ECE 6340
Intermediate EM Waves

Fall 2016

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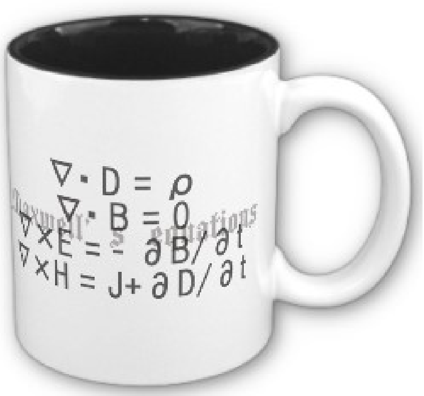

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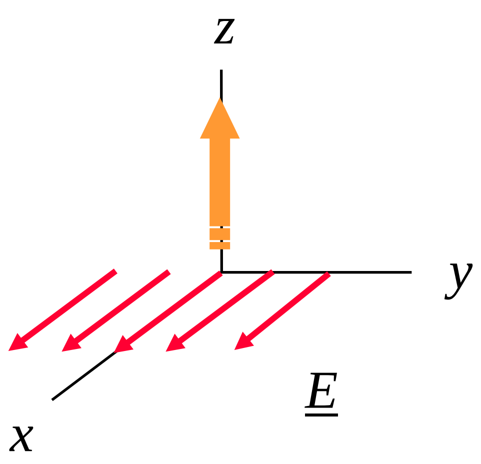
Plane Wave: Lossless Media

Assume

$$\underline{E} = \underline{\hat{x}}\, E_x(z)$$



Then $\nabla^2 E_x + k^2 E_x = 0$

or $\dfrac{d^2 E_x}{dz^2} + k^2 E_x = 0$

Lossless region:

$$\varepsilon_c = \varepsilon = \varepsilon'$$

$$\mu = \mu'$$

Solution:

$$E_x(z) = E_0\, e^{-jkz} \qquad (k = \omega\sqrt{\mu\varepsilon})$$

Plane Wave: Lossless Media (cont.)

The $\underline{H}$ field is found from:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

so

$$\underline{H} = -\frac{1}{j\omega\mu}\nabla \times \left(\underline{\hat{x}}\, E_x(z)\right)$$

$$= -\frac{1}{j\omega\mu}\left(\underline{\hat{y}}\,\frac{dE_x}{dz}\right)$$

$$= -\frac{1}{j\omega\mu}(-jk)\underline{\hat{y}}\,E_x$$

Plane Wave: Lossless Media (cont.)

or $$\underline{H} = \underline{\hat{y}} \left(\frac{k}{\omega \mu} \right) E_x$$

But $$\frac{k}{\omega \mu} = \frac{\omega \sqrt{\mu \varepsilon}}{\omega \mu} = \frac{1}{\eta}$$ (real number)

Hence

$$E_x = E_0 \, e^{-jkz}$$

$$H_y = \frac{1}{\eta} E_0 \, e^{-jkz}$$

TEM_z

Plane Wave: Lossless Media (cont.)

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* = \frac{1}{2}\underline{\hat{z}}\, E_x H_y^*$$

so

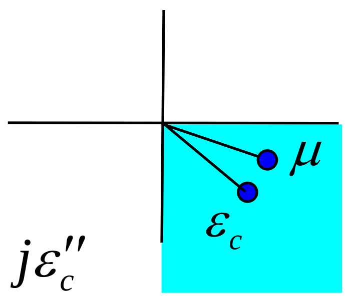
$$\underline{S} = \underline{\hat{z}}\, \frac{|E_0|^2}{2\eta}$$

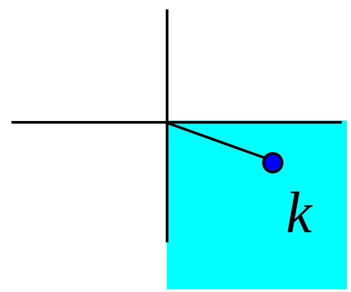
Lossless: η Is real.

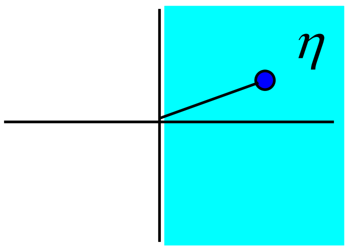
There is no reactive power.

Plane Wave: Lossy Media

$$k = \omega\sqrt{\mu\,\varepsilon_c} = k' - jk''$$







$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c''$$

$$\mu = \mu' - j\mu''$$

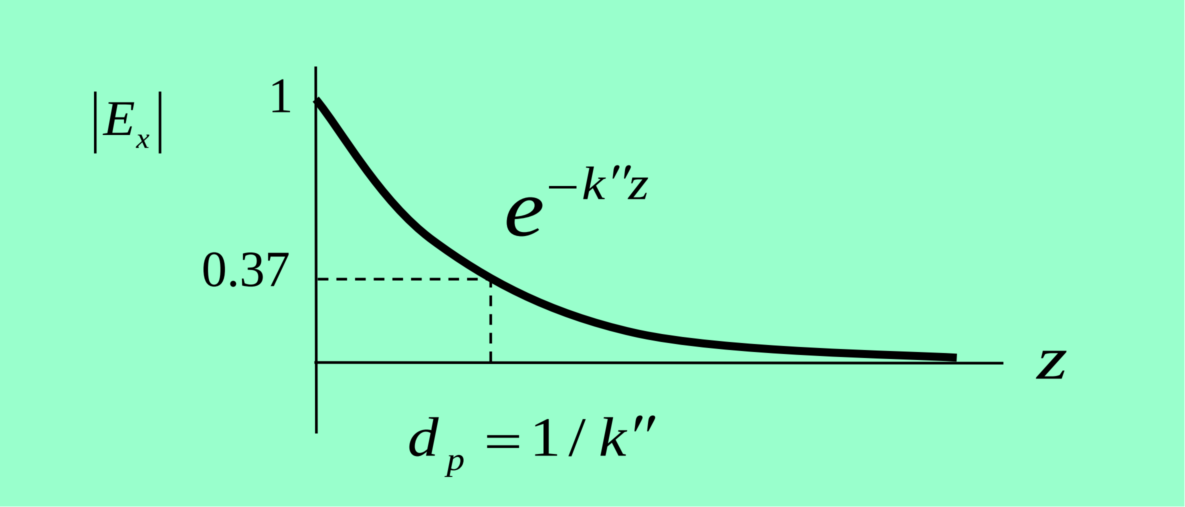
$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = |\eta| e^{j\phi}$$

