ECE 6340
Intermediate EM Waves

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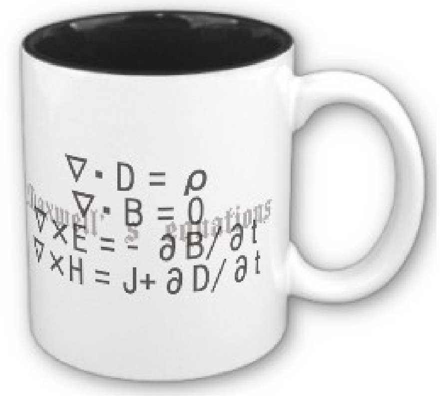


Notes 17

General Plane Waves

General form of plane wave: $\underline{E}(x, y, z) = \underline{E}_0 \, \psi(x, y, z)$

where $\psi(x, y, z) = e^{-j(k_x x + k_y y + k_z z)}$

The wavenumber terms may be complex.

Helmholtz Eq.: $\nabla^2 \underline{E} + k^2 \underline{E} = \underline{0}$

Property of vector Laplacian: $\nabla^2 \left(\underline{E}_0 \psi(x, y, z) \right) = \underline{E}_0 \nabla^2 \psi(x, y, z)$

Hence $\underline{E}_0 \nabla^2 \psi + k^2 \underline{E}_0 \psi = 0$

so $\nabla^2 \psi + k^2 \psi = 0$

General Plane Waves (cont.)

$$\nabla^2 \psi + k^2 \psi = 0$$

This gives

$$(-k_x^{\,2} - k_y^{\,2} - k_z^{\,2} + k^2)\,\psi = 0$$

or

$$k_x^{\,2} + k_y^{\,2} + k_z^{\,2} = k^2$$

(separation equation or wavenumber equation)

General Plane Waves (cont.)

Denote

$$\underline{k} = \underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y + \underline{\hat{z}}\, k_z$$

$$\underline{r} = \underline{\hat{x}}\, x + \underline{\hat{x}}\, y + \underline{\hat{x}}\, z$$

Then

$$\psi(x, y, z) = e^{-j\underline{k}\cdot\underline{r}}$$

and

$$\underline{k} \cdot \underline{k} = k^2$$

(wavenumber equation)

Note: For complex $\underline{k}$ vectors, this is not the same as saying that the magnitude of the $\underline{k}$ vector is equal to k.

General Plane Waves (cont.)

We can also write

$$\underline{k} = \underline{\beta} - j\underline{\alpha}$$

Wavenumber vector

so

$$\psi(x, y, z) = e^{-j\underline{\beta}\cdot\underline{r}}\, e^{-\underline{\alpha}\cdot\underline{r}}$$

The $\underline{\beta}$ vector gives the direction of most rapid phase change (decrease).

The $\underline{\alpha}$ vector gives the direction of most rapid attenuation.

To illustrate, consider the phase of the plane wave:

$$\Phi(x, y, z) = -\underline{\beta}\cdot\underline{r} = -\beta_x x - \beta_y y - \beta_z z$$

$$\Rightarrow \quad \nabla\Phi(x, y, z) = -\beta_x \underline{\hat{x}} - \beta_y \underline{\hat{y}} - \beta_z \underline{\hat{z}} = -\underline{\beta}$$

Similarly, $\nabla|\Psi|(x, y, z) = -\underline{\alpha}|\Psi|$

General Plane Waves (cont.)

