ECE 6340
Intermediate EM Waves

Fall 2025

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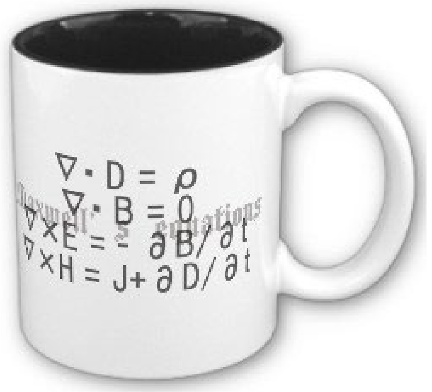
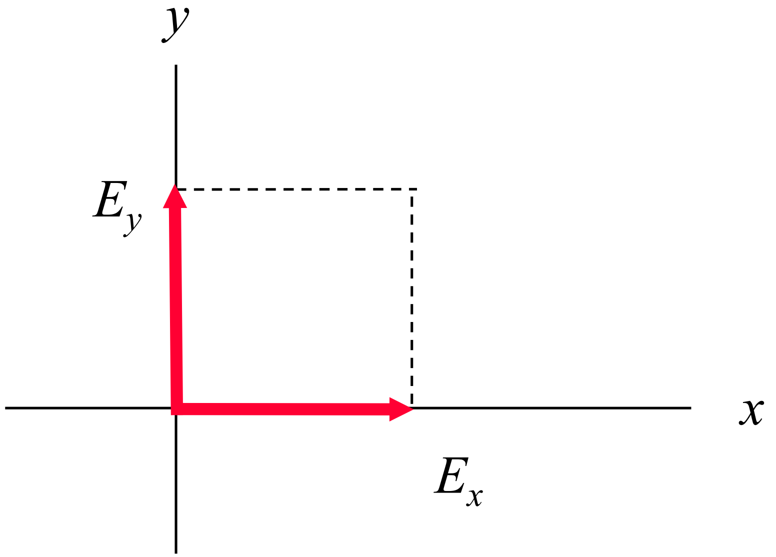


Notes 16

Polarization of Plane Waves

Polarization of Plane Waves

Consider a plane wave with both x and y components



$$\underline{E}(x,y,z) = (\underline{\hat{x}}\, E_{x0} + \underline{\hat{y}}\, E_{y0})\, e^{-jkz}$$

Assume $\quad E_{x0} = a e^{j0} = a$

$$E_{y0} = b\, e^{j\beta}$$

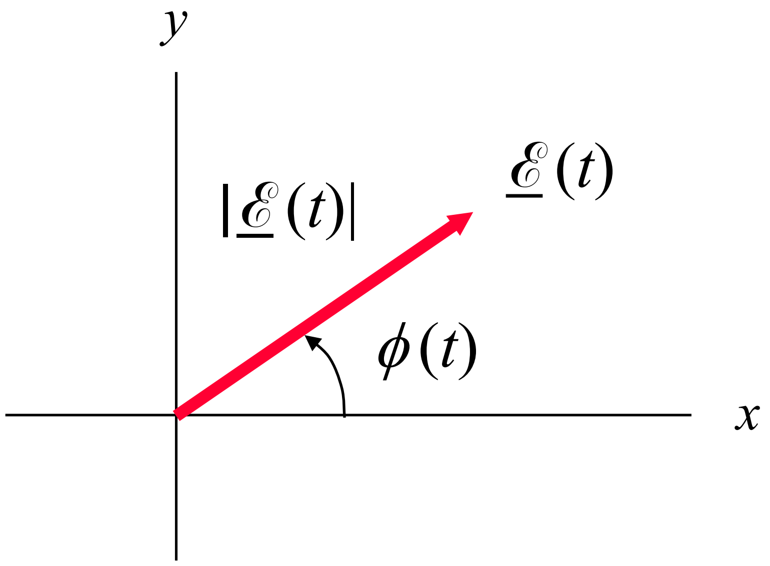
(polar form of complex numbers)

(In general, β = phase of E_{y0} – phase of E_{x0}.)

Polarization of Plane Waves (cont.)

Time Domain:

At $z = 0$



$$\mathcal{E}_x = \mathrm{Re}\left(a\, e^{j\omega t}\right) = a\cos\left(\omega t\right)$$

$$\mathcal{E}_y = \mathrm{Re}\left(b e^{j\beta}\, e^{j\omega t}\right) = b\cos\left(\omega t + \beta\right)$$

Depending on b and β, three different cases arise.

Linear Polarization

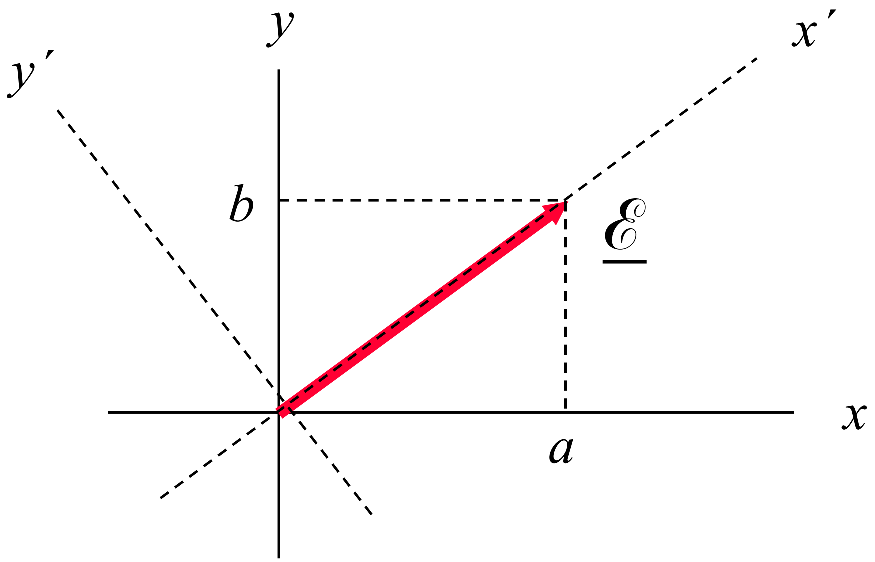
$\beta = 0$ or π

At $z = 0$:

$$\mathcal{E}_x = a\cos\omega t$$

$$\mathcal{E}_y = b\cos\left(\omega t + \beta\right) = \pm b\cos\omega t \qquad (+\text{ for } 0,\ -\text{ for } \pi)$$

$$\underline{\mathcal{E}} = (\hat{\underline{x}}\,a \pm \hat{\underline{y}}\,b)\cos\omega t$$



(shown for $\beta = 0$)

Circular Polarization

$$b = a \text{ AND } \beta = \pm \pi/2$$

At $z = 0$:

$$\mathcal{E}_x = a\cos\omega t$$

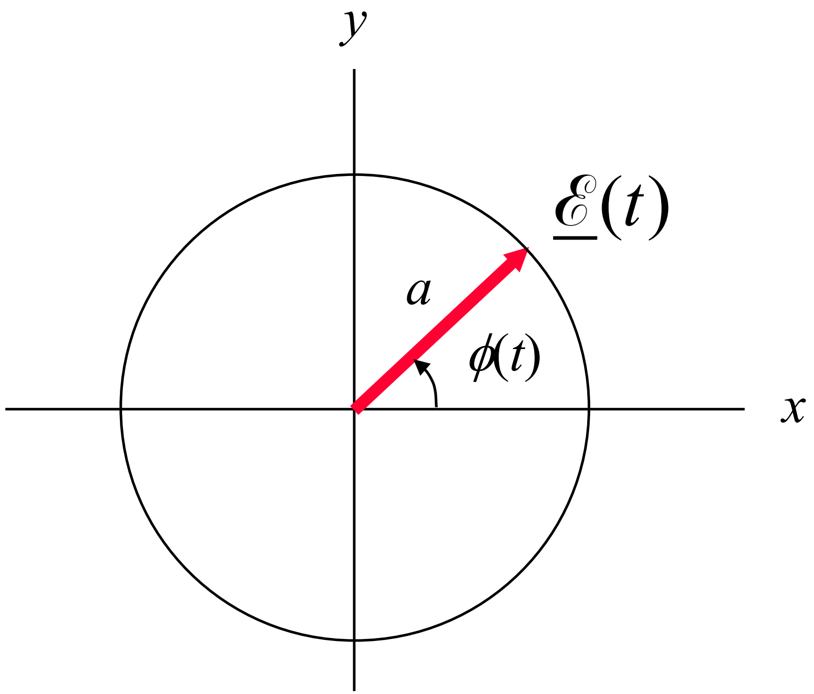
$$\mathcal{E}_y = b\cos(\omega t + \beta) = a\cos(\omega t \pm \pi/2) = \mp a\sin\omega t$$

$$|\underline{\mathcal{E}}|^2 = \mathcal{E}_x^{\,2} + \mathcal{E}_y^{\,2} = a^2\cos^2\omega t + a^2\sin^2\omega t$$

$$= a^2$$

Circular Polarization (cont.)

$$\beta = \pm\pi/2$$



$$\phi = \tan^{-1}\left(\frac{\mathscr{E}_y}{\mathscr{E}_x}\right) = \tan^{-1}(\mp\tan\omega t) = \tan^{-1}(\tan(\mp\omega t))$$