$$E_x = E_0\, e^{-jk'z} e^{-k''z}$$

$$H_y = \frac{1}{\eta} E_0\, e^{-jk'z} e^{-k''z} = \frac{1}{|\eta|} E_0 e^{-j\phi}\, e^{-jk'z} e^{-k''z}$$

Plane Wave: Lossy Media (cont.)

$$E_x = E_0\, e^{-jk'z} e^{-k''z}$$



The "depth of penetration" d_p is defined.

$$d_p \equiv \frac{1}{k''}$$

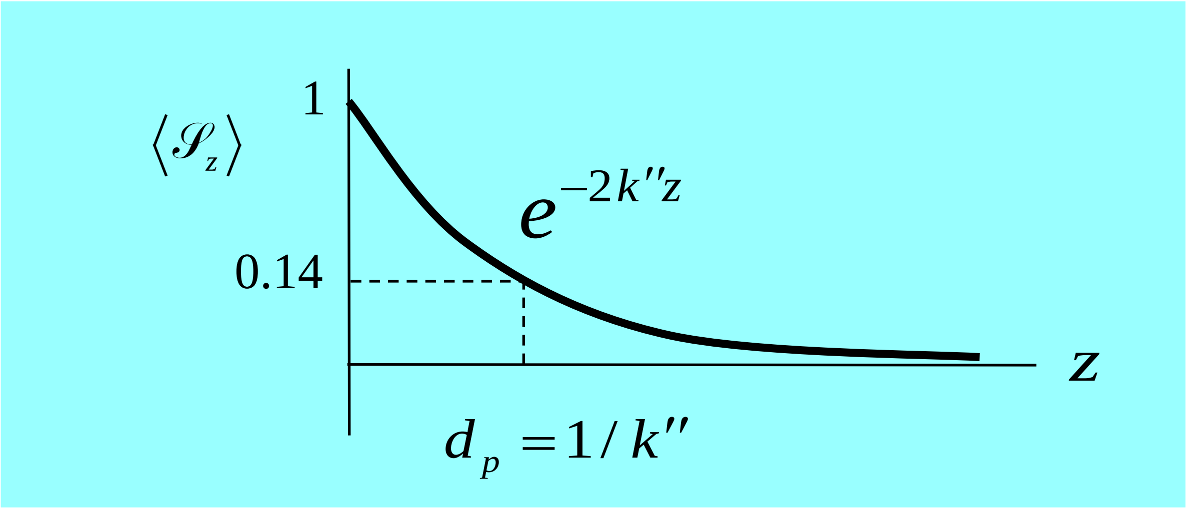
Hence

$$E_x = E_0\, e^{-jk'z} e^{-z/d_p}$$

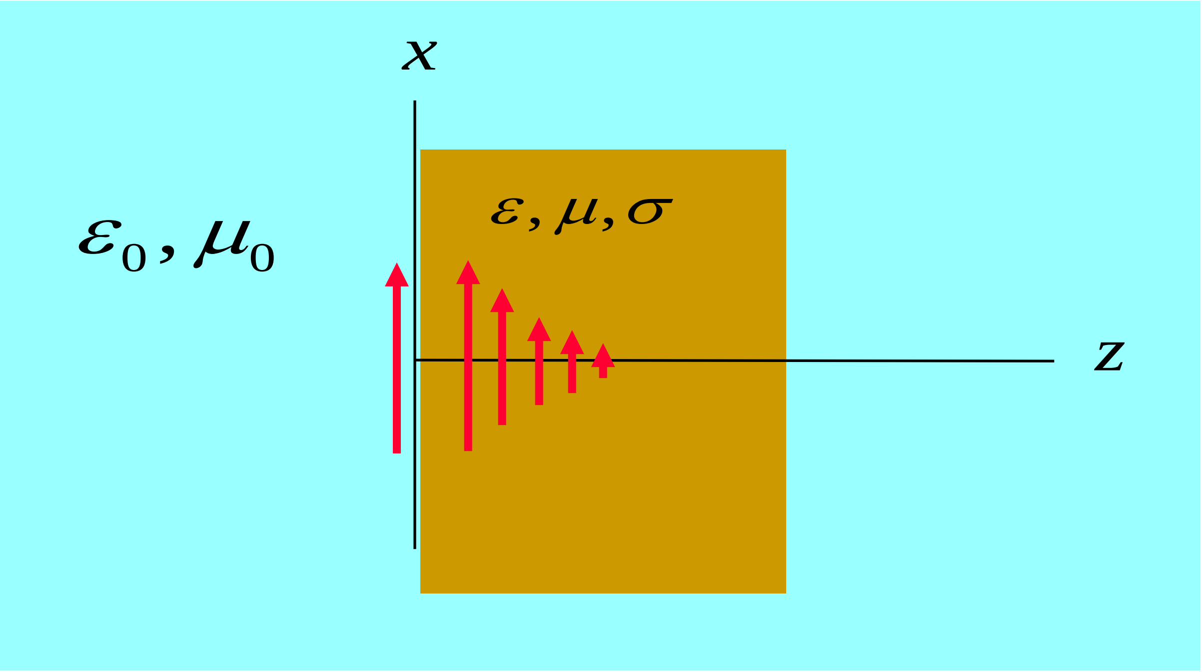
Plane Wave: Lossy Media (cont.)

$$\underline{S} = \underline{\hat{z}}\frac{1}{2}E_x H_y^* = \underline{\hat{z}}\frac{|E_0|^2}{2\eta^*}e^{-2k''z} = \underline{\hat{z}}\frac{|E_0|^2}{2|\eta|}e^{j\phi}e^{-2k''z}$$

$$\langle \mathscr{S}_z \rangle = \operatorname{Re} S_z = \frac{|E_0|^2}{2|\eta|}\cos\phi\, e^{-2k''z}$$



Plane Wave in a Good Conductor



$$E_x(z) = E_{x0}\, e^{-jkz}$$

$$k = \omega\sqrt{\mu\,\varepsilon_c} = \omega\sqrt{\mu}\,\sqrt{\varepsilon - j\frac{\sigma}{\omega}}$$

Plane Wave in Good Conductor (cont.)