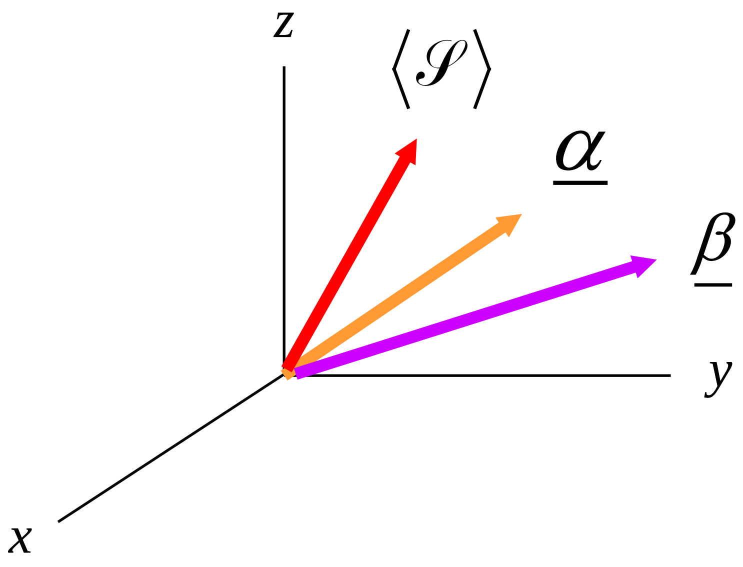
A general plane wave:

$$\psi(x,y,z) = e^{-j\underline{k}\cdot\underline{r}}$$

$$\underline{k} = \underline{\beta} - j\underline{\alpha}$$

$$\langle \underline{\mathscr{S}} \rangle = \operatorname{Re} \underline{S}$$

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^{*}$$

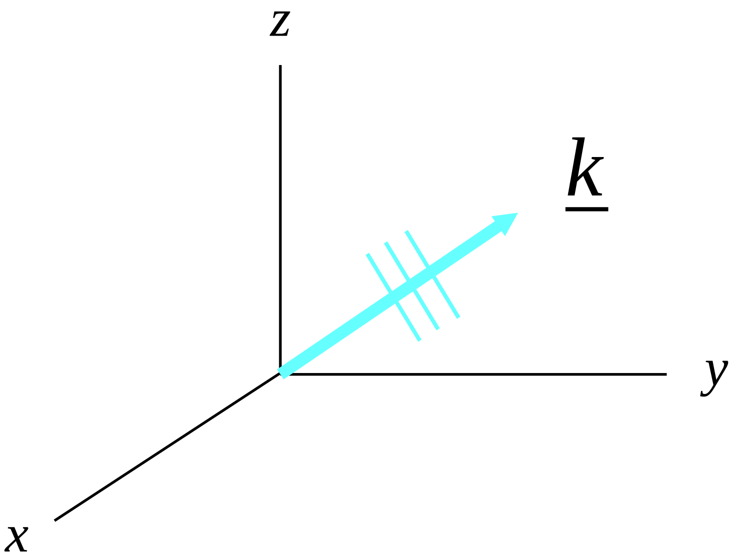


In the most general case, all three vectors may be in different directions.

General Plane Waves (cont.)

Symbol for a plane wave:

In the most general case, the $\underline{k}$ vector may be complex. In this case it is not possible to actually visualize it as vector in 3D space. The blue arrow is still used as a symbol for the $\underline{k}$ vector.



General Plane Waves (cont.)

Next, look at Maxwell's equations for a plane wave:

$$\nabla \times \underline{E} = -j\omega\mu\,\underline{H} \qquad \nabla \times \underline{H} = j\,\omega\,\varepsilon_c\,\underline{E}$$

$$\begin{aligned}\nabla &= \underline{\hat{x}}\frac{\partial}{\partial x} + \underline{\hat{y}}\frac{\partial}{\partial y} + \underline{\hat{z}}\frac{\partial}{\partial z} \\ &= \underline{\hat{x}}(-j\,k_x) + \underline{\hat{y}}(-j\,k_y) + \underline{\hat{z}}(-j\,k_z) \\ &= -\,j\,\underline{k}\end{aligned}$$

$$\nabla = -\,j\,\underline{k}$$

Hence

$$-\,j\,\underline{k} \times \underline{E} = -\,j\omega\mu\underline{H} \qquad -j\,\underline{k} \times \underline{H} = j\,\omega\,\varepsilon_c\,\underline{E}$$

General Plane Waves (cont.)

