

Fields in the waveguides

$$\underline{E}_t(x,y,z) = \underline{\hat{e}}_{TE}(x,y)\,\psi_t(x,y)\,V(z)$$

$$\underline{H}_t(x,y,z) = \underline{\hat{h}}_{TE}(x,y)\,\psi_t(x,y)\,I(z)$$

TE$_{10}$ mode:

$$\underline{\hat{e}}_{TE}(x,y) = \underline{\hat{y}}$$

$$\underline{\hat{h}}_{TE}(x,y) = -\underline{\hat{x}}$$

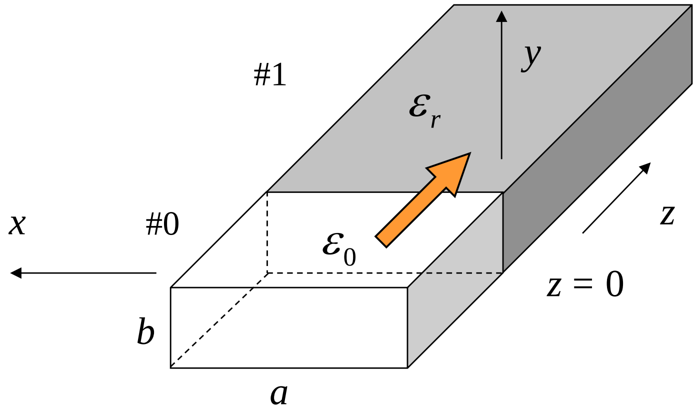
$$\psi_t(x,y) = \sin\left(\frac{\pi x}{a}\right)$$

$$\underline{\hat{h}}_{TE}(x,y) \equiv \underline{\hat{z}} \times \underline{\hat{e}}_{TE}(x,y)$$

This choice of ψ_t does not correspond to equal powers for the WG and the TEN.

$$P^{WG} = P^{TEN}\left(\frac{ab}{2}\right)$$

$V(z) = E_y$ at the center of the WG



Example (cont.)

$$\underline{E}_t(x,y,z)=\underline{\hat{e}}_{TE}(x,y)\psi_t(x,y)V(z)$$
$$\underline{H}_t(x,y,z)=\underline{\hat{h}}_{TE}(x,y)\psi_t(x,y)I(z)$$

$$\underline{\hat{e}}_{TE}(x,y)=\underline{\hat{y}}$$
$$\underline{\hat{h}}_{TE}(x,y)=-\underline{\hat{x}}$$
$$\psi_t(x,y)=\sin\left(\frac{\pi x}{a}\right)$$

Air region:

TL part

$$\underline{E}_t(x,y,z)=\underline{\hat{y}}\sin\left(\frac{\pi x}{a}\right)\left(e^{-jk_{z0}z}+\Gamma e^{jk_{z0}z}\right)$$

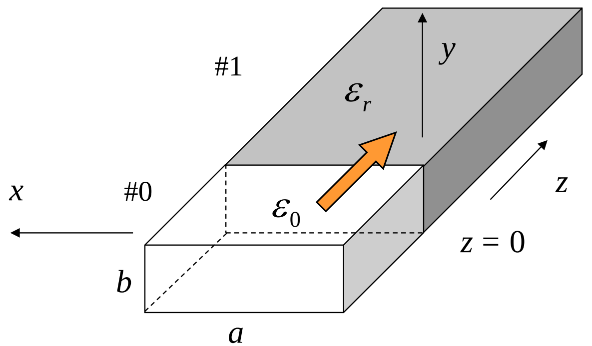
$$\underline{H}_t(x,y,z)=-\underline{\hat{x}}\sin\left(\frac{\pi x}{a}\right)\frac{1}{Z_{00}^{TE}}\left(e^{-jk_{z0}z}-\Gamma e^{jk_{z0}z}\right)$$

$$V^{air}(z)=e^{-jk_{z0}z}+\Gamma e^{jk_{z0}z}$$
$$I^{air}(z)=\frac{1}{Z_{00}^{TE}}\left(e^{-jk_{z0}z}-\Gamma e^{jk_{z0}z}\right)$$
$$V^{diel}(z)=Te^{-jk_{z1}z}$$
$$I^{diel}(z)=\frac{1}{Z_{01}^{TE}}\left(Te^{-jk_{z1}z}\right)$$

Dielectric region:

$$\underline{E}_t(x,y,z)=\underline{\hat{y}}\sin\left(\frac{\pi x}{a}\right)\left(Te^{-jk_{z1}z}\right)$$

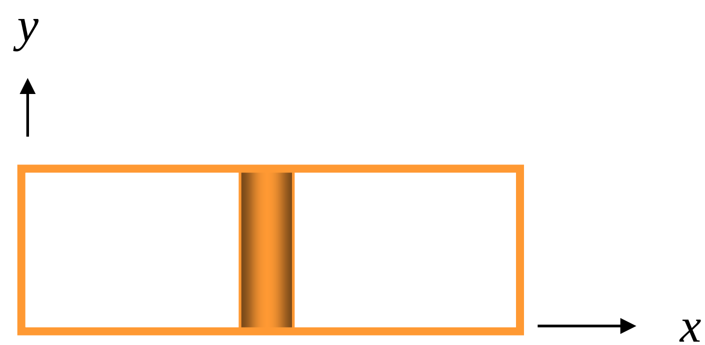
$$\underline{H}_t(x,y,z)=-\underline{\hat{x}}\sin\left(\frac{\pi x}{a}\right)\frac{1}{Z_{01}^{TE}}\left(Te^{-jk_{z1}z}\right)$$