Assume $\left|\frac{\sigma}{\omega\varepsilon}\right| \gg 1$

The we have

$$k \approx \omega\sqrt{\mu}\sqrt{-j\frac{\sigma}{\omega}} = \omega\sqrt{\frac{\mu\sigma}{\omega}}\left(\frac{1-j}{\sqrt{2}}\right) = \sqrt{\frac{\omega\mu\sigma}{2}}(1-j)$$

Hence

$$k = k' - jk'' \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-j)$$

Therefore

$$k' \approx k'' \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

Plane Wave in Good Conductor (cont.)

$$k' \approx k'' \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

Denote $\delta \equiv d_p = \dfrac{1}{k''}$ "skin depth"

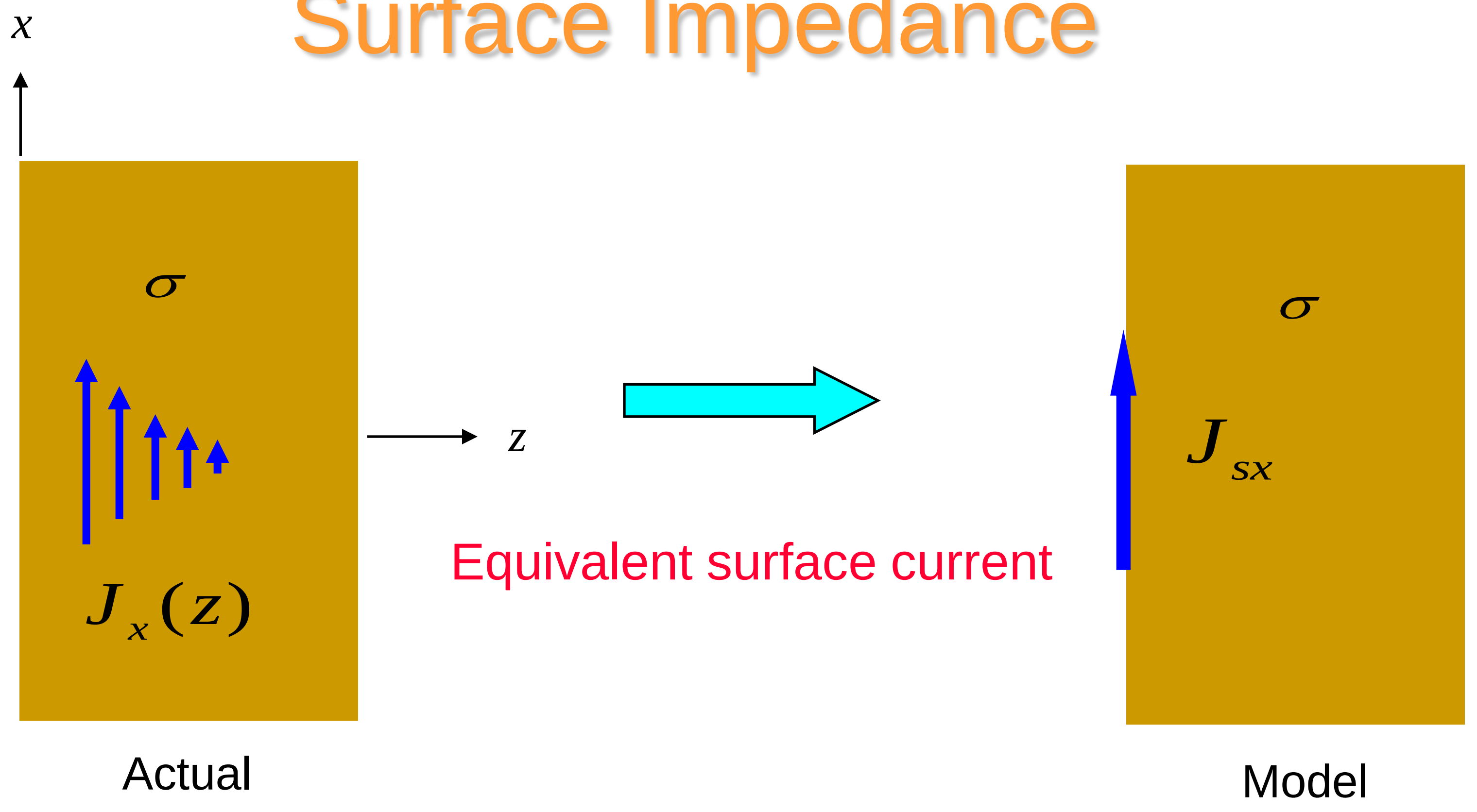
$$\left|E_x(z)\right| = \left|E_{x0}\right| e^{-z/\delta}$$