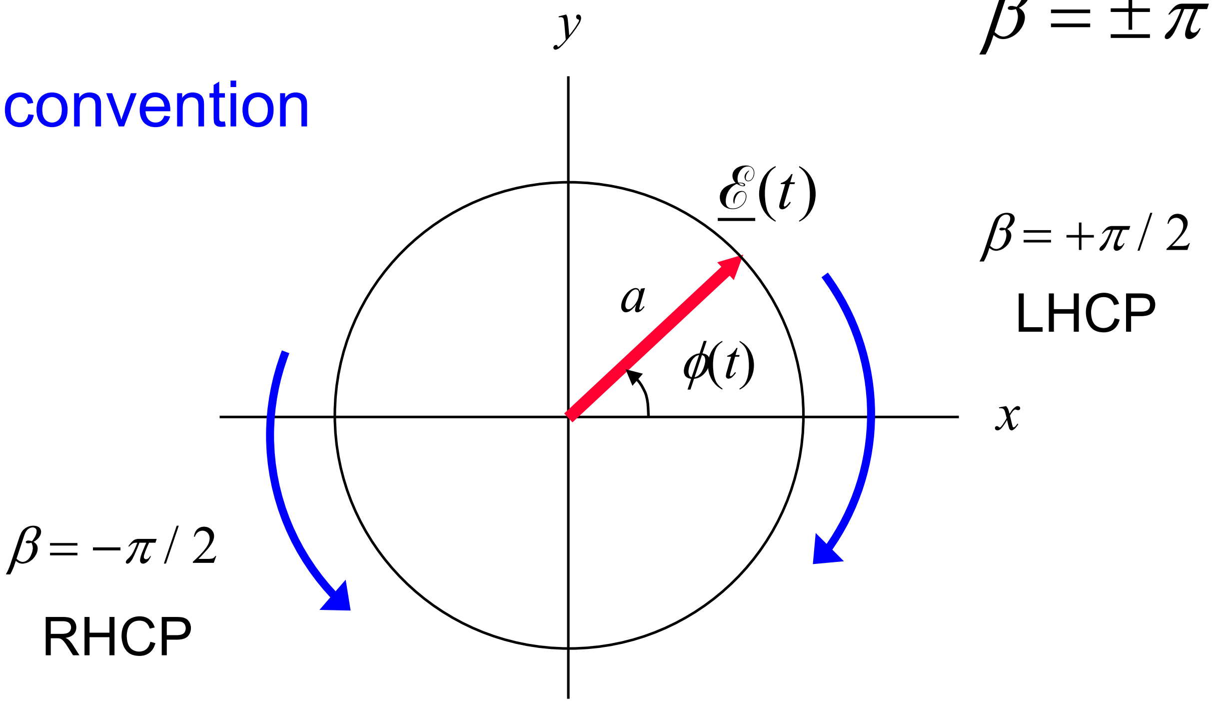
so $\phi = \mp\omega t$

Note: ω_{wave} = angular velocity of field vector $= \left|\frac{d\phi}{dt}\right| = \omega$ $\quad \omega_{\text{wave}} = \omega$

Circular Polarization (cont.)

IEEE convention

$$\beta = \pm \pi / 2$$



$$\phi = \mp \omega t$$

$$\underline{\mathscr{E}} = a\left(\underline{\hat{x}} \cos \omega t \mp \underline{\hat{y}} \sin \omega t\right)$$

Circular Polarization (cont.)

Note: The rotation is space is opposite to that in time.

$$\underline{E}(x,y,z) = (\underline{\hat{x}}\, E_{x0} + \underline{\hat{y}}\, E_{y0})\, e^{-jkz}$$

$$E_x = ae^{j0} = a$$

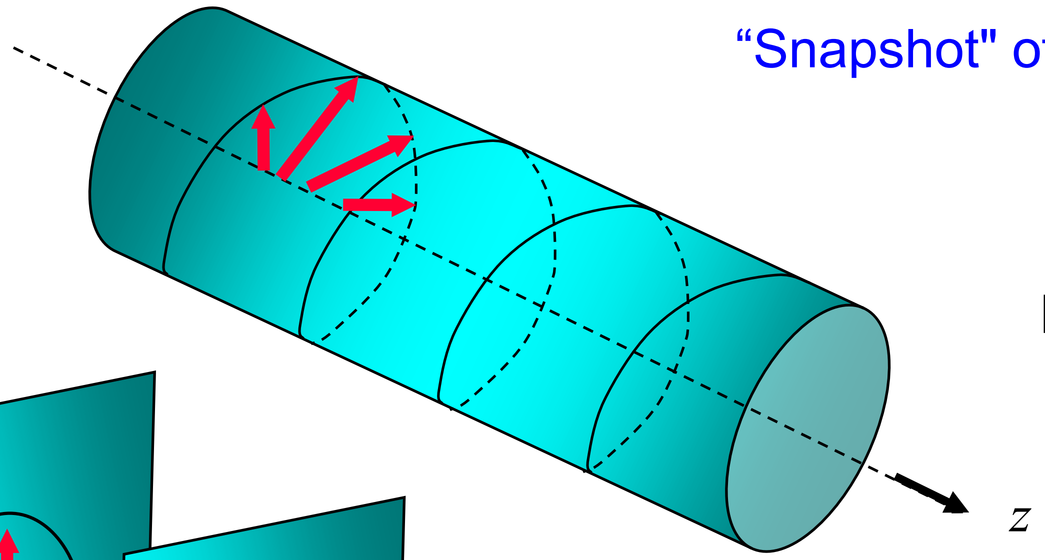
$$E_y = b\, e^{j\beta}$$

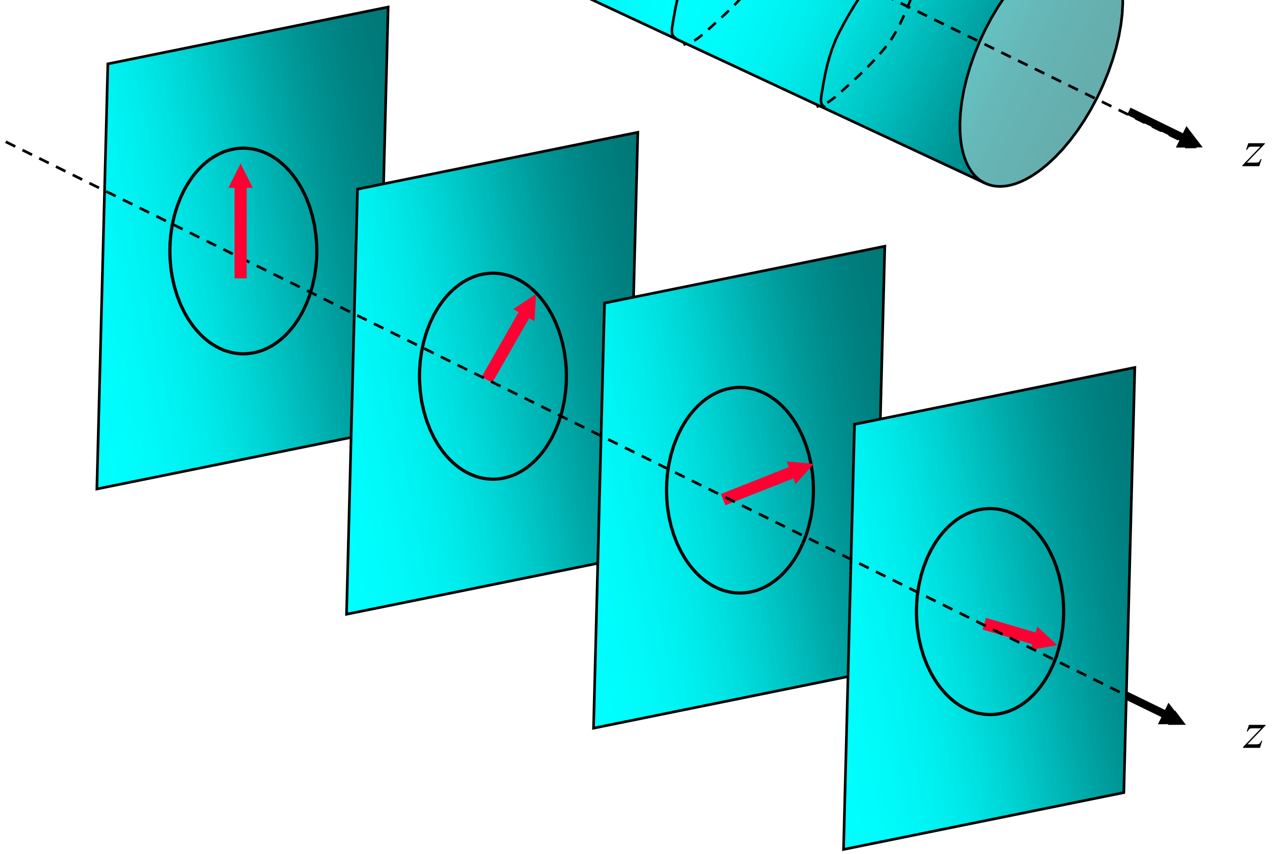
$$\mathscr{E}_x = \mathrm{Re}\left(a\, e^{-jkz} e^{j\omega t}\right) = a\cos\left(\omega t - kz\right)$$

$$\mathscr{E}_y = \mathrm{Re}\left(be^{j\beta} e^{-jkz}\, e^{j\omega t}\right) = b\cos\left(\omega t - kz + \beta\right)$$

Note the minus sign!

Circular Polarization (cont.)