Summary of Maxwell's curl equations for a plane wave:

$$\underline{k} \times \underline{E} = \omega\mu\underline{H}$$

$$\underline{k} \times \underline{H} = -\omega\varepsilon_c\underline{E}$$

General Plane Waves (cont.)

Gauss law (divergence) equations:

$$\nabla \cdot \underline{D} = \rho_v$$

$$\nabla \cdot \underline{B} = 0$$

$$-j\,\underline{k} \cdot \left(\varepsilon \underline{E} \right) = 0$$

$$-j\,\underline{k} \cdot \left(\mu \underline{H} \right) = 0$$

$$\underline{k} \cdot \underline{E} = 0$$

$$\underline{k} \cdot \underline{H} = 0$$

Reminder: The volume charge density is zero in the sinusoidal steady state for a homogeneous source-free region.

General Plane Waves (cont.)

Furthermore, we have from Faraday's law

$$\underline{k} \times \underline{E} = \omega\mu \underline{H}$$

Dot multiply both sides with $\underline{E}$.

$$\underline{E} \cdot \underline{H} = 0$$

Note : $\underline{E} \cdot \left(\underline{k} \times \underline{E} \right) = 0$ (for any vectors $\underline{k}, \underline{E}$)

General Plane Waves (cont.)

Summary of dot products:

$$\underline{k} \cdot \underline{E} = 0$$

$$\underline{k} \cdot \underline{H} = 0$$

$$\underline{E} \cdot \underline{H} = 0$$

Note:
If the dot product of two vectors is zero, we can say that the vectors are perpendicular for the case of real vectors.

For complex vectors, we need a conjugate (which we don't have) to say that the vectors are "orthogonal".

Power Flow

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^*$$

$$\underline{E} = \underline{E}_0 \psi$$

$$\underline{H} = -\frac{1}{j\omega\mu}\nabla \times \underline{E}$$

$$= -\frac{1}{j\omega\mu}(-j\underline{k} \times \underline{E})$$

$$= \frac{1}{\omega\mu}\underline{k} \times \underline{E}$$

$$= \frac{1}{\omega\mu}\psi(\underline{k} \times \underline{E}_0)$$

Power Flow (cont.)

so

$$\underline{S} = \frac{1}{2}\left(\psi \underline{E}_0\right) \times \left(\left[\underline{k} \times \underline{E}_0\right]^* \left(\frac{1}{\omega\mu}\psi^*\right)\right)$$

$$= \frac{1}{2\omega\mu}\left|\psi\right|^2 \underline{E}_0 \times \left(\underline{k}^* \times \underline{E}_0^*\right)$$

Note: μ is assumed to be real here.

Use $\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}\left(\underline{A} \cdot \underline{C}\right) - \underline{C}\left(\underline{A} \cdot \underline{B}\right)$

so that $\underline{E}_0 \times \left(\underline{k}^* \times \underline{E}_0^*\right) = \left(\underline{E}_0 \cdot \underline{E}_0^*\right)\underline{k}^* - \left(\underline{E}_0 \cdot \underline{k}^*\right)\underline{E}_0^*$

and hence

$$\underline{S} = \frac{1}{2\omega\mu}\left|\psi\right|^2 \left[\left(\underline{E}_0 \cdot \underline{E}_0^*\right)\underline{k}^* - \left(\underline{E}_0 \cdot \underline{k}^*\right)\underline{E}_0^*\right]$$

Power Flow (cont.)

$$\underline{S} = \frac{1}{2\omega\mu}|\psi|^2\left[\left(\underline{E}_0 \cdot \underline{E}_0^*\right)\underline{k}^* - \left(\underline{E}_0 \cdot \underline{k}^*\right)\underline{E}_0^*\right]$$

Assume $\underline{E}_0$ = real vector.

(The same conclusion holds if it is a real vector times a complex constant.)

$$\underline{E}_0 \cdot \underline{k}^* = \left(\underline{E}_0^* \cdot \underline{k}\right)^* = \left(\underline{E}_0 \cdot \underline{k}\right)^* = 0$$

(All of the components of the vector are in phase.)

