ECE 6340
Intermediate EM Waves

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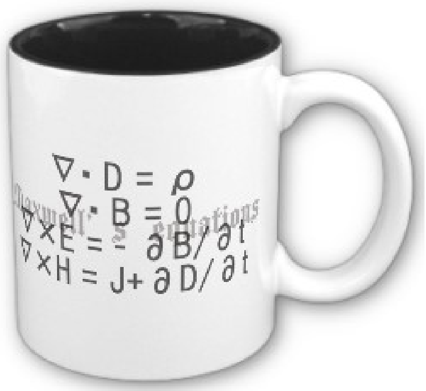

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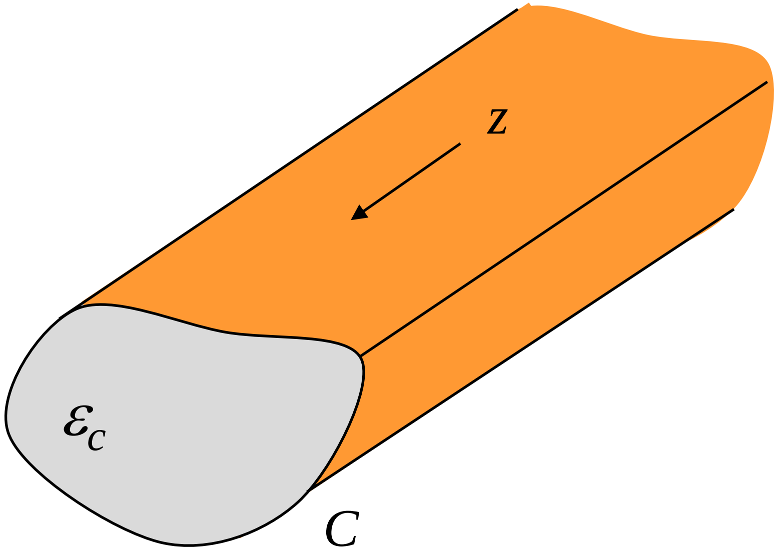
Mode Orthogonality

Mode "m" ($\underline{E}_m$, $\underline{H}_m$)

Mode "n" ($\underline{E}_n$, $\underline{H}_n$)



Waveguide modes are "orthogonal" (in the complex power sense) if:

$$\int_S \left(\underline{E}_m \times \underline{H}_n^*\right) \cdot \underline{\hat{z}}\, dS = 0$$

Mode Orthogonality (cont.)

Assume two modes are orthogonal, and examine the complex power flowing down the guide when <u>two</u> modes are present:

$$P_z = \int_S \frac{1}{2}(\underline{E} \times \underline{H}^*) \cdot \underline{\hat{z}}\, dS$$

$$= \frac{1}{2}\int_S (\underline{E}_m + \underline{E}_n) \times (\underline{H}_m^* + \underline{H}_n^*) \cdot \underline{\hat{z}}\, dS$$

$$= \frac{1}{2}\int_S (\underline{E}_m \times \underline{H}_m^*) \cdot \underline{\hat{z}}\, dS + \frac{1}{2}\int_S (\underline{E}_n \times \underline{H}_n^*) \cdot \underline{\hat{z}}\, dS$$

$$+ \cancel{\frac{1}{2}\int_S (\underline{E}_m \times \underline{H}_n^*) \cdot \underline{\hat{z}}\, dS} + \cancel{\frac{1}{2}\int_S (\underline{E}_n \times \underline{H}_m^*) \cdot \underline{\hat{z}}\, dS}$$

Mode Orthogonality (cont.)

Hence

$$P_z = P_z^{(m)} + P_z^{(n)}$$

If two modes are orthogonal, the <u>total</u> complex power is the <u>sum</u> of the two complex powers of the individual modes.

Reference

To see a derivation of the orthogonality theorems presented here, and others, please see the following reference:

R. E. Collin, *Field Theory of Guided Waves*, IEEE Press, 1991.

Waveguides with PEC Walls

(There may be a lossy material inside the waveguide.)

Theorem 1

A TE_z mode is always orthogonal to a TM_z mode.