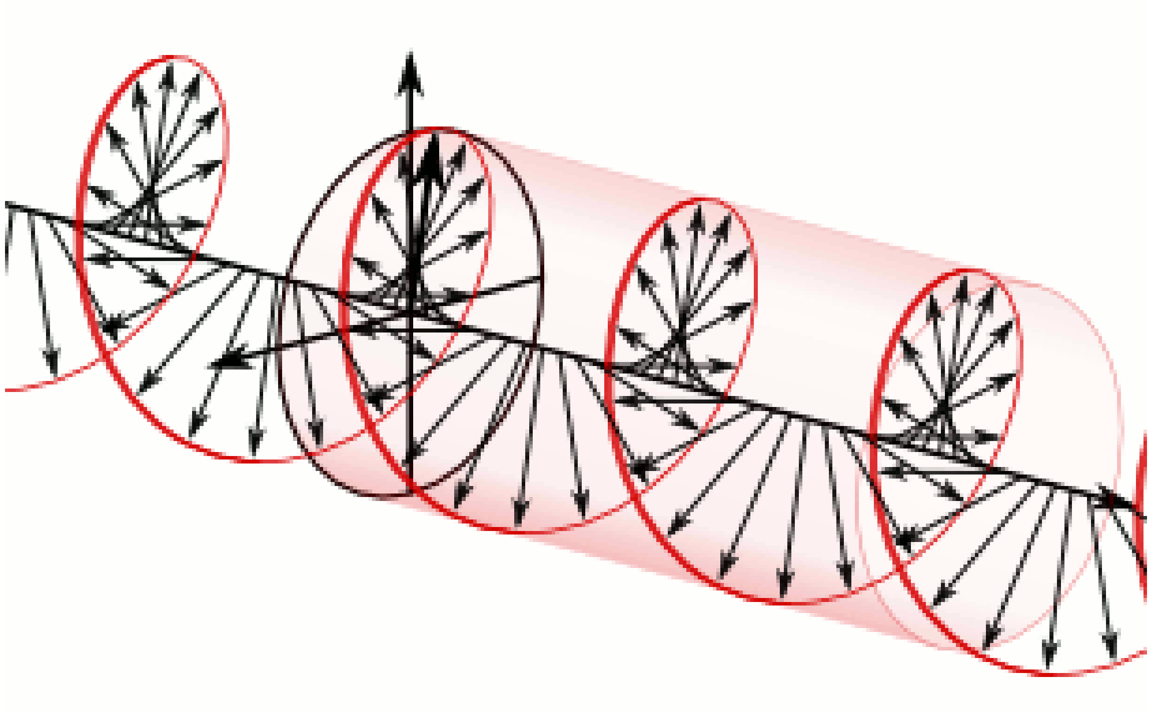




Circular Polarization (cont.)

Animation of LHCP wave

(Use pptx version in full-screen mode to see motion.)



http://en.wikipedia.org/wiki/Circular_polarization

Unit Rotation Vectors

$$\underline{\hat{r}} = \frac{1}{\sqrt{2}}(\underline{\hat{x}} - j\underline{\hat{y}})$$

RHCP ($\beta = -\ \pi / 2$)

$$\underline{\hat{l}} = \frac{1}{\sqrt{2}}(\underline{\hat{x}} + j\underline{\hat{y}})$$

LHCP ($\beta = +\ \pi / 2$)

Add: $$\underline{\hat{x}} = \frac{1}{\sqrt{2}}(\underline{\hat{r}} + \underline{\hat{l}})$$

Subtract: $$\underline{\hat{y}} = \frac{j}{\sqrt{2}}(\underline{\hat{r}} - \underline{\hat{l}})$$

General Wave Representation

$$\underline{E} = \underline{\hat{x}}\, E_x + \underline{\hat{y}}\, E_y$$

$$= E_x \frac{1}{\sqrt{2}}(\underline{\hat{r}} + \underline{\hat{l}}) + E_y \frac{1}{\sqrt{2}}\, j(\underline{\hat{r}} - \underline{\hat{l}})$$

$$\underline{E} = \underline{\hat{r}}\, A_{\mathrm{RH}} + \underline{\hat{l}}\, A_{\mathrm{LH}}$$

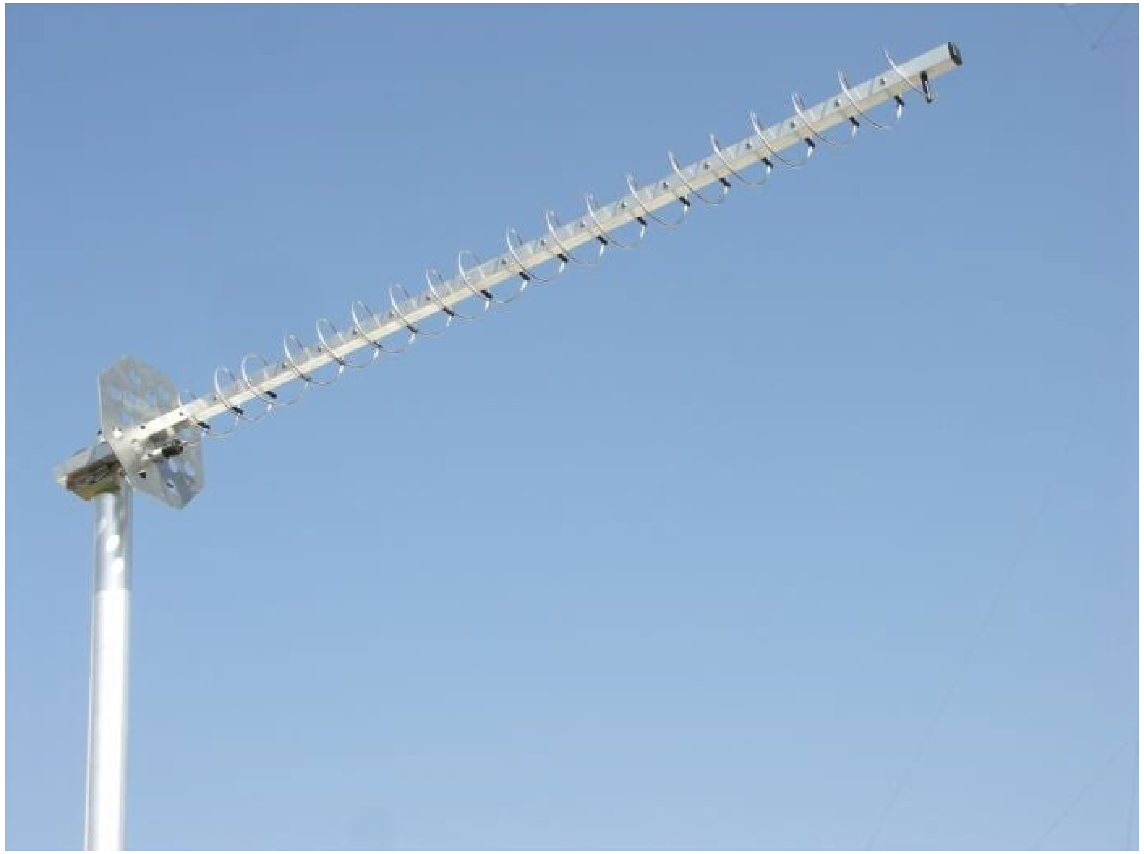
$$A_{\mathrm{RH}} = \frac{1}{\sqrt{2}}(E_x + j\, E_y) \qquad A_{\mathrm{LH}} = \frac{1}{\sqrt{2}}(E_x - j\, E_y)$$

Note: Any polarization can be written as combination of RHCP and LHCP waves.

This could be useful in dealing with CP antennas.

Circularly Polarized Antennas

Method 1: The antenna is fundamentally CP.



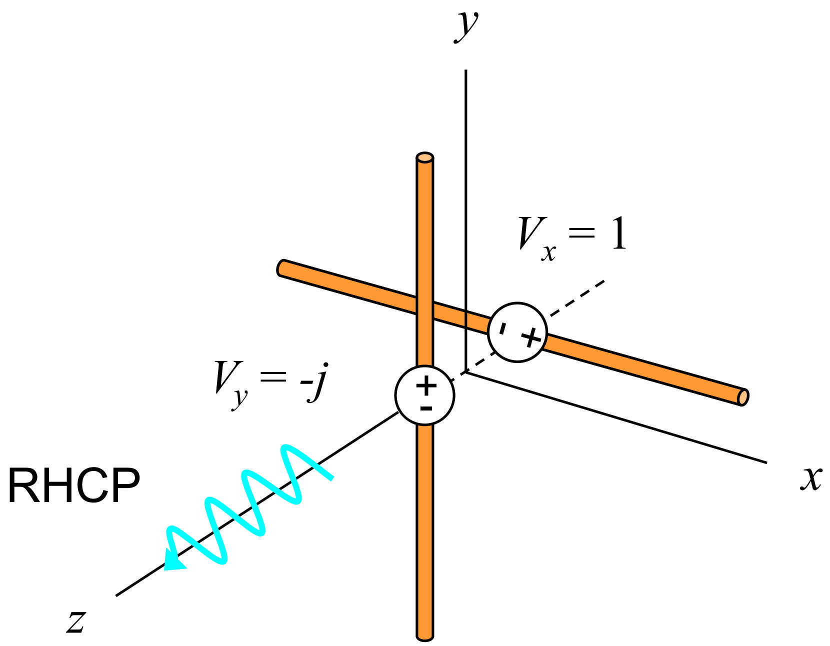
A helical antenna for satellite reception is shown here.

The helical antenna is shown with a circular ground plane.

Circularly Polarized Antennas (cont.)

Method 2: Two perpendicular antennas are used, fed 90° out of phase.

A more complicated version using four antennas (omni CP)

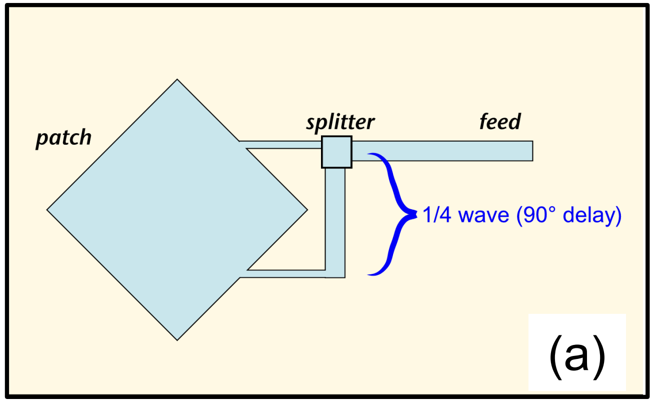


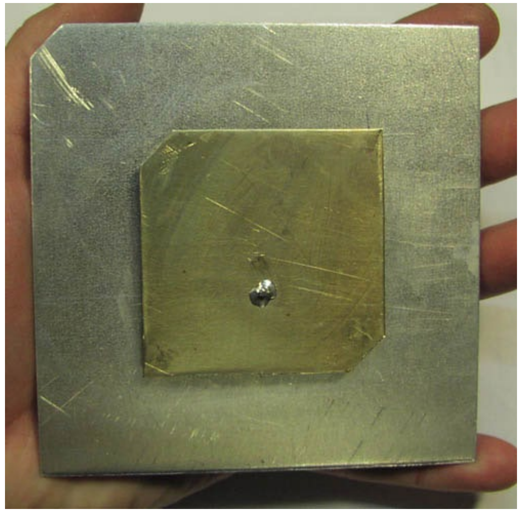
Two antennas are shown here being fed by a common feed point.