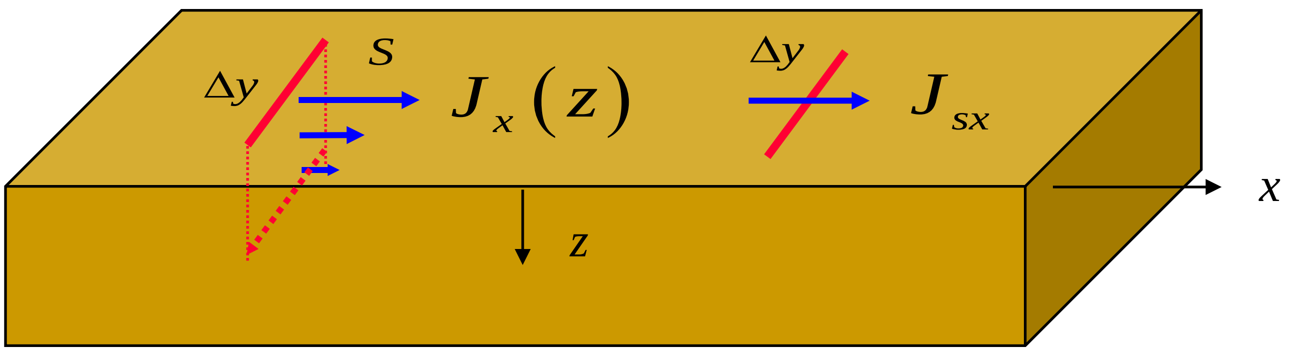
Then we have

$$\delta = d_p = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$k' \approx k'' \approx \frac{1}{\delta}$$

Surface Impedance





Surface Impedance (cont.)

$$I = \int_S J_x(z)\, dS = \Delta y \int_0^{\infty} J_x(z)\, dz$$ Actual current through Δy

$$I = J_{sx} \Delta y$$ Surface current model

Hence

$$J_{sx} = \int_0^{\infty} J_x(z)\, dz$$

Surface Impedance (cont.)

Define

$$Z_s \equiv \frac{E_{x0}}{J_{sx}}$$

$$E_x(z) = E_{x0}\, e^{-jkz}$$

$$\begin{aligned} J_{sx} &= \int_0^\infty J_x(z)\, dz \\ &= \int_0^\infty \sigma E_x(z)\, dz \\ &= \int_0^\infty \sigma E_{x0}\, e^{-jkz} dz \\ &= \sigma E_{x0} \int_0^\infty e^{-jkz} dz \end{aligned}$$

Integrating, we have

$$\begin{aligned} J_{sx} &= \sigma E_{x0} \left(-\frac{1}{jk} e^{-jkz} \right) \Bigg|_0^\infty \\ &= \sigma E_{x0} \left(\frac{1}{jk} \right) \end{aligned}$$

Surface Impedance (cont.)

Hence

$$J_{sx} = \sigma E_{x0}\left[\frac{1}{j(k' - jk'')}\right]$$

$$= \sigma E_{x0}\left[\frac{1}{(k'' + jk')}\right]$$

$$= \sigma E_{x0}\left[\frac{1}{k''(1+j)}\right]$$

$$= \sigma \delta E_{x0}\left[\frac{1}{1+j}\right]$$

Hence,

$$Z_s = \frac{E_{x0}}{J_{sx}} = \left(\frac{1}{\sigma\delta}\right)(1+j)$$

Surface Impedance (cont.)

$$Z_s = \left(\frac{1}{\sigma\delta}\right)(1+j)$$

Define "surface resistance" and "surface reactance"