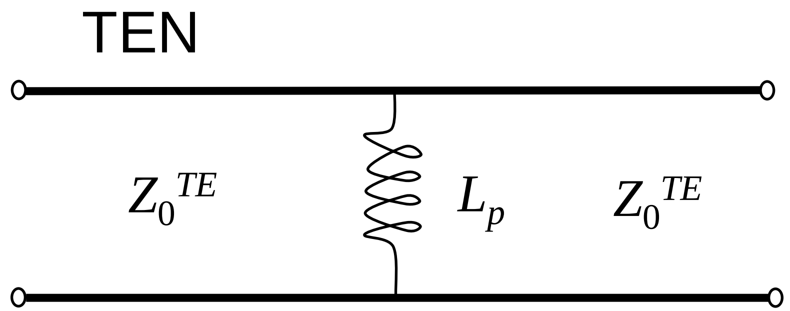
Matching Elements

A couple of commonly used matching elements:

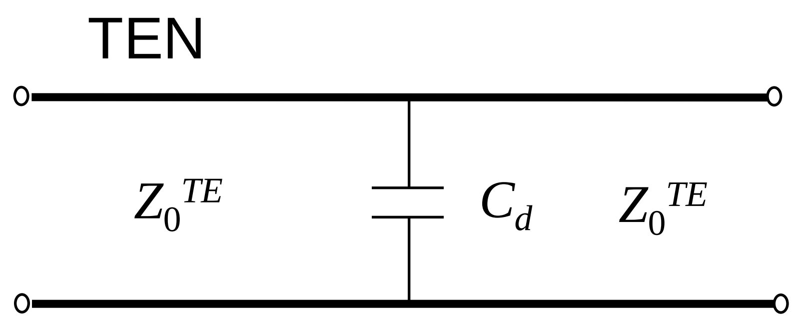
$$Z_0^{TE} = \frac{\omega\mu_0}{k_z^{10}}$$



Inductive post (narrow)



Capacitive diagraphragm (iris)

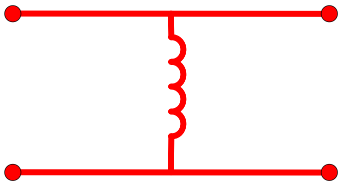


Matching Elements (cont.)

Note: Planar discontinuities are modeled as purely <u>shunt</u> elements.

End view

⇒



Inductive iris

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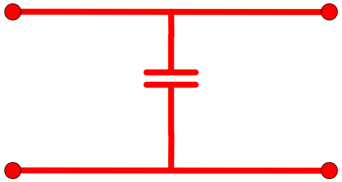

Capacitive iris

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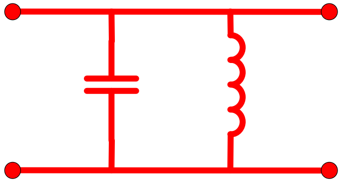
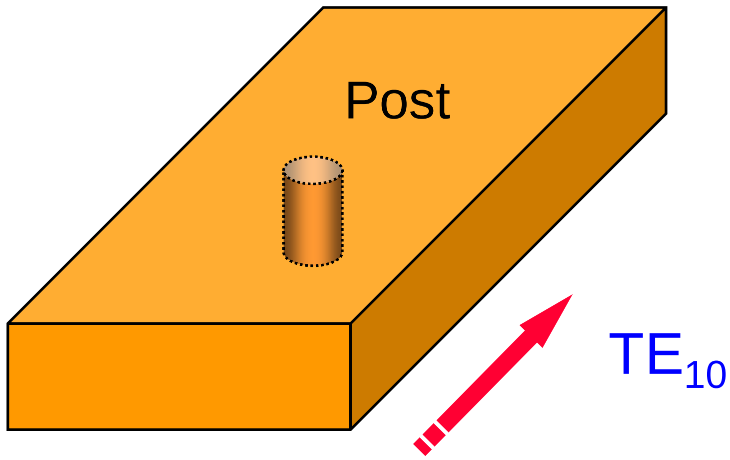


Resonant iris

The equivalent circuit gives us the correct reflection and transmission for the dominant TE_{10} mode.