Note: LHCP is radiated in the negative z direction.

Circularly Polarized Antennas (cont.)

Method 3: A single antenna is used, but two perpendicular modes are excited 90° out of phase.





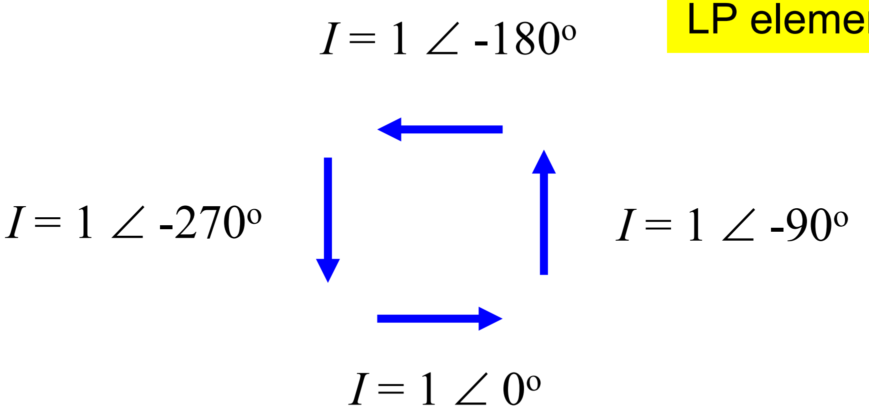
(b)

(a) A square microstrip antenna with two perpendicular modes being excited.

(b) A square patch with two corners chopped off, fed at 45° from the edge.

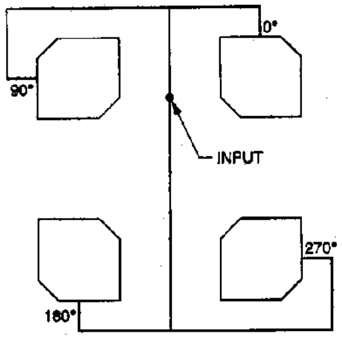
Circularly Polarized Antennas (cont.)

Method 4: Sequential rotation of antennas is used (shown for $N = 4$).



LP elements

CP elements



The elements may be either LP or CP.

- This is an extension of method 2 when used for LP elements.
- Using multiple CP elements instead of one CP element often gives better CP in practice (due to symmetry).

Pros and Cons of CP

Pros

- Alignment between transmit and receive antennas is not important.
- The reception does not depend on rotation of the electric field vector that might occur[†].
- When a RHCP bounces off an object, it mainly changes to a LHCP wave. Therefore, a RHCP receive antenna will not be as sensitive to “multipath” signals that bound from objects.

Cons

- A CP system is usually more complicated and expensive.
- Often, simple wire antennas (LP) are used for reception of signals in wireless communications, and using a CP transmitted signal will result in 50% of the transmitted power being wasted, or a 3 dB drop in the received signal.

[†] Satellite transmission often uses CP because of Faraday rotation in the ionosphere.

Pros and Cons of CP (cont.)

Effects of alignment errors

α = alignment angle error

TX-RX	Gain Loss
LP-LP	$\cos^2\alpha$: The received signal can vary from zero dB to $-\infty$ dB
CP-LP	-3 dB, no matter what the alignment
LP-CP	-3 dB, no matter what the alignment
CP-CP	0 dB (polarizations same) or $-\infty$ dB (polarizations opposite), no matter what the alignment

Examples of Polarization

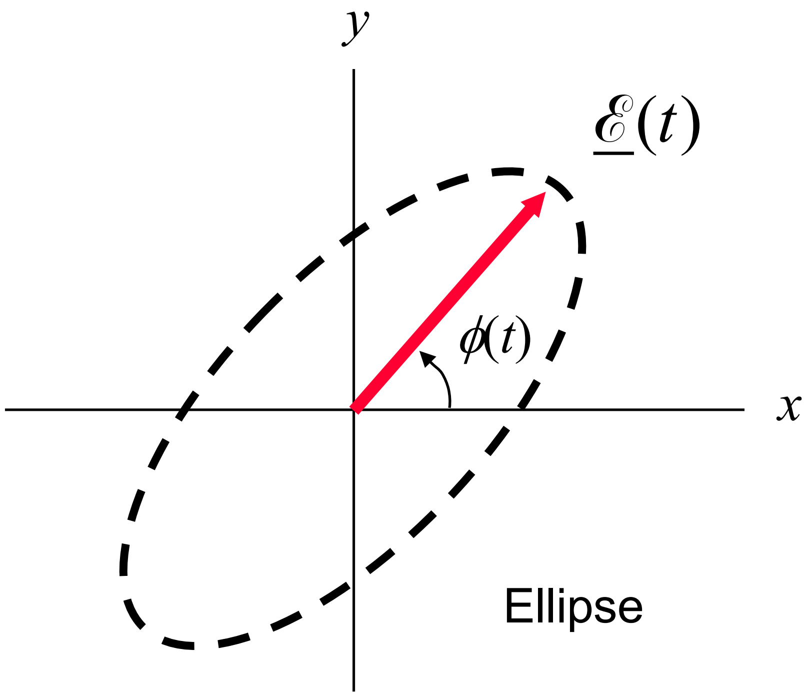
- AM Radio (0.540-1.6 MHz): linearly polarized (vertical) (Note 1)
- FM Radio (88-108 MHz): linearly polarized (horizontal)
- TV (VHF, 54-216 MHz; UHF, 470-698 MHz: linearly polarized (horizontal) (some transmitters are CP)
- Cell phone antenna (about 2 GHz, depending on system): linearly polarized (direction arbitrary)
- Cell phone base-station antennas (about 2 GHz, depending on system): often linearly polarized (dual-linear slant 45^o) (Note 2)
- DBS Satellite TV (11.7-12.5 GHz): transmits both LHCP and RHCP (frequency reuse) (Note 3)
- GPS (1.574 GHz): RHCP (Note 3)

Notes:

1) Low-frequency waves travel better along the earth when they are polarized vertically instead of horizontally.
2) Slant linear is used to switch between whichever polarization is stronger.
3) Satellite transmission often uses CP because of Faraday rotation in the ionosphere.

Elliptical Polarization

Includes all other cases



Elliptical Polarization (cont.)

Note: The rotation is not at a constant speed.

Assume

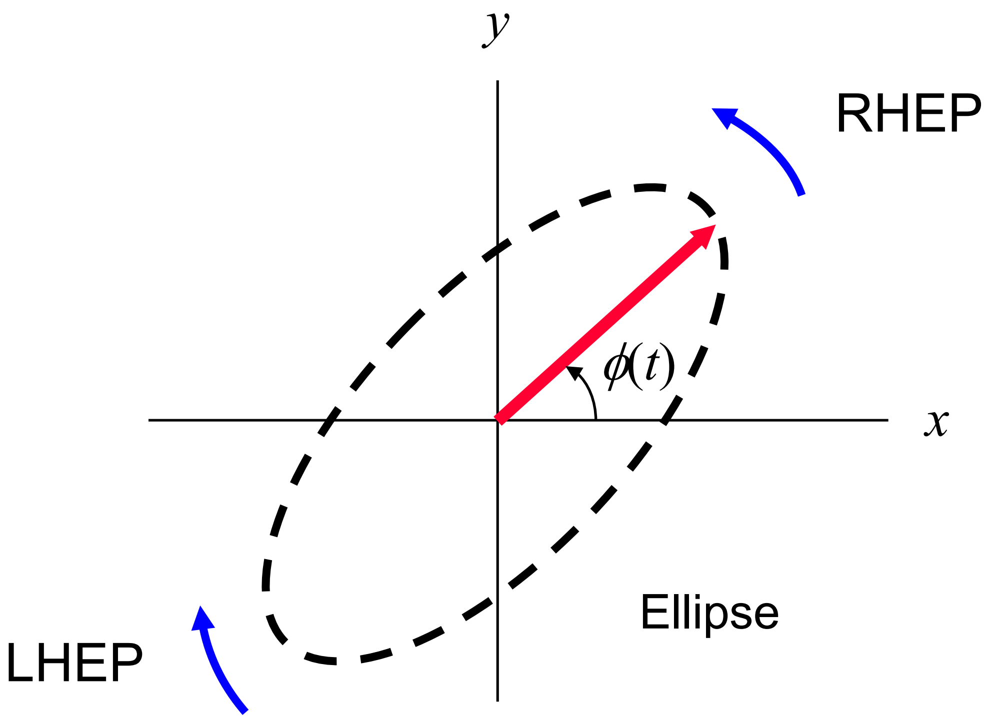
$-\pi < \beta < \pi$

$0 < \beta < \pi$ LHEP

$-\pi < \beta < 0$ RHEP

$$E_{x0} = a e^{j0} = a$$

$$E_{y0} = b\, e^{j\beta}$$



Proof of Ellipse Property

$$\mathscr{E}_x = a\cos\omega t \qquad \mathscr{E}_y = b\cos(\omega t+\beta) = b\cos\omega t\cos\beta - b\sin\omega t\sin\beta$$

so

$$\mathscr{E}_y = b\cos\beta\left[\frac{\mathscr{E}_x}{a}\right] - b\sin\beta\left[\sqrt{1-\left(\frac{\mathscr{E}_x}{a}\right)^2}\right]$$

$$\Rightarrow \quad \mathscr{E}_y - \mathscr{E}_x\left[\frac{b}{a}\cos\beta\right] = -\sin\beta\sqrt{b^2-\left(\frac{b}{a}\right)^2\mathscr{E}_x^{\,2}}$$

$$\Rightarrow \quad \mathscr{E}_y^{\,2} + \mathscr{E}_x^{\,2}\left[\frac{b}{a}\cos\beta\right]^2 - 2\mathscr{E}_x\mathscr{E}_y\left(\frac{b}{a}\cos\beta\right) = \sin^2\beta\left[b^2-\left(\frac{b}{a}\right)^2\mathscr{E}_x^{\,2}\right]$$