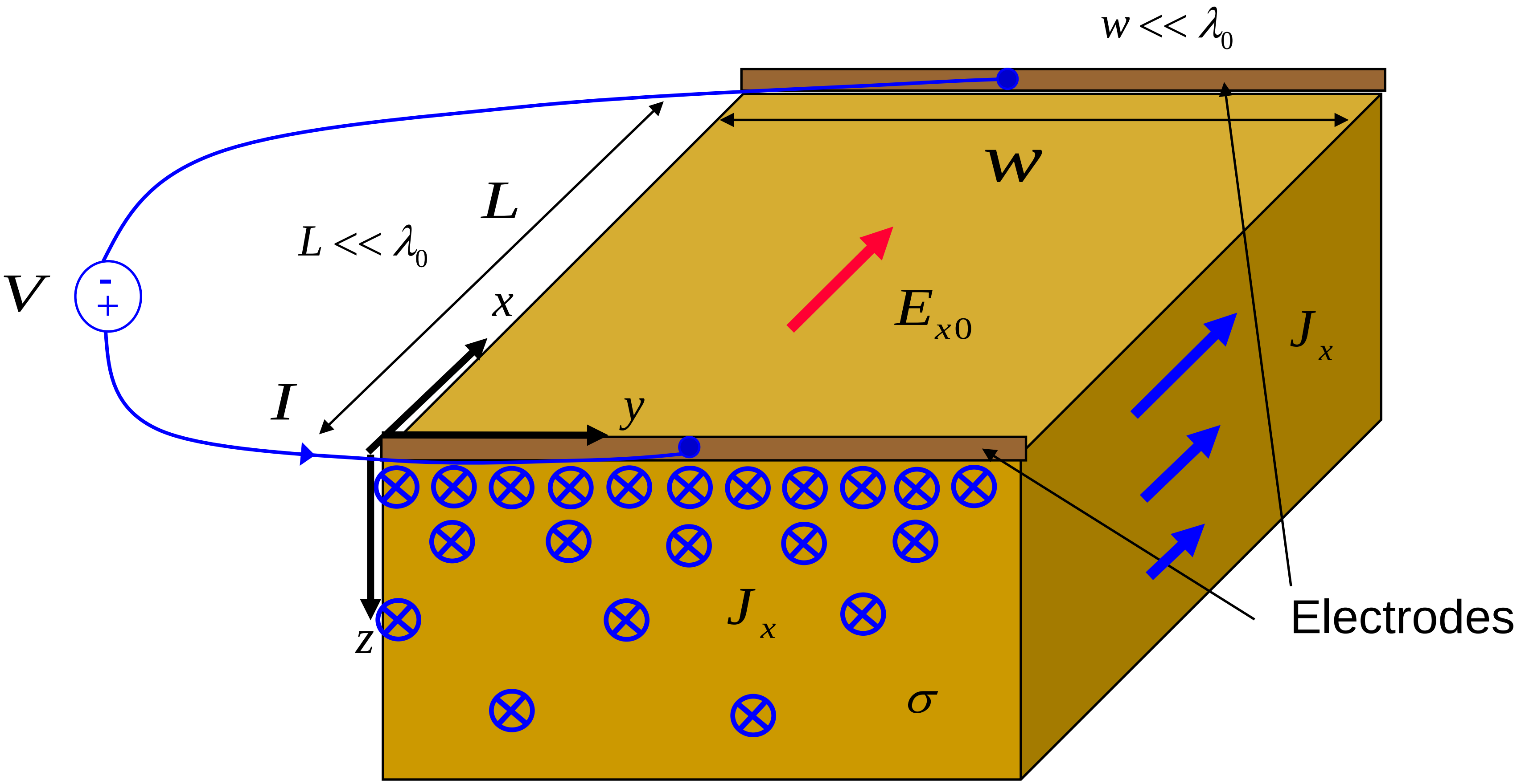
$$Z_s = R_s + jX_s$$

so

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$X_s = R_s$$

Impedance of a Bulk Conductor



Assume no x or y variation

Electrodes are attached at the outer (top) surface.

$$V = E_{x0} L$$

$$I = J_{sx} w$$

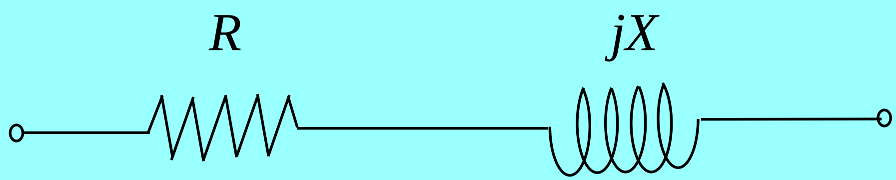
$$Z_{in} = \frac{V}{I} = \left(\frac{E_{x0}}{J_{sx}}\right)\left(\frac{L}{w}\right) = Z_s \frac{L}{w}$$

Impedance of a Bulk Conductor (cont.)

$$Z_{in} = Z_s \left(\frac{L}{w} \right)$$

Assumption: $(w, L) << \lambda_0$

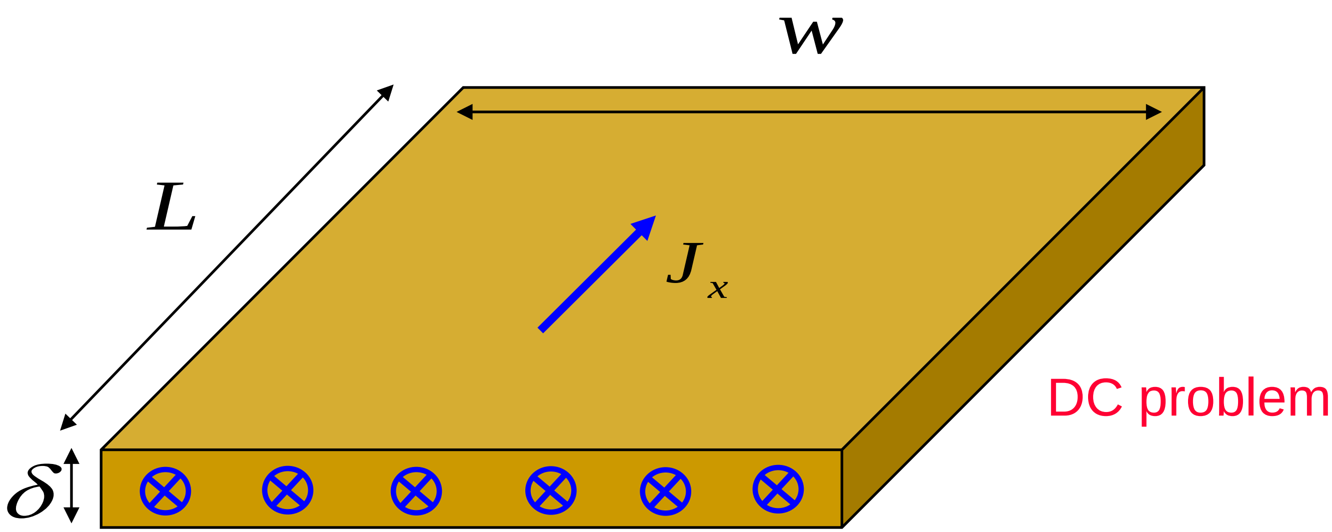
$$Z_{in} = R + jX = R_s (1 + j) \left(\frac{L}{w} \right) = \frac{1}{\sigma \delta} (1 + j) \left(\frac{L}{w} \right)$$



$$R = R_s \left(\frac{L}{w} \right)$$

$$X = R$$

DC Equivalent Model

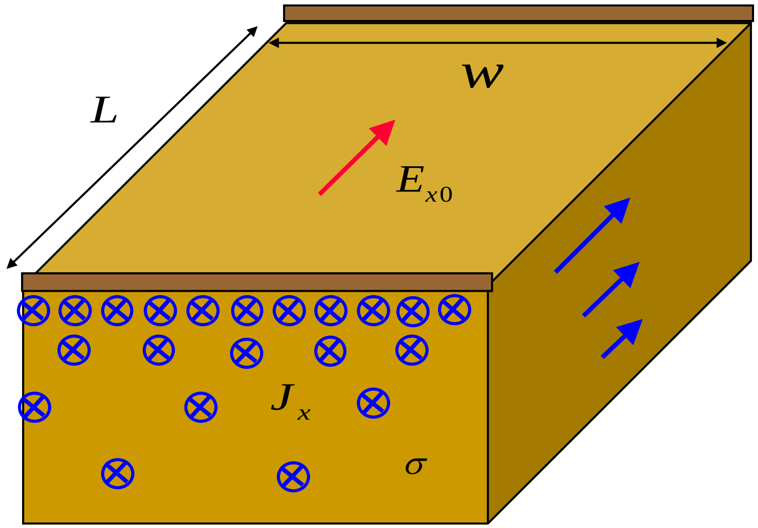


The thickness in the DC problem is chosen as the skin depth in the high-frequency bulk conductor problem.