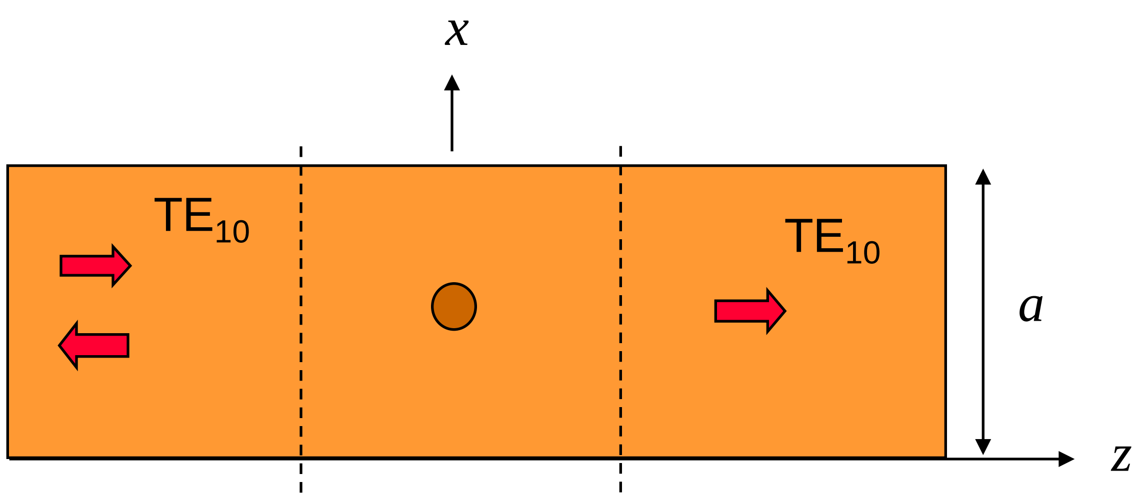
Discontinuity

Rectangular waveguide with a post:



a_p = radius of post

Top view:

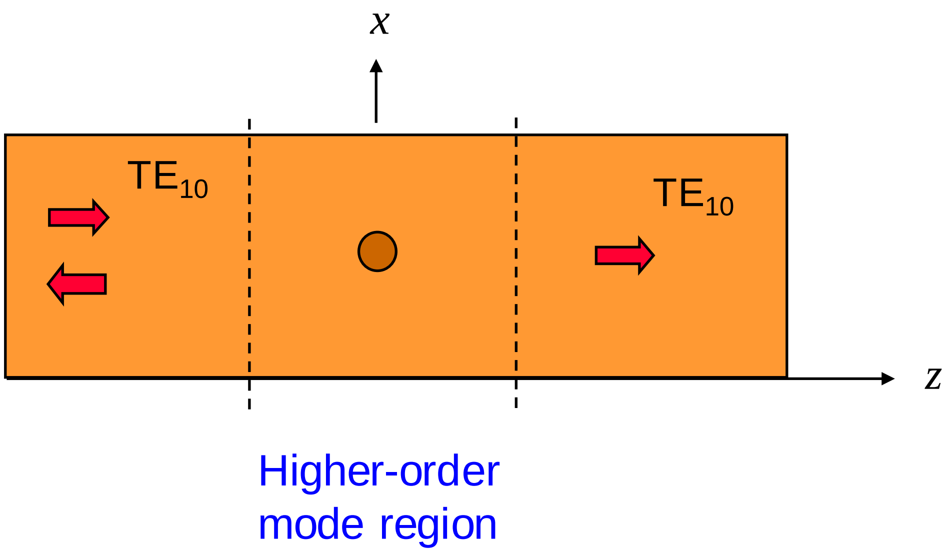


Assumption: Only the TE_{10} mode can propagate.

Higher-order mode region

Discontinuity (cont.)

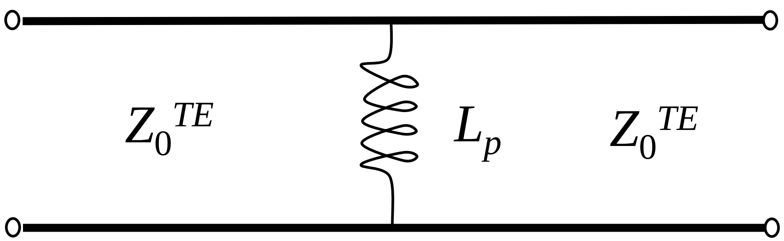
Top view:



$$Z_0^{TE} = \frac{\omega\mu_0}{k_z^{10}}$$

$$= \frac{\eta_0}{\sqrt{1-\left(\frac{\pi}{k_0 a}\right)^2}}$$

TEN :



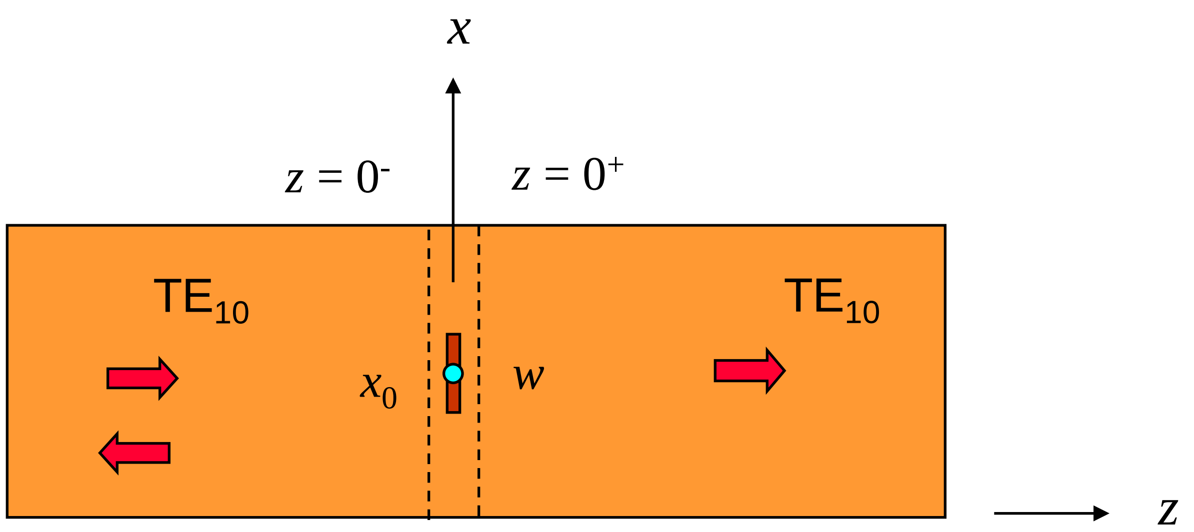
The element is chosen to give the same reflected and transmitted TE_{10} waves as in the actual waveguide.