Theorem 2

Two TM_z modes (or two TE_z modes) are orthogonal to each other if they are not degenerate:

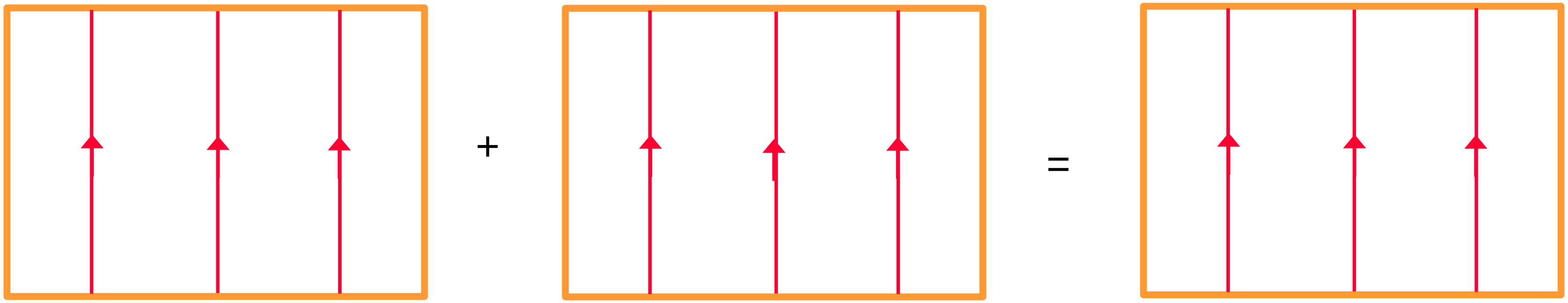
$$k_{zm} \neq \pm k_{zn}$$

Degenerate Modes

Degenerate modes are <u>not</u> in general orthogonal.

A simple example is two modes that are really the same mode.

TE$_{10}$ mode



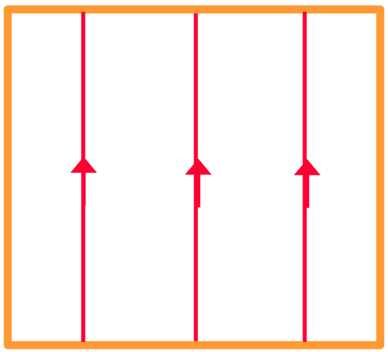
1 Watt 1 Watt 4 Watt

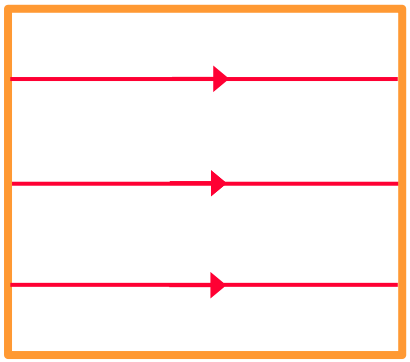
If the mode doubles, the fields are twice as strong,
and hence the power increases by a factor of four.

Degenerate Mode (cont.)

<u>Sometimes</u> the modes are orthogonal, even when they are degenerate.