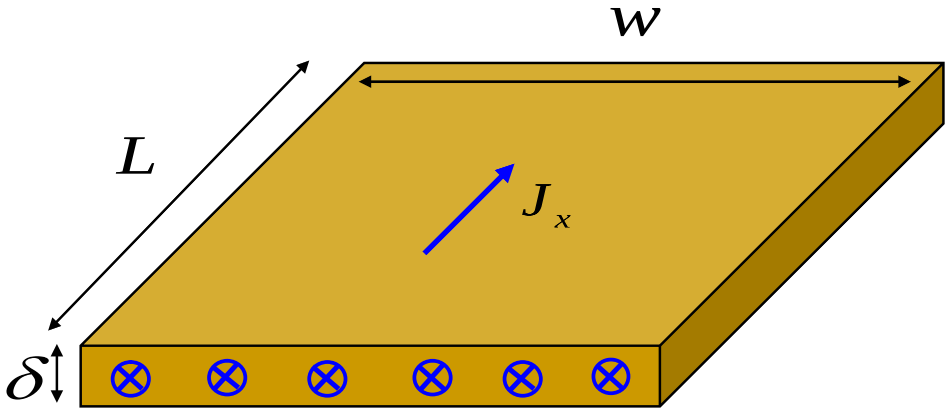
$$R_\delta^{DC} = \frac{L}{\sigma A} = \frac{1}{\sigma}\left(\frac{L}{\delta w}\right) = \frac{1}{\sigma\delta}\left(\frac{L}{w}\right) = R_s\left(\frac{L}{w}\right) = R$$

Hence $$R = R_\delta^{DC}$$

DC Equivalent Model (cont.)



High-frequency problem (R)

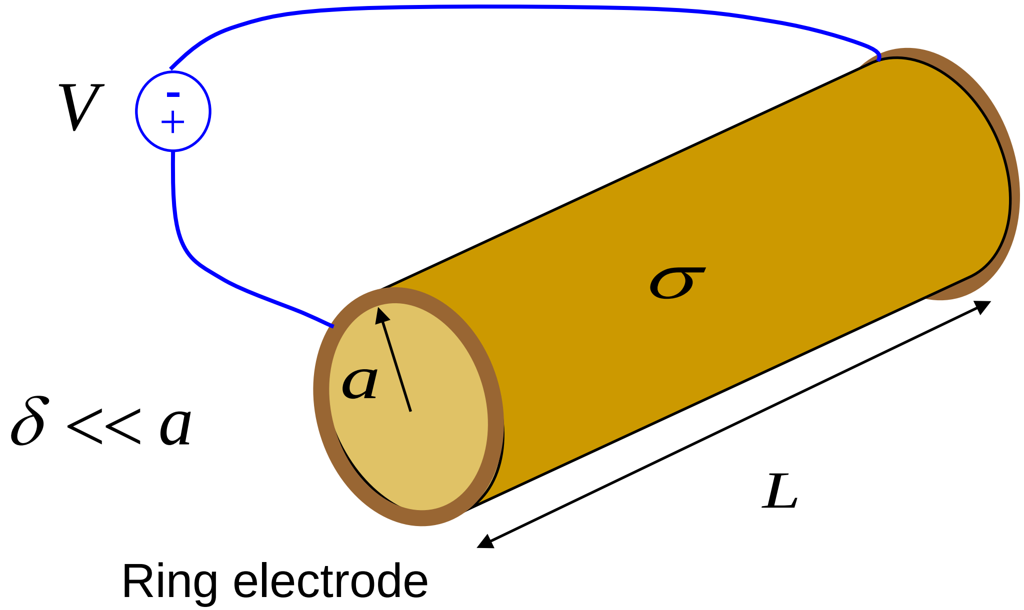


DC current problem (R_δ^{DC})

$$R = R_\delta^{DC}$$

Example

Find the high-frequency resistance and inductance for a solid wire.



$$Z_{in} = Z_s\left(\frac{L}{w}\right)$$

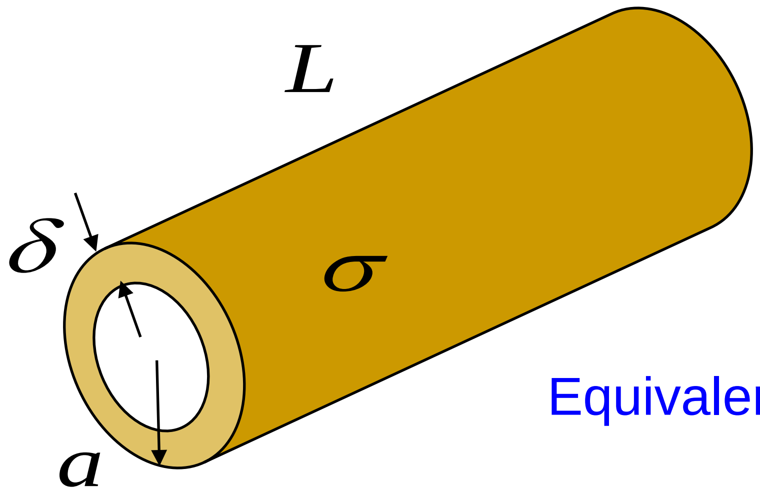
$$= Z_s\left(\frac{L}{2\pi a}\right)$$

$$Z_{in} = Z_s\left(\frac{L}{2\pi a}\right)$$

The current is uniform along the boundary (ϕ direction), and therefore we can treat this as a "rolled-up" version of a bulk conductor ($w = 2\pi a$).

Example (cont.)

We can also get the same result by using the equivalent DC model.