Note: The discontinuity is approximately a shunt load because the tangential electric field of the dominant mode is approximately continuous, while the tangential magnetic field of the dominant mode is not (shown later).

Discontinuity (cont.)

Flat strip model (strip of width w)

Top view:



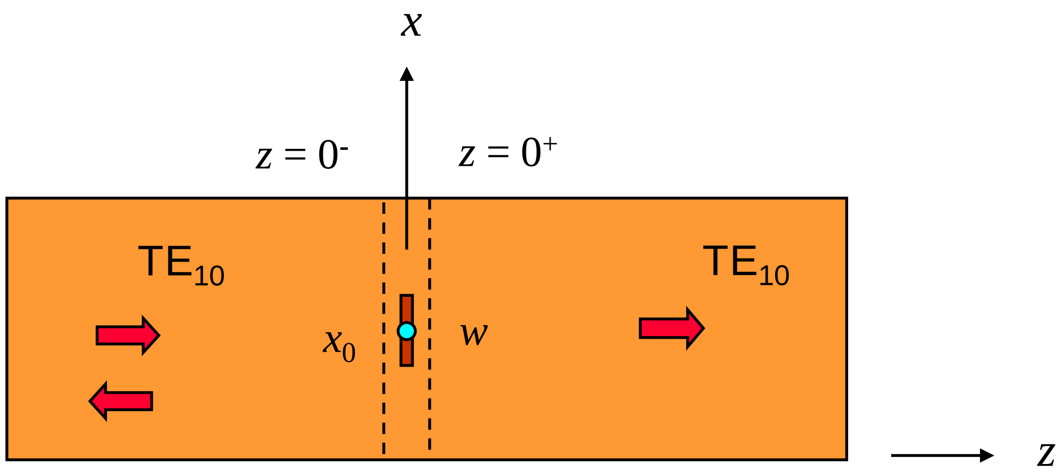
Narrow strip model for post ($w = 4a_p$)

Assume center of post is at $x = x_0$

In this model (flat strip model) the equivalent circuit is exactly a shunt inductor (proof given later).

Discontinuity (cont.)

Top view:



$$E_y = E_y^{inc} + E_y^{sca}$$

$$E_y^{inc} = \sin\left(\frac{\pi x}{a}\right) e^{-jk_z^{10} z}$$

E_y^{sca} = field produced by strip current

(that is, the field scattered (or radiated) by the strip)

Discontinuity (cont.)

Transverse fields radiated by the post current (no y variation):

$$E_y^{sca+}(x,y,z)=\sum_{m=1}^{\infty}A_m^{+}\sin\left(\frac{m\pi x}{a}\right)e^{-jk_z^{m0}z}$$

$$E_y^{sca-}(x,y,z)=\sum_{m=1}^{\infty}A_m^{-}\sin\left(\frac{m\pi x}{a}\right)e^{+jk_z^{m0}z}$$

$$H_x^{sca+}(x,y,z)=\sum_{m=1}^{\infty}-\left(\frac{A_m^{+}}{Z_{m0}^{TE}}\right)\sin\left(\frac{m\pi x}{a}\right)e^{-jk_z^{m0}z}$$

$$H_x^{sca-}(x,y,z)=\sum_{m=1}^{\infty}\left(\frac{A_m^{-}}{Z_{m0}^{TE}}\right)\sin\left(\frac{m\pi x}{a}\right)e^{+jk_z^{m0}z}$$

By symmetry, the scattered field should have no y variation.

$$k_z^{m0}=\left(k_0^2-\left(\frac{m\pi}{a}\right)^2\right)^{1/2}$$

$$\underline{H}_t^{\pm}=\pm\frac{1}{Z_{m0}^{TE}}\left(\underline{\hat{z}}\times\underline{E}_t^{\pm}\right)$$