Square waveguide: TE_{10} and TE_{01} modes



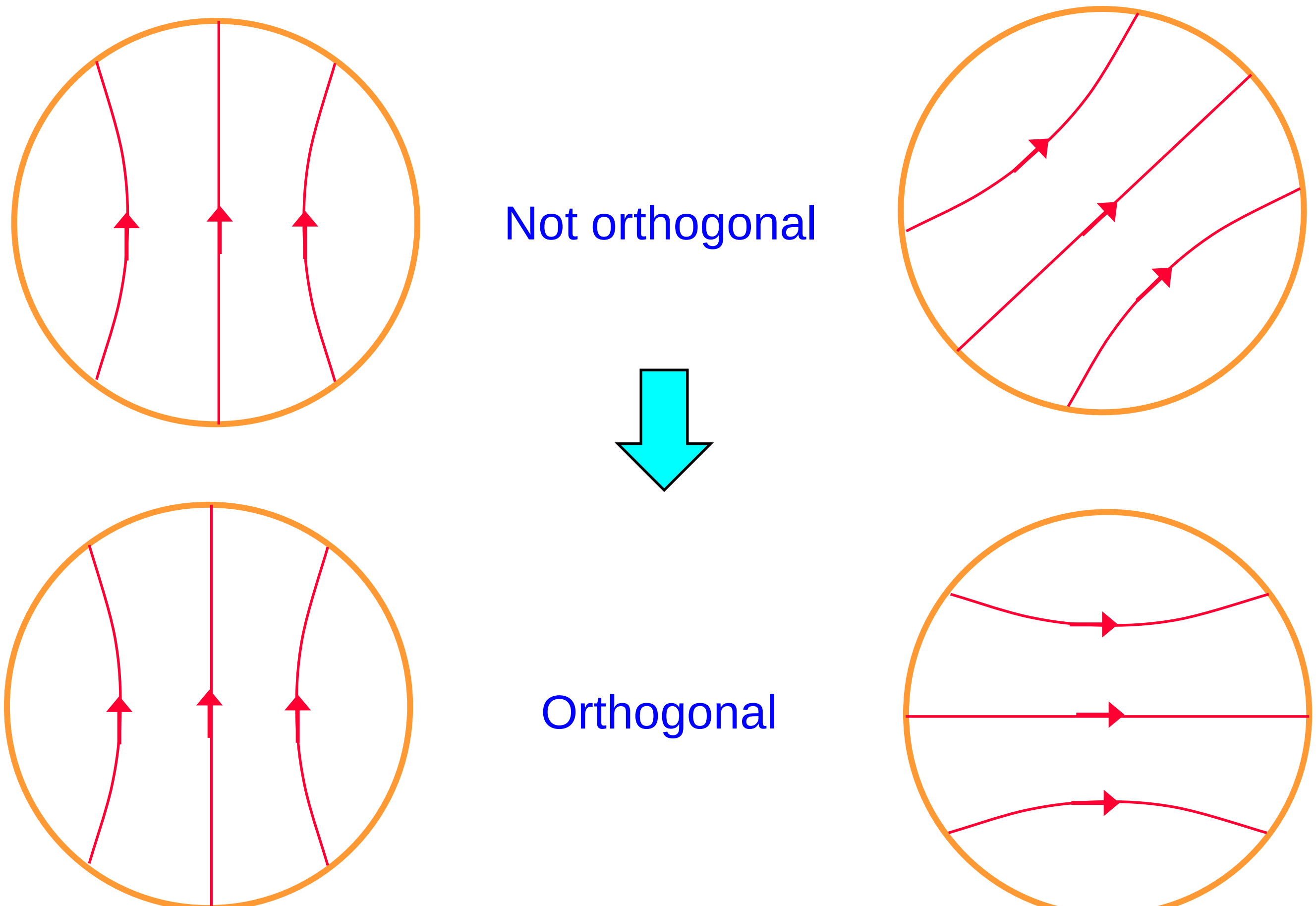


These two modes are orthogonal (even though they are degenerate).

Degenerate Mode (cont.)

If two modes of the same type are degenerate, but they <u>are linearly independent</u>, we can always choose a combination of them that will correspond to two orthogonal modes.

Circular waveguide: Two TE_{11} modes



Rectangular Waveguide

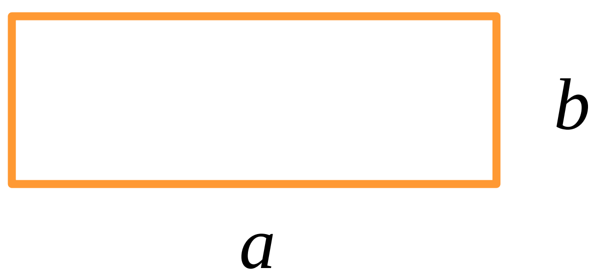
For a <u>rectangular waveguide</u>, two modes of the same type (TE_z or TM_z) will be orthogonal provided that the mode indices are not exactly the same.

$$\left(m_1, n_1\right) \neq \left(m_2, n_2\right)$$

TE_{mn}

or

TM_{mn}



The proof is left as a homework problem.

(Use orthogonality of the sin and cosine functions that make up the field variations in the x and y directions.)