Note: This conclusion also holds if $\underline{k}$ is real, or is a real vector times a complex constant.

Hence

$$\underline{S} = \frac{1}{2\omega\mu}|\psi|^2 \left|\underline{E}_0\right|^2 \underline{k}^*$$

Power Flow (cont.)

$$\underline{S} = \frac{1}{2\omega\mu}|\psi|^2 \left|\underline{E}_0\right|^2 \underline{k}^*$$

The power flow is: $<\underline{\mathscr{S}}> = \text{Re}\,\underline{S}$

so $$<\underline{\mathscr{S}}> = \frac{1}{2\omega\mu}|\psi|^2 \left|\underline{E}_0\right|^2 \text{Re}\,\underline{k}$$

(assuming that μ is real)

Recall: $\underline{k} = \underline{\beta} - j\underline{\alpha}$

Power Flow (cont.)

Then

$$< \underline{\mathscr{S}} > = \left[\frac{|\psi|^2}{2\omega\mu} |\underline{E}_0|^2 \right] \underline{\beta}$$

Power flows in the direction of $\underline{\beta}$.

Assumption:

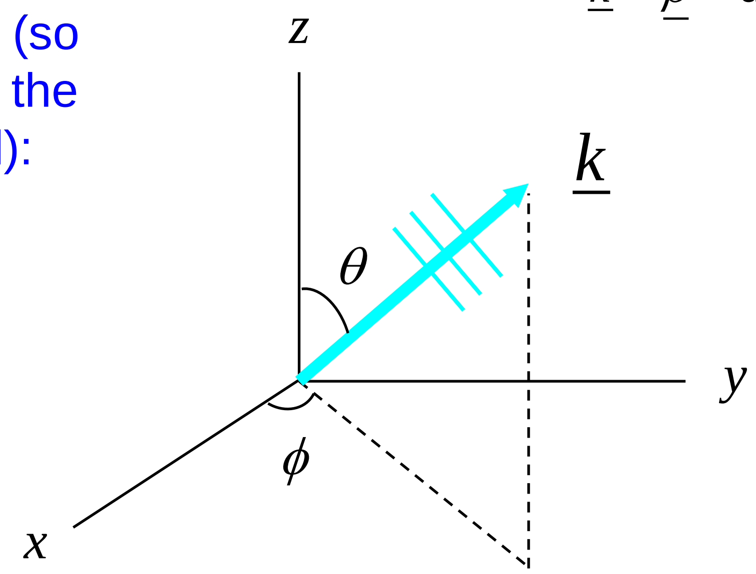
Either the electric field vector or the wavenumber vector is a real vector times a complex constant.

(This assumption is true for most of the common plane waves.)

Direction Angles

$\underline{k} = \underline{\beta} =$ direction of power flow

First assume $\underline{k}$ = real vector (so that we can visualize it) and the medium is lossless (k is real):



The direction angles (θ, ϕ) are defined by:

$$k_x = k \sin\theta \cos\phi$$
$$k_y = k \sin\theta \sin\phi$$
$$k_z = k \cos\theta$$

Note:

$$k_x^2 + k_y^2 + k_z^2$$
$$= k^2 \sin^2\theta + k^2 \cos^2\theta$$
$$= k^2$$

Direction Angles (cont.)

From the direction angle equations we have:

$$\cos\theta = \frac{k_z}{k}$$

$$\tan\phi = \frac{k_y}{k_x}$$

Even when (k_x, k_y, k_z) become complex, and k is also complex, these equations are used to <u>define</u> the direction angles, which may be complex.