Proof of Ellipse Property (cont.)

$$\mathscr{E}_y{}^2 + \mathscr{E}_x{}^2\left[\frac{b}{a}\cos\beta\right]^2 - 2\mathscr{E}_x\mathscr{E}_y\left(\frac{b}{a}\cos\beta\right) = \sin^2\beta\left[b^2 - \left(\frac{b}{a}\right)^2\mathscr{E}_x{}^2\right]$$

$$\Rightarrow \quad \mathscr{E}_x{}^2\left[\left(\frac{b}{a}\right)^2\left(\cos^2\beta + \sin^2\beta\right)\right] + \mathscr{E}_y{}^2 - 2\mathscr{E}_x\mathscr{E}_y\left(\frac{b}{a}\cos\beta\right) = b^2\sin^2\beta$$

$$\Rightarrow \quad \mathscr{E}_x{}^2\left(\frac{b}{a}\right)^2 + \mathscr{E}_x\mathscr{E}_y\left[-2\frac{b}{a}\cos\beta\right] + \mathscr{E}_y{}^2 = b^2\sin^2\beta$$

Consider the following quadratic form:

$$A\,\mathscr{E}_x{}^2 + B\,\mathscr{E}_x\mathscr{E}_y + C\,\mathscr{E}_y{}^2 = D$$

Proof of Ellipse Property (cont.)

Discriminant:

$$\Delta = B^2 - 4AC$$

$$= 4\left(\frac{b}{a}\right)^2 \cos^2\beta - 4\left(\frac{b}{a}\right)^2$$

$$= 4\left(\frac{b}{a}\right)^2 \left[\cos^2\beta - 1\right]$$

so

$$\Delta = -4\left(\frac{b}{a}\right)^2 \sin^2\beta < 0$$

Hence, this is an ellipse.

From analytic geometry:

$\Delta > 0$, hyperbola
$\Delta = 0$, line
$\Delta < 0$, ellipse

If $\beta = 0$ or π we have $\Delta = 0$, and this is linear polarization.

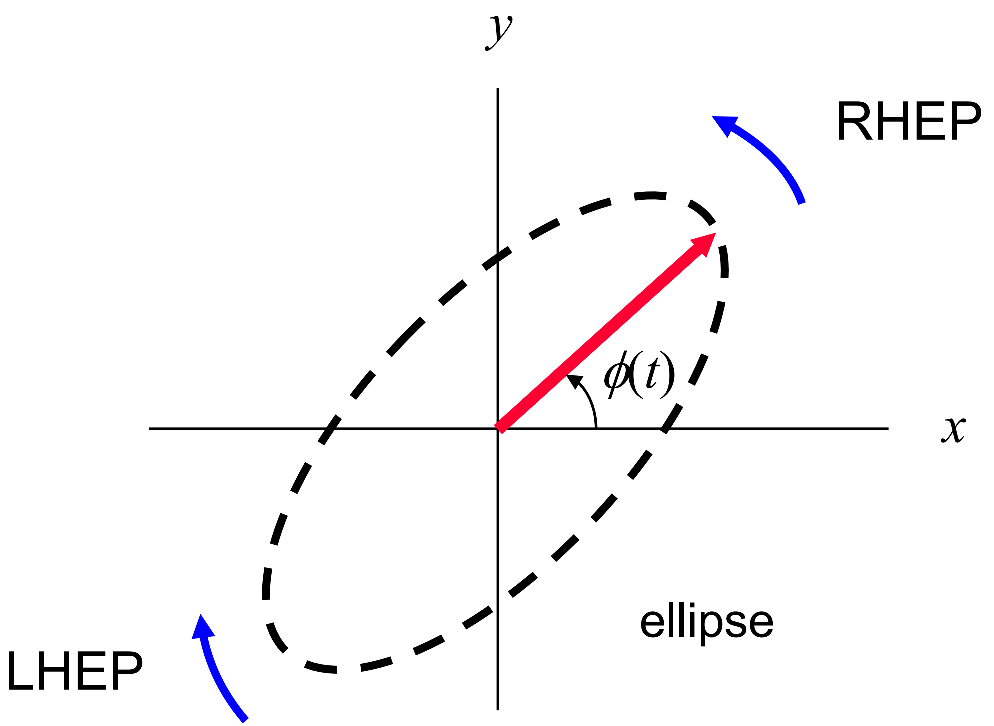
Rotation Property

We now prove the rotation property:

$$0 < \beta < \pi \qquad \text{LHEP}$$
$$-\pi < \beta < 0 \qquad \text{RHEP}$$

Assume

$$-\pi < \beta < \pi$$



Recall that

$$\mathscr{E}_x = a\cos\omega t$$
$$\mathscr{E}_y = b\cos(\omega t + \beta)$$

$$\mathscr{E}_y = b\cos\omega t\cos\beta - b\sin\omega t\sin\beta$$

Rotation Property (cont.)

Hence

$$\tan\phi = \frac{\mathcal{E}_y}{\mathcal{E}_x} = \frac{b}{a}\left[\cos\beta - \tan\omega t \sin\beta\right]$$

Take the derivative:

$$\sec^2\phi \frac{d\phi}{dt} = \left(\frac{b}{a}\right)\left[-\sec^2(\omega t)(\omega)\sin\beta\right]$$

so

$$\frac{d\phi}{dt} = -\sin\beta\left[\left(\frac{b}{a}\right)\cos^2\phi \sec^2(\omega t)(\omega)\right]$$

$$(a) \quad 0 < \beta < \pi \;\Rightarrow\; \frac{d\phi}{dt} < 0$$ LHEP

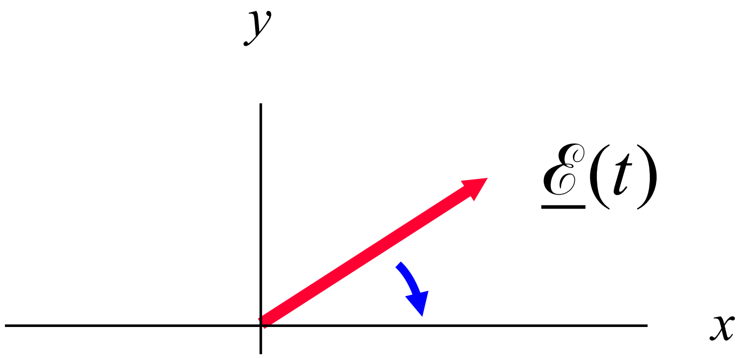
$$(b) \quad -\pi < \beta < 0 \;\Rightarrow\; \frac{d\phi}{dt} > 0$$ RHEP (proof complete)

Phasor Picture

Assume

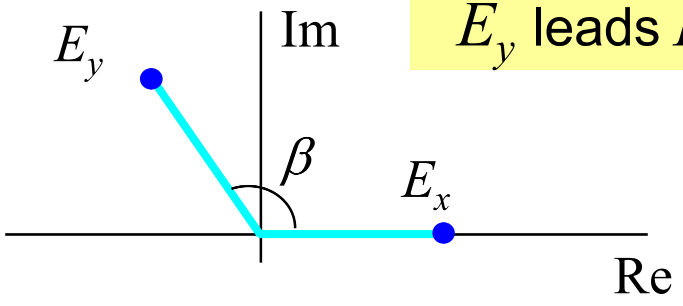
$-\pi < \beta < \pi$

$(a) \quad 0 < \beta < \pi$ LHEP



Rule:

The electric field vector rotates in time from the leading axis to the lagging axis.



E_y leads E_x

LHEP

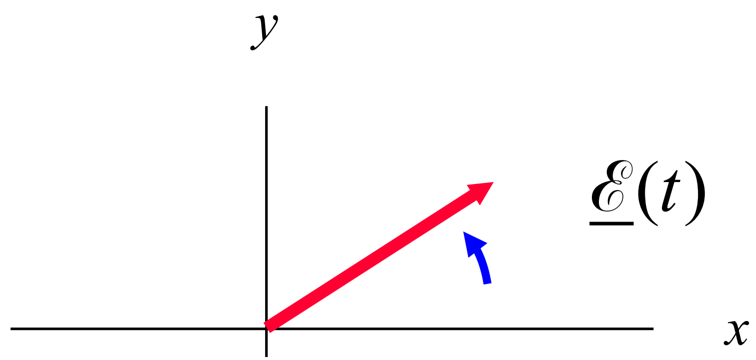
- A phase angle $0 < \phi < \pi$ is a leading phase angle (leading with respect to zero degrees).
- A phase angle $-\pi < \phi < 0$ is a lagging phase angle (lagging with respect to zero degrees).

Phasor Picture (cont.)