Orthogonality Between Incident and Reflected Waves

Assume a single mode, but waves going <u>in both directions</u>:

$$P_z = \int_S \frac{1}{2} \left(\underline{E} \times \underline{H}^* \right) \cdot \underline{\hat{z}} \, dS$$

$$= \frac{1}{2} \int_S \left(\underline{E}_t^+ + \underline{E}_t^- \right) \times \left(\underline{H}_t^+ + \underline{H}_t^- \right)^* \cdot \underline{\hat{z}} \, dS$$

$$= \frac{1}{2} \int_S \left(\underline{E}_t^+ \times \underline{H}_t^{+*} \right) \cdot \underline{\hat{z}} \, dS + \frac{1}{2} \int_S \left(\underline{E}_t^- \times \underline{H}_t^{-*} \right) \cdot \underline{\hat{z}} \, dS$$

$$+ \frac{1}{2} \int_S \left(\underline{E}_t^+ \times \underline{H}_t^{-*} \right) \cdot \underline{\hat{z}} \, dS + \frac{1}{2} \int_S \left(\underline{E}_t^- \times \underline{H}_t^{+*} \right) \cdot \underline{\hat{z}} \, dS$$

Orthogonality Between Incident and Reflected Waves (cont.)

The real part of the complex power (watts) is

$$\operatorname{Re} P_z = \operatorname{Re} P_z^{+} + \operatorname{Re} P_z^{-}$$
$$+ \operatorname{Re} \frac{1}{2} \int_S \left(\underline{E}_t^{+} \times \underline{H}_t^{-*} \right) \cdot \underline{\hat{z}} \, dS + \operatorname{Re} \frac{1}{2} \int_S \left(\underline{E}_t^{-} \times \underline{H}_t^{+*} \right) \cdot \underline{\hat{z}} \, dS$$

Examine the following term:

$$\underline{E}_t^{+} \times \underline{H}_t^{-*} = \underline{E}_t^{+} \times \left(-\frac{1}{Z^{TE}} \underline{\hat{z}} \times \underline{E}_t^{-} \right)^{*}$$

(assume TE_z)

$$= -\frac{1}{Z^{TE*}} \underline{E}_t^{+} \times \left(\underline{\hat{z}} \times \underline{E}_t^{-*} \right)$$

Orthogonality Between Incident and Reflected Waves (cont.)

We then have

$$\underline{E}_t^{+} \times \underline{H}_t^{-*} = -\frac{1}{Z^{TE*}}\underline{E}_t^{+} \times \left(\underline{\hat{z}} \times \underline{E}_t^{-*}\right)$$

$$= -\frac{1}{Z^{TE*}}\left(\underline{E}_t^{+} \cdot \underline{E}_t^{-*}\right)\underline{\hat{z}}$$

Note : $\underline{A} \times \left(\underline{B} \times \underline{C}\right) = \underline{B}\left(\underline{A} \cdot \underline{C}\right) - \underline{C}\left(\underline{A} \cdot \underline{B}\right)$

Similarly, we have

$$\underline{E}_t^{-} \times \underline{H}_t^{+*} = +\frac{1}{Z^{TE*}}\underline{E}_t^{-} \times \left(\underline{\hat{z}} \times \underline{E}_t^{+*}\right)$$

$$= +\frac{1}{Z^{TE*}}\left(\underline{E}_t^{-} \cdot \underline{E}_t^{+*}\right)\underline{\hat{z}}$$

Orthogonality Between Incident and Reflected Waves (cont.)

We then have

$$\operatorname{Re} P_z = \operatorname{Re} P_z^{+} + \operatorname{Re} P_z^{-}$$

$$+\operatorname{Re}\frac{1}{2}\int_S \left(-\frac{1}{Z^{TE*}}\underline{E}_t^{+}\cdot\underline{E}_t^{-*}\right)dS + \operatorname{Re}\frac{1}{2}\int_S \left(+\frac{1}{Z^{TE*}}\underline{E}_t^{-}\cdot\underline{E}_t^{+*}\right)dS$$

If the waveguide is lossless and we are above cutoff : $Z^{TE} =$ real

$$\Rightarrow \quad \operatorname{Re} P_z = \operatorname{Re} P_z^{+} + \operatorname{Re} P_z^{-}$$

$$-\frac{1}{Z^{TE}}\operatorname{Re}\frac{1}{2}\int_S \left(\underline{E}_t^{+}\cdot\underline{E}_t^{-*}\right)-\left(\underline{E}_t^{+}\cdot\underline{E}_t^{-*}\right)^{*} dS$$

Orthogonality Between Incident and Reflected Waves (cont.)