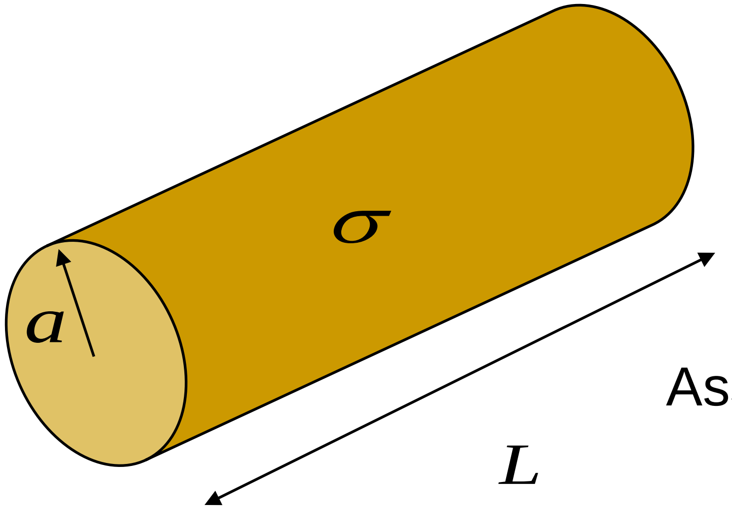


Equivalent DC Model:

$$R = R_{\delta}^{DC} = \frac{L}{\sigma A} \approx \frac{L}{\sigma 2\pi a \delta} = R_s \frac{L}{2\pi a}$$

$$Z_{in} = R + jX = R(1+j) = R_s \frac{L}{2\pi a}(1+j) = R_s(1+j)\frac{L}{2\pi a} = Z_s\left(\frac{L}{2\pi a}\right)$$

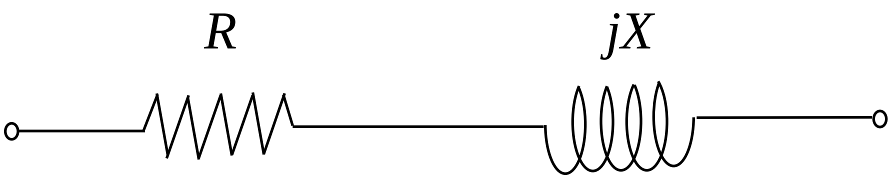
Example (cont.)



High-frequency equivalent circuit:

$\delta << a$

Assumption: $L << \lambda_0$



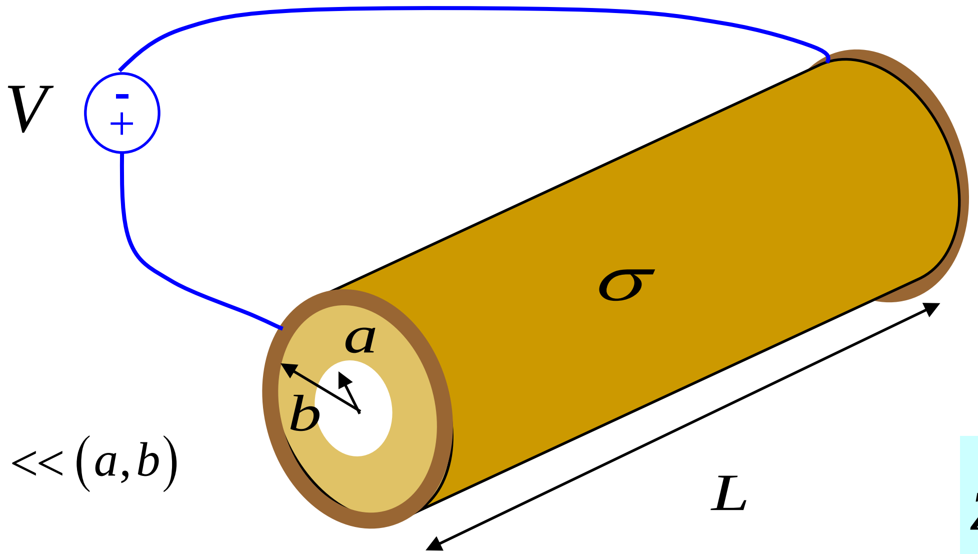
$$Z_{in} = R + jX$$

$$R = X = R_s \left(\frac{L}{2\pi a} \right)$$

Example

Find the high-frequency resistance and inductance for a hollow tube.

The electrodes are attached from the outside.



$\delta << (a,b)$

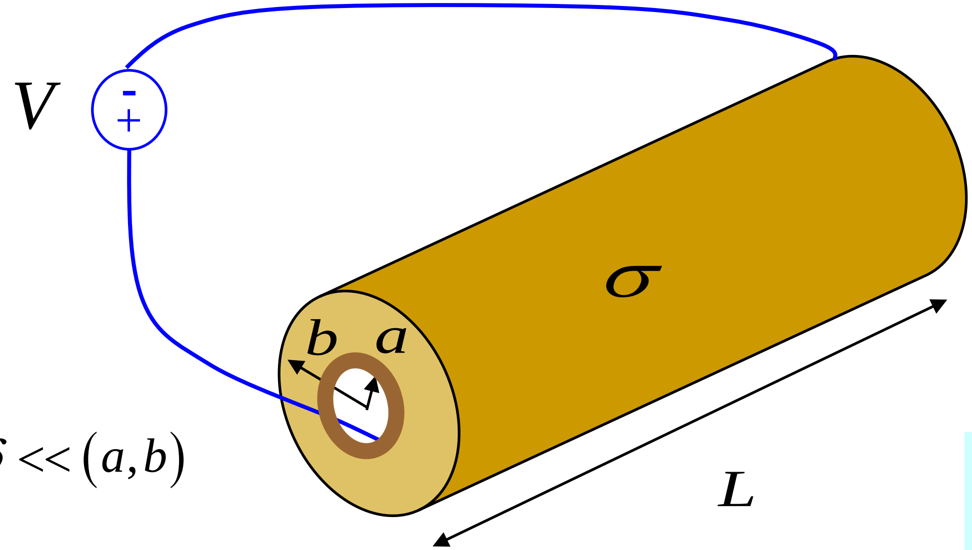
Ring electrode

$$Z_{in} = Z_s \left(\frac{L}{2\pi b} \right)$$

Example

Find the high-frequency resistance and inductance for a hollow tube.