Assume

$-\pi < \beta < \pi$

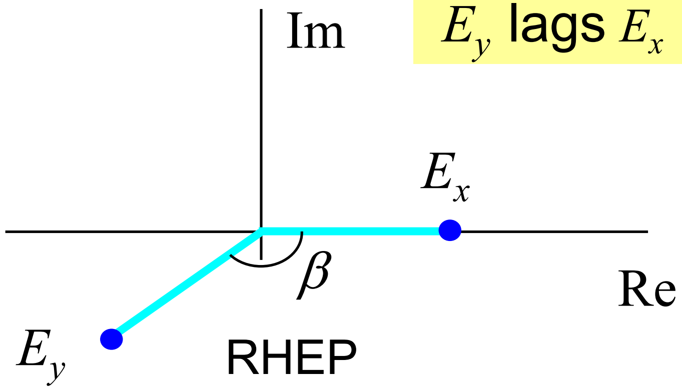
$(b) \quad -\pi < \beta < 0$ RHEP



Rule:

The electric field vector rotates in time from the leading axis to the lagging axis.

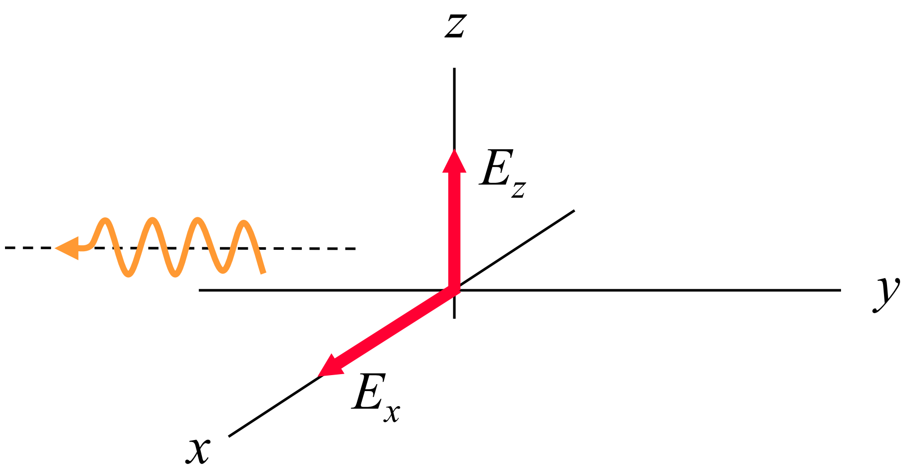
E_y lags E_x



- A phase angle $0 < \phi < \pi$ is a leading phase angle (leading with respect to zero degrees).
- A phase angle $-\pi < \phi < 0$ is a lagging phase angle (lagging with respect to zero degrees).

Example

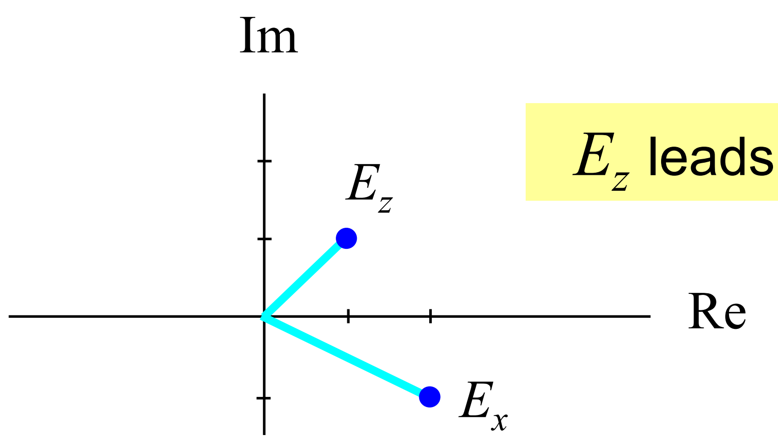
$$\underline{E} = \left[\underline{\hat{z}}(1+j) + \underline{\hat{x}}(2-j)\right]e^{jky}$$



What is this wave’s polarization?

Example (cont.)

$$\underline{E} = \left[\underline{\hat{z}}\,(1+j) + \underline{\hat{x}}\,(2-j) \right] e^{jky}$$

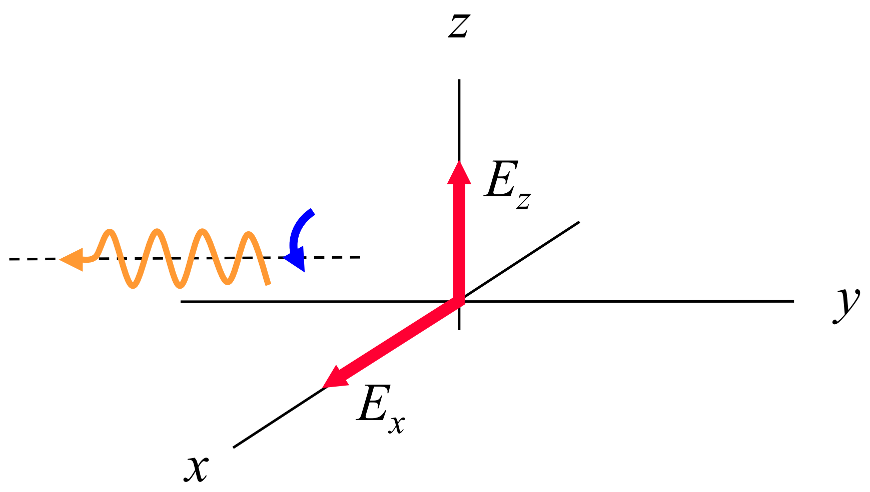


E_z leads E_x

Therefore, in time, the wave rotates from the z axis to the x axis.

Example (cont.)

$$\underline{E} = \left[\underline{\hat{z}}(1+j) + \underline{\hat{x}}(2-j) \right] e^{jky}$$

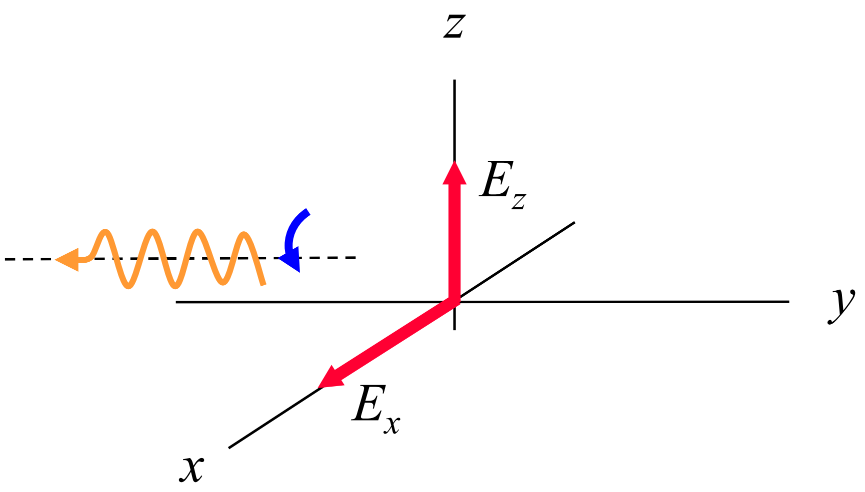


$\Rightarrow$ LHEP or LHCP

Note: $|E_x| \neq |E_z|$ and $\beta \neq \pm \dfrac{\pi}{2}$ (so this is not LHCP)

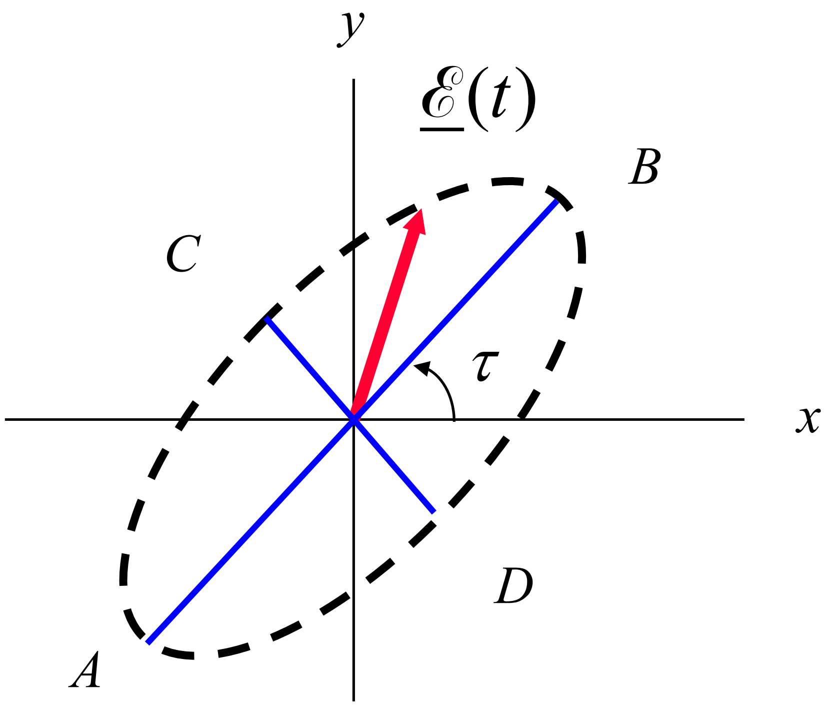
Example (cont.)

$$\underline{E} = \left[\underline{\hat{z}}\,(1 + j) + \underline{\hat{x}}\,(2 - j)\right] e^{jky}$$



LHEP

Axial Ratio (AR) and Tilt Angle (τ)



$$\mathrm{AR} \equiv \frac{\text{major axis}}{\text{minor axis}} = \frac{\mathrm{AB}}{\mathrm{CD}} \geq 1$$

$$\mathrm{AR}_{\mathrm{dB}} = 20\log_{10}\left(\mathrm{AR}\right)$$

Axial Ratio (AR) and Tilt Angle (τ)

$$E_x = ae^{j0} = a$$
$$E_y = b\,e^{j\beta}$$

We first calculate γ:

$$\gamma = \tan^{-1}\left(\frac{b}{a}\right)$$

Tilt Angle

$$\tan 2\tau = \tan 2\gamma \cos\beta$$

$(-90^{o} < \tau < 90^{o})$

Note:
The tilt angle τ is ambiguous by the addition of $\pm 90^{o}$.
(We cannot tell the difference between the major and minor axes.)