or

$$\operatorname{Re} P_z = \operatorname{Re} P_z^+ + \operatorname{Re} P_z^- - \frac{1}{Z^{TE}} \operatorname{Re} \frac{1}{2} \int_S 2j \operatorname{Im}\left(\underline{E}_t^+ \cdot \underline{E}_t^{-*} \right) dS$$

or

$$\boxed{\operatorname{Re} P_z = \operatorname{Re} P_z^+ + \operatorname{Re} P_z^-}$$

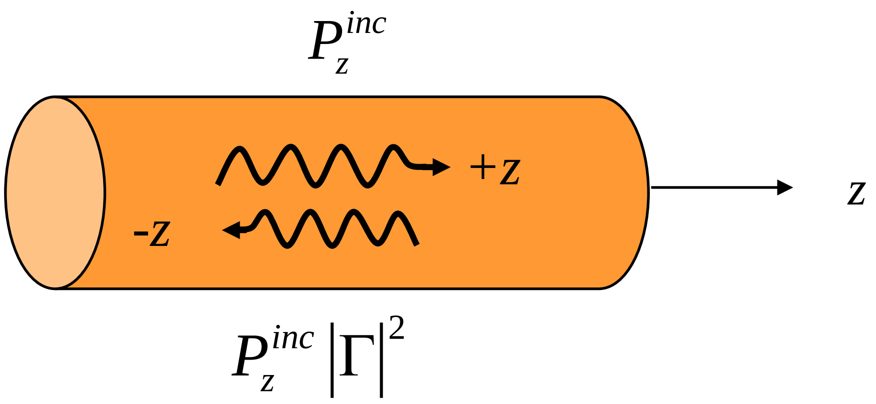
The total power (watts) in a lossless waveguide is the <u>sum</u> of the two individual powers.

Note:
For a lossless waveguide above cutoff, the powers will all be real (so we don't actually need to take the real part).

Orthogonality Between Incident and Reflected Waves (cont.)

For a <u>lossless</u> waveguide with a reflected wave, we can then say:

$$P_z = P_z^{inc}\left(1-|\Gamma|^2\right)$$



Appendix

Here we mention some other types of orthogonality relations that exist for waveguides:

- Orthogonality for waveguides with lossy walls
- Orthogonality for the longitudinal fields
- Orthogonality for the transverse electric or magnetic fields

Waveguides with Lossy Walls

The previous two theorems (theorem 1 and theorem 2) hold for a waveguide with lossy walls if we change the definition of orthogonality to be:

$$\int_S \left(\underline{E}_m \times \underline{H}_n\right) \cdot \underline{\hat{z}}\, dS = 0$$

(The lossy walls are modeled as an impedance surface.)

(Note that there is no conjugate here.)

However, in this case, we can no longer say that the total power flowing down the waveguide is the sum of the individual mode powers.

Orthogonality for Longitudinal Fields

Consider two non-degenerate modes that are either both TM_z or both TE_z.
Then we have that

TM_z $$\int_S \left(E_z^{(m)} E_z^{(n)}\right) dS = 0 \qquad k_{zm} \neq k_{zn}$$

TE_z $$\int_S \left(H_z^{(m)} H_z^{(n)}\right) dS = 0 \qquad k_{zm} \neq k_{zn}$$

Assumption: lossless walls

Note: If the two modes are degenerate, but they are linearly independent, we can always choose a combination of them that will correspond to two orthogonal modes.

Orthogonality for Transverse Fields

Consider two non-degenerate modes that are either both TE_z or both TM_z.
Then we have that for either case,

$$\int_S \left(\underline{E}_t^{(m)} \cdot \underline{E}_t^{(n)} \right) dS = 0 \qquad k_{zm} \neq k_{zn}$$

$$\int_S \left(\underline{H}_t^{(m)} \cdot \underline{H}_t^{(n)} \right) dS = 0 \qquad k_{zm} \neq k_{zn}$$

Assumption: lossless walls

Note: If the two modes are degenerate, but they are linearly independent, we can always choose a combination of them that will correspond to two orthogonal modes.

Orthogonality for Transverse Fields (cont.)

Consider one mode that is TE_z and one mode that is TM_z. Then we have that

$$\int_S \left(\underline{E}_t^{TM} \cdot \underline{E}_t^{TE} \right) dS = 0$$

$$\int_S \left(\underline{H}_t^{TM} \cdot \underline{H}_t^{TE} \right) dS = 0$$

Assumption: lossless walls

This orthogonality is true whether the modes are degenerate or not.