$$Z_{m0}^{TE}=\frac{\omega\mu_0}{k_z^{m0}}$$

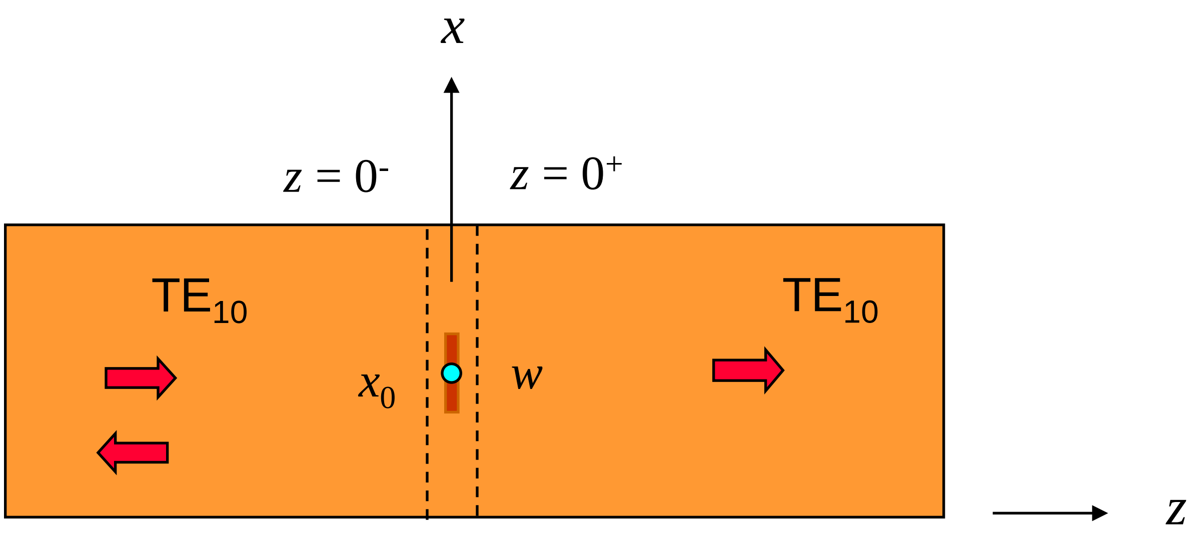
We assume a field representation in terms of TE_{m0} waveguide modes (therefore, there is only a y component of the electric field).

$$E_{x0}=\frac{-j\omega\mu}{k^2-k_z^2}\frac{\partial H_{z0}}{\partial y}-\frac{jk_z}{k^2-k_z^2}\frac{\partial E_{z0}}{\partial x}$$

Note: TM_{m0} modes do not exist.

Discontinuity (cont.)

Top view:



$$E_y^{inc} = \sin\left(\frac{\pi x}{a}\right) e^{-jk_z^{10} z}$$

Note: The incident field is continuous at $z = 0$.

From boundary conditions:

$$E_y\left(z = 0^-\right) = E_y\left(z = 0^+\right)$$

Hence $$E_y^{sca}\left(z = 0^-\right) = E_y^{sca}\left(z = 0^+\right)$$

Discontinuity (cont.)

$$E_y^{sca}\left(z=0^+\right)=E_y^{sca}\left(z=0^-\right)$$

$$\sum_{m=1}^{\infty} A_m^+ \sin\left(\frac{m\pi x}{a}\right)=\sum_{m=1}^{\infty} A_m^- \sin\left(\frac{m\pi x}{a}\right)$$

Equate terms of the Fourier series:

$$A_m^- = A_m^+ = A_m \;\Rightarrow\; A_1^- = A_1^+ = A_1 \;\Rightarrow\; E_y^{sca\,10}\left(0^-\right)=E_y^{sca\,10}\left(0^+\right)$$

$$\Rightarrow\; E_y^{10}\left(0^-\right)=E_y^{10}\left(0^+\right)$$

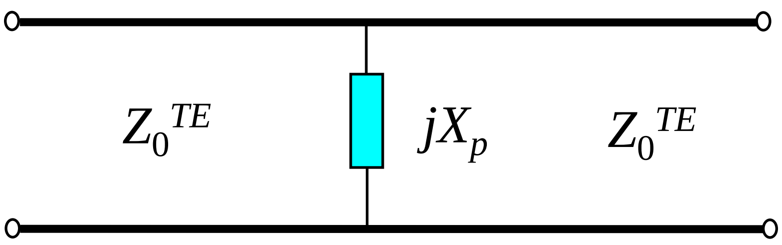
Modeling equation:

$$\underline{E}_t^{10}=\underline{\hat{y}}\sin\left(\frac{\pi x}{a}\right)V\left(z\right) \;\Rightarrow\; V\left(0^-\right)=V\left(0^+\right)$$

Discontinuity (cont.)

$$V\left(0^{-}\right)=V\left(0^{+}\right)$$

This establishes that the circuit model for the flat strip must be a parallel (shunt) element.



$$X_p = \omega L_p$$

Discontinuity (cont.)

The fields inside the waveguide are then:

$$E_y^{sca+}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{-jk_z^{m0} z}$$

$$E_y^{sca-}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{+jk_z^{m0} z}$$

$$H_x^{sca+}(x,y,z) = \sum_{m=1}^{\infty} -\left(\frac{A_m}{Z_{m0}^{TE}}\right) \sin\left(\frac{m\pi x}{a}\right) e^{-jk_z^{m0} z}$$

$$H_x^{sca-}(x,y,z) = \sum_{m=1}^{\infty} \left(\frac{A_m}{Z_{m0}^{TE}}\right) \sin\left(\frac{m\pi x}{a}\right) e^{+jk_z^{m0} z}$$

$$k_z^{m0} = \left(k_0^2 - \left(\frac{m\pi}{a}\right)^2\right)^{1/2}$$

$$Z_{m0}^{TE} = \frac{\omega\mu_0}{k_z^{m0}}$$

There is <u>one</u> set of unknown coefficients A_m.

Discontinuity (cont.)