Axial Ratio

$$\sin 2\xi = \sin 2\gamma \sin\beta$$

$$\left(-45^{\circ} \le \xi \le +45^{\circ}\right)$$

$$\mathrm{AR} = \cot\left(\left|\xi\right|\right)$$

$\xi > 0$ implies LHEP

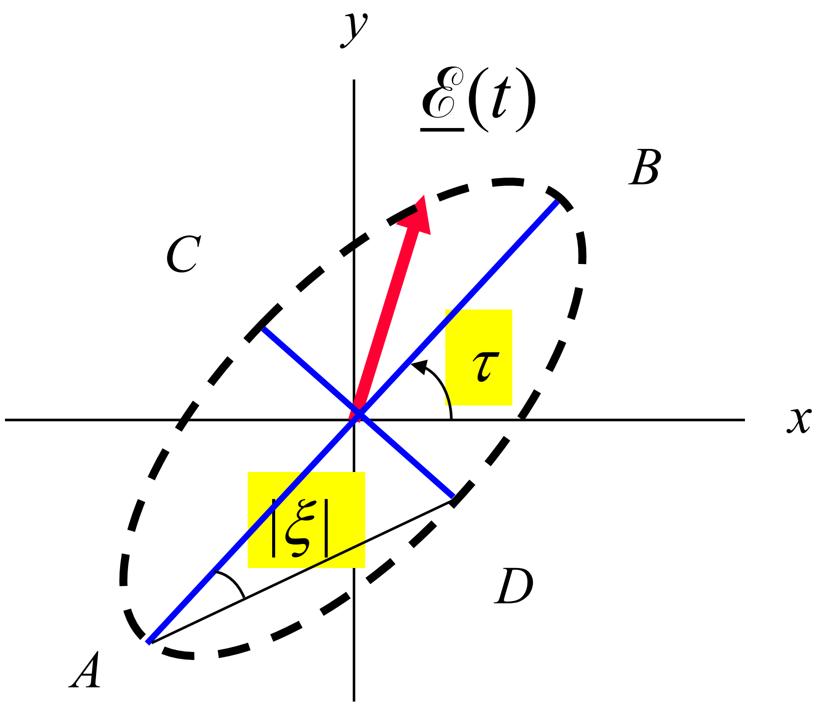
$\xi < 0$ implies RHEP

Axial Ratio (AR) and Tilt Angle (τ) (cont.)

$$\mathrm{AR} = \cot\left(\left|\xi\right|\right)$$

Physical interpretation of the angle $|\xi|$

Note : $\underbrace{0}_{\mathrm{LP}} < \left|\xi\right| < \underbrace{45^{\circ}}_{\mathrm{CP}}$



Special Case

The tilt angle is zero or 90º if: $\beta = \pm\pi / 2$

Tilt Angle: $\tan 2\tau = \tan 2\gamma \cos\beta = 0$

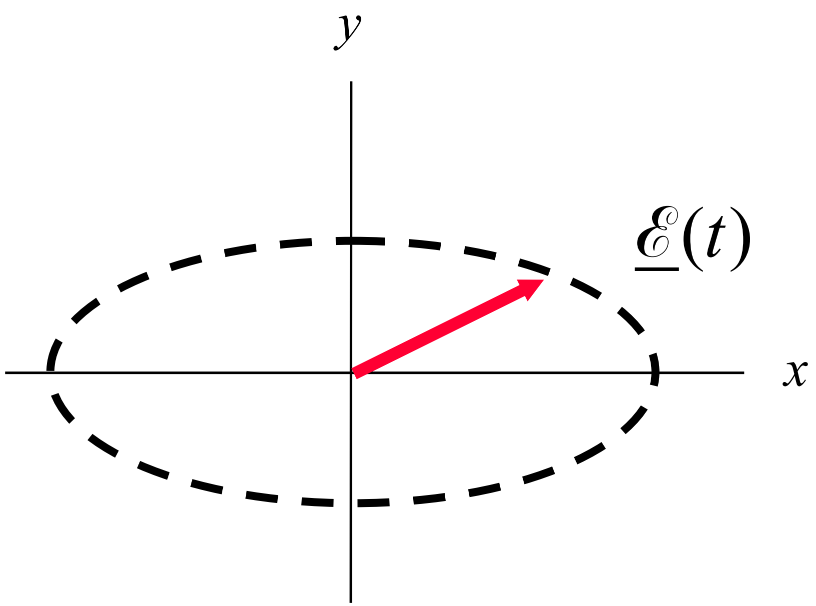
$\Rightarrow \quad \tau = 0$ or $\pi / 2$

$$\mathcal{E}_{x0} = a\cos\omega t$$

$$\mathcal{E}_{y0} = b\cos(\omega t \pm \pi / 2)$$

$$= \mp b\sin\omega t$$

$$\frac{\mathcal{E}_x^2}{a^2} + \frac{\mathcal{E}_y^2}{b^2} = 1$$

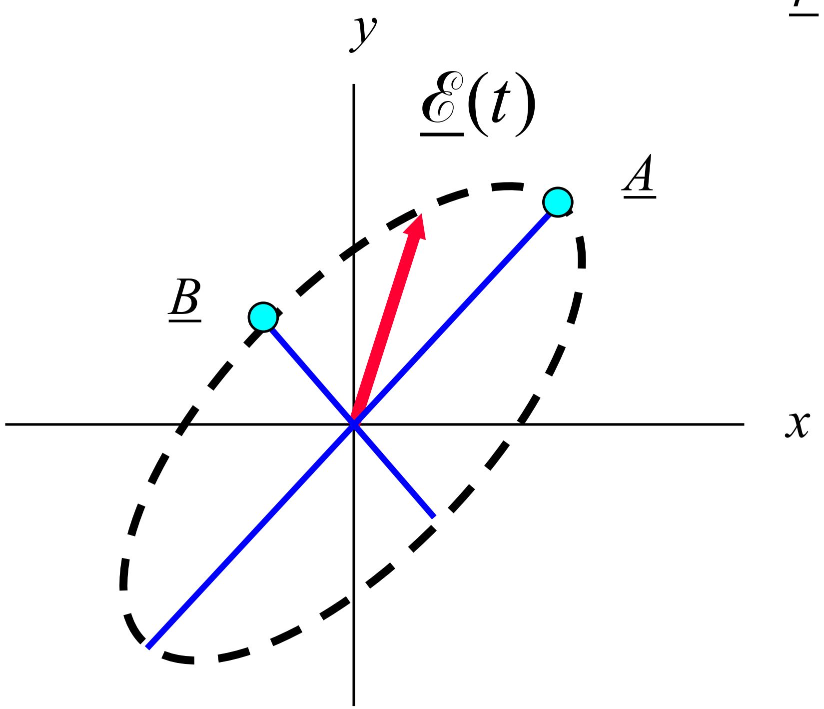


LHCP/RHCP Ratio

Sometimes it is useful to know the ratio of the LHCP and RHCP wave amplitudes.

$$\underline{E} = \underline{\hat{r}}\, A_{\mathrm{RH}} + \underline{\hat{l}}\, A_{\mathrm{LH}} = \underline{\hat{r}} \left| A_{\mathrm{RH}} \right| e^{j\phi_{\mathrm{RH}}} + \underline{\hat{l}} \left| A_{\mathrm{LH}} \right| e^{j\phi_{\mathrm{LH}}}$$

$$\underline{\hat{r}} = \frac{1}{\sqrt{2}}(\underline{\hat{x}} - j\underline{\hat{y}}) \qquad \underline{\hat{l}} = \frac{1}{\sqrt{2}}(\underline{\hat{x}} + j\underline{\hat{y}})$$



Point $\underline{A}$: $\left| \underline{\mathscr{E}}(t) \right| = \left| A_{\mathrm{RH}} \right| + \left| A_{\mathrm{LH}} \right|$

Point $\underline{B}$: $\left| \underline{\mathscr{E}}(t) \right| = \left\| A_{\mathrm{RH}} \right| - \left| A_{\mathrm{LH}} \right\|$

Hence, we have:

$$\mathrm{AR} = \frac{\left| A_{\mathrm{RH}} \right| + \left| A_{\mathrm{LH}} \right|}{\left\| A_{\mathrm{RH}} \right| - \left| A_{\mathrm{LH}} \right\|} = \left| \frac{\left| A_{\mathrm{RH}} \right| + \left| A_{\mathrm{LH}} \right|}{\left| A_{\mathrm{RH}} \right| - \left| A_{\mathrm{LH}} \right|} \right|$$

LHCP/RHCP Ratio (Cont.)

Hence

$$\mathrm{AR} = \left| \frac{1 + |A_{\mathrm{LH}}| / |A_{\mathrm{RH}}|}{1 - |A_{\mathrm{LH}}| / |A_{\mathrm{RH}}|} \right|$$

or

$$\mathrm{AR} = \pm \left(\frac{1 + |A_{\mathrm{LH}}| / |A_{\mathrm{RH}}|}{1 - |A_{\mathrm{LH}}| / |A_{\mathrm{RH}}|} \right)$$

(Use whichever sign gives $\mathrm{AR} > 0$.)

From this we can also solve for the ratio of the CP components, if we know the axial ratio:

$$\frac{|A_{\mathrm{LH}}|}{|A_{\mathrm{RH}}|} = \frac{\mathrm{AR} - 1}{\mathrm{AR} + 1} \quad \text{or} \quad \frac{\mathrm{AR} + 1}{\mathrm{AR} - 1}$$

(We can’t tell which one is correct from knowing only the AR.)