The electrodes are attached from the inside.

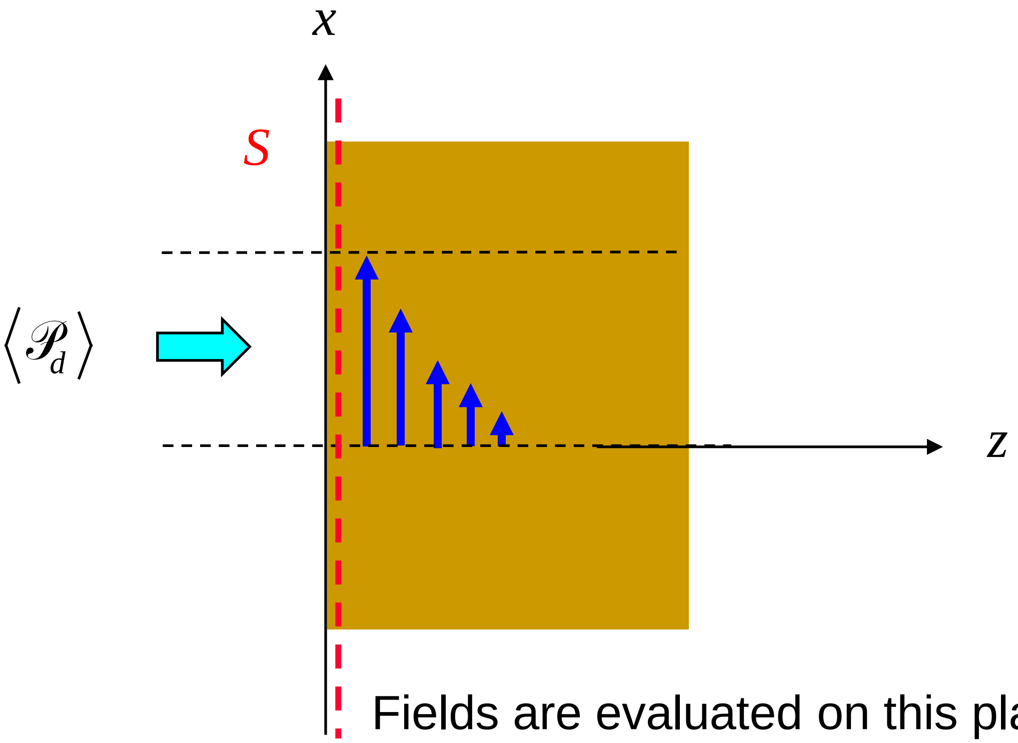


$\delta << (a,b)$

Ring electrode

$$Z_{in} = Z_s \left(\frac{L}{2\pi a} \right)$$

Power Dissipation



$\langle \mathscr{P}_d \rangle$ = time-average power dissipated / m^2 on S

$$\langle \mathscr{P}_d \rangle = \frac{1}{2}\mathrm{Re}\left(\underline{E} \times \underline{H}^*\right)_{z=0} \cdot \underline{\hat{z}} = \frac{1}{2}\mathrm{Re}\left(E_x H_y^*\right)_{z=0} = \frac{1}{2}\mathrm{Re}\left(E_{x0} H_{y0}^*\right)$$

Power Dissipation (cont.)

Use

$$E_{x0} = \eta H_{y0}$$

$$E_x = E_{x0}\, e^{-jkz}$$
$$H_y = H_{y0}\, e^{-jkz}$$
$$\Rightarrow\ E_{x0} / H_{y0} = E_x / H_y = \eta$$

where

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} \approx \sqrt{\frac{\mu}{-j\dfrac{\sigma}{\omega}}} = \sqrt{j}\sqrt{\frac{\omega\mu}{\sigma}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$$

$$= (1+j)\, R_s$$

$$= Z_s$$

$$\eta \approx Z_s$$

Note:

$$\eta = \frac{E_x}{H_y}$$

$$Z_s = \frac{E_x}{J_{sx}}$$

$$\eta \approx Z_s$$

$$\Rightarrow\ J_{sx} \approx H_y$$

Power Dissipation (cont.)

We then have

$$\left\langle \mathscr{P}_d \right\rangle = \frac{1}{2}\mathrm{Re}\left(E_x H_y^*\right)_{z=0} = \frac{1}{2}\mathrm{Re}\left(\eta\right)\left|H_{y0}\right|^2$$

$$= \frac{1}{2} R_s \left|H_{y0}\right|^2$$

In general,

$$\left\langle \mathscr{P}_d \right\rangle = \frac{1}{2} R_s \left|\underline{H}_{t0}\right|^2$$

Power Dissipation (cont.)

For a good conductor,

$$\underline{J}_s \approx \hat{\underline{z}} \times \underline{H}_{t0} \qquad \text{(exactly true for a PEC)}$$

Hence, using this approximation, we have

$$\langle \mathscr{P}_d \rangle = \frac{1}{2} R_s \left| \underline{J}_s \right|^2$$

Power Dissipation (cont.)

For the reactive power absorbed by the conducting surface (VARS/m²) we have:

$$\operatorname{Im} S_z = \frac{1}{2}\operatorname{Im}\left(E_x H_y^*\right)_{z=0} = \frac{1}{2}\operatorname{Im}(\eta)\left|H_{y0}\right|^2$$

$$= \frac{1}{2} X_s \left|H_{y0}\right|^2$$

Hence

$$\operatorname{Im} S_z = \frac{1}{2} X_s \left|\underline{J}_s\right|^2 = \frac{1}{2} R_s \left|\underline{J}_s\right|^2 = \operatorname{Re} S_z = \left\langle \mathscr{P}_d \right\rangle$$

$$\text{VARS / m}^2 = \text{Watts / m}^2$$