Magnetic field:

$$H_x\left(z=0^+\right)-H_x\left(z=0^-\right)=J_{sy}\left(x\right)$$

From the Fourier series for the magnetic field, we then have:

$$\sum_{m=1}^{\infty}\frac{1}{Z_{m0}^{TE}}\left(-A_m-A_m\right)\sin\left(\frac{m\pi x}{a}\right)=J_{sy}\left(x\right)$$

or

$$\sum_{m=1}^{\infty}\frac{-1}{Z_{m0}^{TE}}\left(2A_m\right)\sin\left(\frac{m\pi x}{a}\right)=J_{sy}\left(x\right)$$

Note:
We can use the scattered field here, since he incident field is continuous.

Represent the strip current as

$$J_{sy}\left(x\right)=\sum_{m=1}^{\infty}j_m\sin\left(\frac{m\pi x}{a}\right)$$

$$j_m=\frac{2}{a}\int_0^a J_{sy}\left(x\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

Discontinuity (cont.)

Therefore

$$\sum_{m=1}^{\infty}\left(\frac{-2}{Z_{m0}^{TE}}\right)A_m \sin\left(\frac{m\pi x}{a}\right)=\sum_{m=1}^{\infty} j_m \sin\left(\frac{m\pi x}{a}\right)$$

Hence

$$A_m = j_m\left(\frac{-Z_{m0}^{TE}}{2}\right)$$

In order to solve for j_m, we need to enforce the condition that $E_y = 0$ on the strip.

Discontinuity (cont.)

Assume that the strip is narrow, so that a single "Maxwell" function describes accurately the shape of the current on the strip:

$$J_{sy}(x) = I_0 \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - (x - x_0)^2}} \right) \qquad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$$

I_0 is now the unknown.

(I_0 is the total current on the strip.)

Basis function
$B(x)$

$$j_m = \frac{2}{a} \int_0^a J_{sy}(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

Hence

$$j_m = I_0 \left(\frac{2}{a}\right) \int_{x_0 - w/2}^{x_0 + w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - (x - x_0)^2}} \right) \sin\left(\frac{m\pi x}{a}\right) dx$$

Discontinuity (cont.)

Hence $$j_m = I_0\left(\frac{2}{a}\right)c_m$$

where

$$c_m \equiv \int_{x_0-w/2}^{x_0+w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - \left(x-x_0\right)^2}} \right) \sin\left(\frac{m\pi x}{a}\right) dx$$

Note: c_m can be evaluated in closed form (in terms of the Bessel function J_0).

$$c_m = J_0\left(\frac{m\pi w}{2a}\right)\sin\left(\frac{m\pi x_0}{a}\right)$$

(Please see the Appendix.)

Discontinuity (cont.)

The fields inside the waveguide are now given by:

$$E_y^{sca+}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{-jk_z^{m0} z}$$

$$E_y^{sca-}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{+jk_z^{m0} z}$$

$$H_x^{sca+}(x,y,z) = \sum_{m=1}^{\infty} -\left(\frac{A_m}{Z_{m0}^{TE}}\right) \sin\left(\frac{m\pi x}{a}\right) e^{-jk_z^{m0} z}$$

$$H_x^{sca-}(x,y,z) = \sum_{m=1}^{\infty} \left(\frac{A_m}{Z_{m0}^{TE}}\right) \sin\left(\frac{m\pi x}{a}\right) e^{+jk_z^{m0} z}$$

$$A_m = j_m \left(\frac{-Z_{m0}^{TE}}{2}\right) \qquad j_m = I_0 \left(\frac{2}{a}\right) c_m \qquad c_m = J_0\left(\frac{m\pi w}{2a}\right) \sin\left(\frac{m\pi x_0}{a}\right)$$

We still need to solve for the unknown post current I_0, by enforcing that the electric field vanish on the strip.

Discontinuity (cont.)

To solve for I_0, enforce the electric field integral equation (EFIE):

$$E_y^{inc} + E_y^{sca} = 0$$ on strip $z = 0, \quad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$

$$E_y^{inc}(x,y,z) = \sin\left(\frac{\pi x}{a}\right) e^{-jk_z^{10} z}$$ (unit-amplitude incident mode)

$$E_y^{sca\,\pm}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{\mp jk_z^{m0} z}$$

Hence

$$\sin\left(\frac{\pi x}{a}\right) = -\sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) \qquad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$$

Discontinuity (cont.)

$$\sin\left(\frac{\pi x}{a}\right) = -\sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) \qquad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$$

$$A_m = j_m\left(\frac{-Z_{m0}^{TE}}{2}\right)$$

$$\sin\left(\frac{\pi x}{a}\right) = -\sum_{m=1}^{\infty} j_m\left(\frac{-Z_{m0}^{TE}}{2}\right)\sin\left(\frac{m\pi x}{a}\right) \qquad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$$

$$j_m = I_0\left(\frac{2}{a}\right)c_m$$

$$\sin\left(\frac{\pi x}{a}\right) = -\sum_{m=1}^{\infty} c_m I_0\left(\frac{2}{a}\right)\left(\frac{-Z_{m0}^{TE}}{2}\right)\sin\left(\frac{m\pi x}{a}\right) \qquad x_0 - \frac{w}{2} < x < x_0 + \frac{w}{2}$$

Discontinuity (cont.)