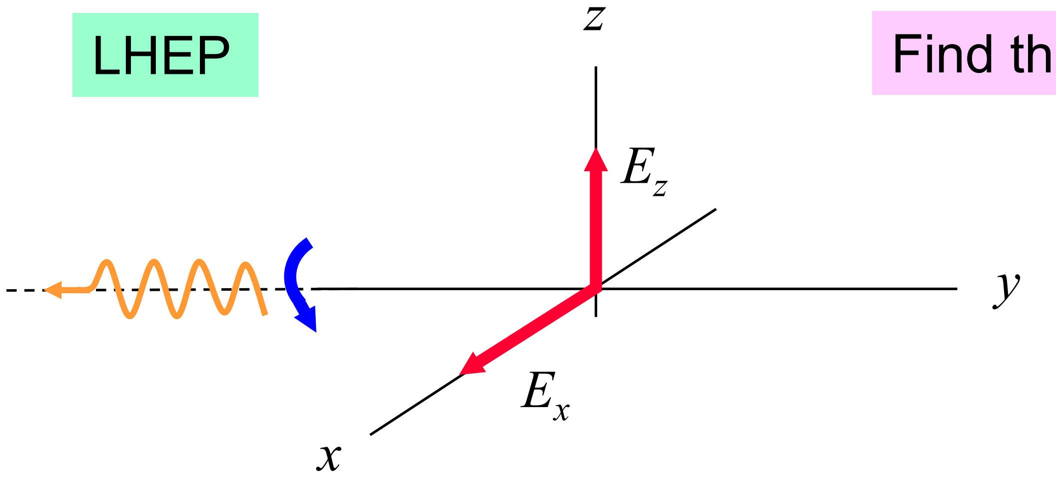
Example

$$\underline{E} = \left[\underline{\hat{z}}(1+j) + \underline{\hat{x}}(2-j) \right] e^{jky}$$

Find the axial ratio and tilt angle.

Find the ratio of the CP amplitudes.

LHEP



Re-label the coordinate system:

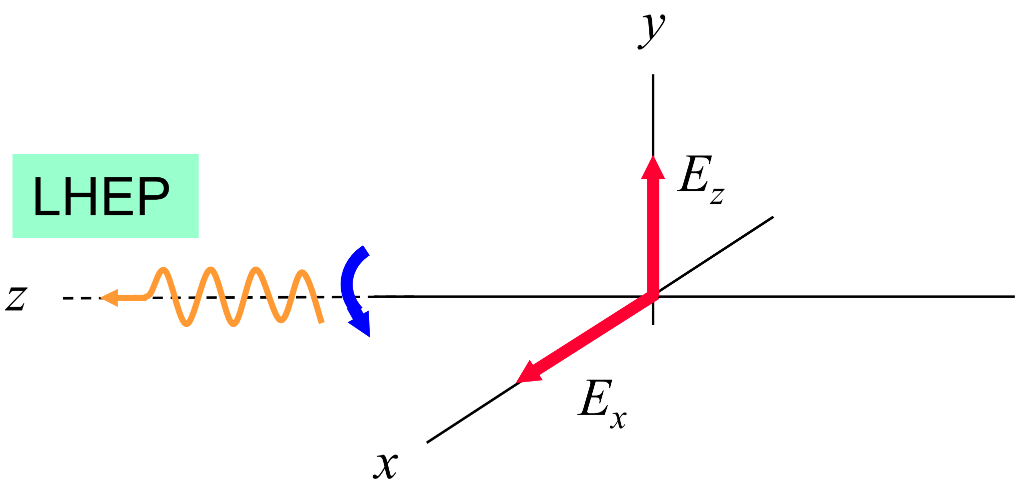
$$x \to x$$

$$z \to y$$

$$y \to -z$$

Example (cont.)

$$\underline{E} = \left[\underline{\hat{x}}(2-j) + \underline{\hat{y}}(1+j) \right] e^{-jkz}$$



$$\frac{E_y}{E_x} = \frac{1+j}{2-j} = 0.2 + j0.6 = 0.6324 e^{j1.249} = 0.6324 \angle 71.565^{\circ}$$

$$\Rightarrow \quad b/a = 0.6324, \quad \beta = 71.565^{\circ}$$

$$\gamma = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(0.632) = 0.564 \text{ radians}$$

Example (cont.)

Formulas

$$\tan 2\tau = \tan 2\gamma \cos\beta$$

$$\sin 2\xi = \sin 2\gamma \sin\beta$$

$$-45^{\circ} \le \xi \le +45^{\circ}$$

$$\mathrm{AR} = \cot|\xi|$$

$$\xi > 0: \quad \mathrm{LHEP}$$

$$\xi < 0: \quad \mathrm{RHEP}$$

Results

$$\tau = 16.845^{\circ}$$

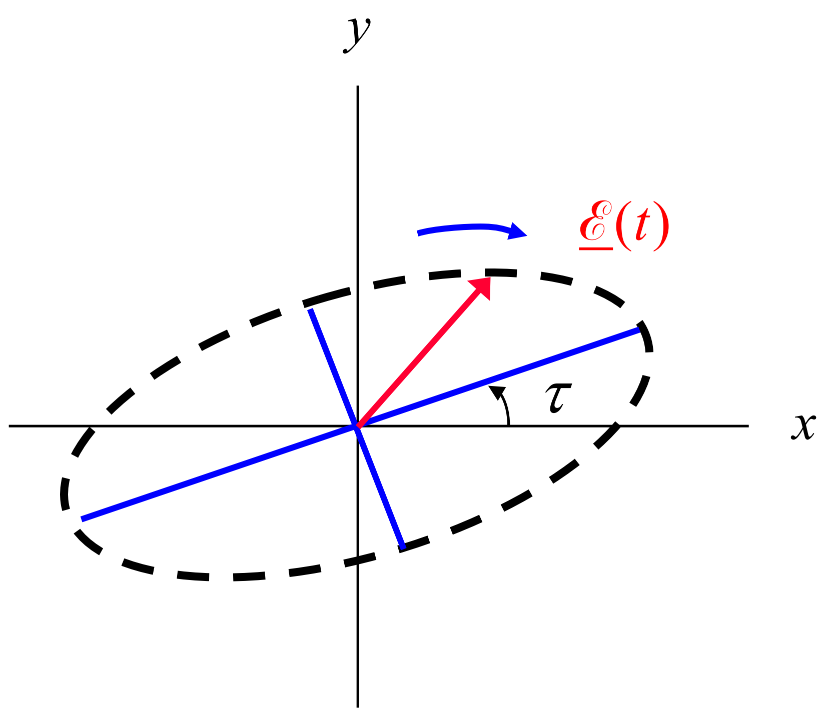
or $16.845^{\circ} \pm 90^{\circ}$

$$\xi = 29.499^{\circ}$$

$$\mathrm{AR} = 1.768$$

LHEP

Example (cont.)



$$\mathrm{AR} = 1.768$$

$$\tau = 16.845^{\circ}$$

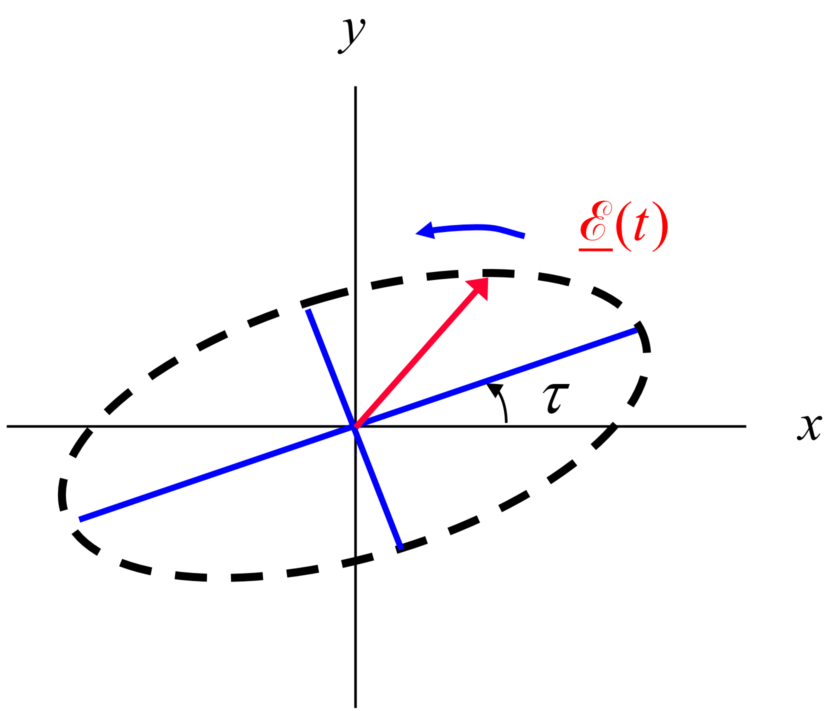
Normalized field ($a = 1$):

$$\mathcal{E}_{x0} = \cos \omega t$$

$$\mathcal{E}_{y0} = (0.6324)\cos(\omega t + 71.565^{\circ})$$

Note: Plotting the ellipse as a function of time will help determine which value is correct for the tilt angle τ.

Example (cont.)



$$\text{AR} = 1.768$$

$$\frac{|A_{\text{LH}}|}{|A_{\text{RH}}|} = \frac{\text{AR}-1}{\text{AR}+1} \text{ or } \frac{\text{AR}+1}{\text{AR}-1}$$

Hence, we have:

$$\frac{|A_{\text{LH}}|}{|A_{\text{RH}}|} = 0.2775 \text{ or } 3.604$$

We know that the polarization is LHEP, so the LHCP amplitude must dominate.

Hence, we have:

$$\frac{|A_{\text{LH}}|}{|A_{\text{RH}}|} = 3.604$$