$$\sin\left(\frac{\pi x}{a}\right) = I_0\left(\frac{2}{a}\right)\sum_{m=1}^{\infty} c_m\left(\frac{Z_{m0}^{TE}}{2}\right)\sin\left(\frac{m\pi x}{a}\right) \qquad x_0-\frac{w}{2}<x<x_0+\frac{w}{2}$$

This is in the form

$$f(x) = I_0\, g(x) \qquad x_0-\frac{w}{2}<x<x_0+\frac{w}{2}$$

To solve for the unknown I_0, we can use the idea of a "testing function." We multiply both sides by a testing function and then integrate over the strip.

$$\int_{x_0-w/2}^{x_0+w/2} T(x)\, f(x)\, dx = I_0 \int_{x_0-w/2}^{x_0+w/2} T(x)\, g(x)\, dx$$

Discontinuity (cont.)

$$\int_{x_0-w/2}^{x_0+w/2} T(x)\sin\left(\frac{\pi x}{a}\right)dx = I_0\left(\frac{2}{a}\right)\sum_{m=1}^{\infty} c_m\left(\frac{Z_{m0}^{TE}}{2}\right)\int_{x_0-w/2}^{x_0+w/2} T(x)\sin\left(\frac{m\pi x}{a}\right)dx$$

Galerkin's method:

The testing function is the <u>same</u> as the basis function:

$$T(x) = B(x) = \frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - (x-x_0)^2}}$$

Discontinuity (cont.)

Hence

$$\int_{x_0-w/2}^{x_0+w/2}\left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2-\left(x-x_0\right)^2}}\right)\sin\left(\frac{\pi x}{a}\right)dx$$

$$=I_0\left(\frac{2}{a}\right)\sum_{m=1}^{\infty}c_m\left(\frac{Z_{m0}^{TE}}{2}\right)\int_{x_0-w/2}^{x_0+w/2}\left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2-\left(x-x_0\right)^2}}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

This integral is c_m.

Discontinuity (cont.)

Hence

$$c_1 = I_0 \left(\frac{2}{a} \right) \sum_{m=1}^{\infty} c_m \left(\frac{Z_{m0}^{TE}}{2} \right) c_m$$

so

$$I_0 = \frac{c_1}{\left(\frac{2}{a} \right) \sum_{m=1}^{\infty} \left(\frac{Z_{m0}^{TE}}{2} \right) c_m^2}$$

or

$$I_0 = \frac{a\, c_1}{\sum_{m=1}^{\infty} Z_{m0}^{TE}\, c_m^2}$$

Discontinuity (cont.)

Summary

$$I_0 = \frac{a\,c_1}{\sum_{m=1}^{\infty} Z_{m0}^{TE}\, c_m^2}$$

$$c_m = J_0\left(\frac{m\pi w}{2a}\right)\sin\left(\frac{m\pi x_0}{a}\right)$$

$$A_m = j_m\left(\frac{-Z_{m0}^{TE}}{2}\right)$$

$$j_m = I_0\left(\frac{2}{a}\right)c_m$$

$$Z_{m0}^{TE} = \frac{\omega\mu_0}{k_z^{m0}} = \frac{\eta_0}{\sqrt{1-\left(\frac{m\pi}{k_0 a}\right)^2}}$$

$$E_y^{sca+}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right)e^{-jk_z^{m0}z}$$

$$E_y^{sca-}(x,y,z) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right)e^{+jk_z^{m0}z}$$

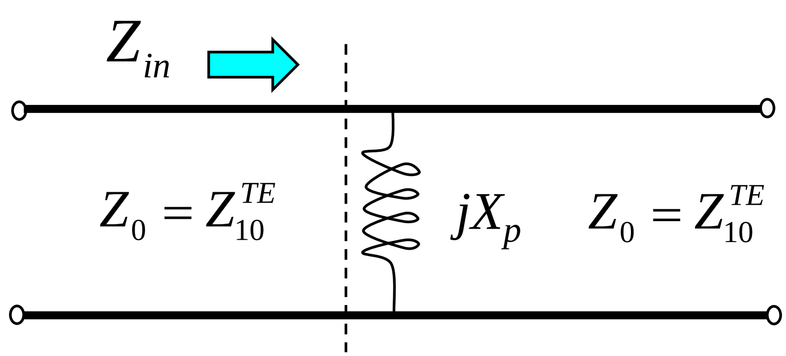
Discontinuity (cont.)

Reflection coefficient:

$$\Gamma = \frac{A_1^-}{1} = A_1^- = A_1 \qquad \text{so} \qquad \Gamma = A_1$$

Post reactance:

$$\Gamma = \frac{Z_{in} - Z_{10}^{TE}}{Z_{in} + Z_{10}^{TE}} \qquad \text{where} \qquad Z_{in} = Z_{10}^{TE} \parallel jX_p = \frac{jX_p Z_{10}^{TE}}{jX_p + Z_{10}^{TE}}$$



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From these two equations, X_p may be found in terms of Γ.

Discontinuity (cont.)

Result:

$$jX_p = -Z_{10}^{TE}\left(\frac{1+\Gamma}{2\Gamma}\right)$$

where

$$\Gamma = A_1 = j_1\left(\frac{-Z_{10}^{TE}}{2}\right)$$

$$j_1 = I_0\left(\frac{2}{a}\right)c_1$$

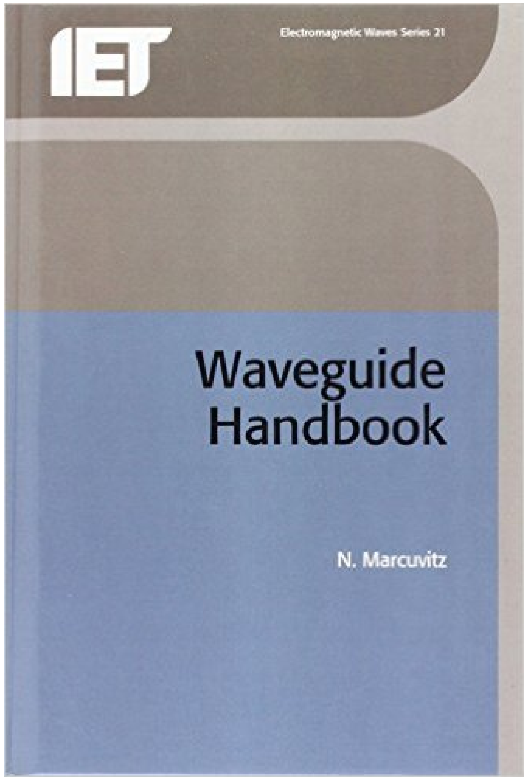
$$I_0 = \frac{a\,c_1}{\sum_{m=1}^{\infty} Z_{m0}^{TE}\,c_m^2}$$

$$Z_{10}^{TE} = \frac{\eta_0}{\sqrt{1-\left(\frac{\pi}{k_0 a}\right)^2}}$$

$$c_m = J_0\left(\frac{m\pi w}{2a}\right)\sin\left(\frac{m\pi x_0}{a}\right)$$

Discontinuity (cont.)

Results for many different types of waveguide discontinuities may be found in:



N. Marcuvitz, *The Waveguide Handbook*, IET (The Institution of Engineering and Technology), IEE Electromagnetic Wave Series, 1985.

(The book was originally published in 1951 as vol. 10 of the MIT Radiation Laboratory series.)

Appendix

In this appendix we evaluate the c_m coefficients.

$$c_m = \int_{x_0-w/2}^{x_0+w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - (x-x_0)^2}} \right) \sin\left(\frac{m\pi x}{a}\right) dx$$

Use $x' = x - x_0$

$$c_m = \frac{1}{\pi} \int_{-w/2}^{w/2} \left(\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (x')^2}} \right) \sin\left(\frac{m\pi \left[x' + x_0\right]}{a}\right) dx$$

Appendix (cont.)

$$c_m = \frac{1}{\pi} \int_{-w/2}^{w/2} \left(\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - \left(x'\right)^2}} \right) \sin\left(\frac{m\pi\left[x' + x_0\right]}{a} \right) dx$$

or

$$c_m = \frac{1}{\pi} \int_{-w/2}^{w/2} \left(\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - \left(x'\right)^2}} \right) \left[\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{m\pi x_0}{a}\right) + \cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{m\pi x_0}{a}\right) \right] dx$$

This term integrates to zero (odd function).

Appendix (cont.)

$$c_m = \frac{1}{\pi}\int_{-w/2}^{w/2}\left(\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2-\left(x'\right)^2}}\right)\sin\left(\frac{m\pi x_0}{a}\right)\cos\left(\frac{m\pi x'}{a}\right)dx$$

or

$$c_m = \frac{2}{\pi}\sin\left(\frac{m\pi x_0}{a}\right)\int_{0}^{w/2}\left(\frac{\cos\left(\frac{m\pi x'}{a}\right)}{\sqrt{\left(\frac{w}{2}\right)^2-\left(x'\right)^2}}\right)dx$$

Appendix (cont.)

Next, use the transformation

$$x' = \frac{w}{2}\sin\theta$$

$$dx' = \frac{w}{2}\cos\theta\, d\theta$$

$$c_m = \frac{2}{\pi}\sin\left(\frac{m\pi x_0}{a}\right)\int_0^{\pi/2}\left(\frac{\cos\left(\frac{m\pi w}{2a}\sin\theta\right)}{\frac{w}{2}\cos\theta}\right)\frac{w}{2}\cos\theta$$

Appendix (cont.)

$$c_m = \frac{2}{\pi}\sin\left(\frac{m\pi x_0}{a}\right)\int_0^{\pi/2} \cos\left(\frac{m\pi w}{2a}\sin\theta\right)d\theta$$

or

$$c_m = \frac{1}{\pi}\sin\left(\frac{m\pi x_0}{a}\right)\int_0^{\pi} \cos\left(\frac{m\pi w}{2a}\sin\theta\right)d\theta$$

Next, use the following integral identify for the Bessel function:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta - n\theta)\,d\theta$$

so that

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta)\,d\theta$$

Appendix (cont.)

Hence

$$c_m = \frac{1}{\pi} \sin\left(\frac{m\pi x_0}{a}\right) \int_0^{\pi} \cos\left(\frac{m\pi w}{2a} \sin\theta\right) d\theta$$

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin\theta)\, d\theta$$

$$c_m = \sin\left(\frac{m\pi x_0}{a}\right) J_0\left(\frac{m\pi w}{2a}\right)$$