Homogeneous Plane Wave

Definition of a homogenous (uniform) plane wave:

(θ, ϕ) are real angles

In the general lossy case (complex k):

$$\underline{k} = \underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y + \underline{\hat{z}} k_z$$
$$= k\left[\, \underline{\hat{x}} \sin\theta \cos\phi + \underline{\hat{y}} \sin\theta \sin\phi + \underline{\hat{z}} \cos\theta \right]$$
$$= k\, \underline{\hat{r}}$$

where

$$\underline{\hat{r}} \equiv \underline{\hat{x}} \sin\theta \cos\phi + \underline{\hat{y}} \sin\theta \sin\phi + \underline{\hat{z}} \cos\theta$$

$\underline{\hat{r}} =$ real unit vector pointing in the direction of power flow

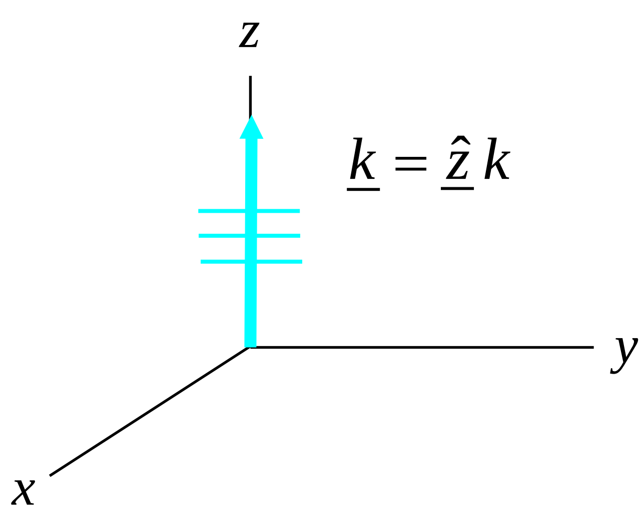
Homogeneous Plane Wave (cont.)

Hence we have

$$\underline{k} = \left(k' - jk''\right)\underline{\hat{r}} \quad \Rightarrow \quad \begin{aligned} \underline{\beta} &= k'\,\underline{\hat{r}} \\ \underline{\alpha} &= k''\,\underline{\hat{r}} \end{aligned}$$

The phase and attenuation vectors point in the same direction.
The amplitude and phase of the wave are both constant (uniform) in a plane perpendicular to the direction of propagation.

Note: A simple plane wave of the form $\psi = \exp(-j\,k\,z)$ is a special case, where $\theta = 0$.

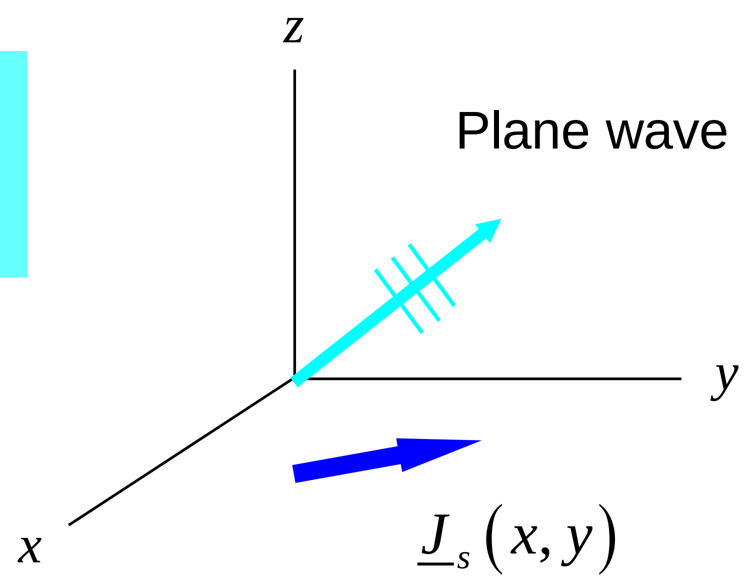


Note:
A homogeneous plane wave in a <u>lossless</u> medium has no $\underline{\alpha}$ vector:

$$\underline{\alpha} = k''\,\underline{\hat{r}} = \underline{0}$$

Infinite Current Sheet

An infinite surface current sheet at $z = 0$ launches a plane wave in free space.



Assume $\underline{J}_s = \underline{A} e^{-j(k_x x + k_y y)}, \quad k_x, k_y \in$ real

The vertical wavenumber is then given by $k_x^2 + k_y^2 + k_z^2 = k_0^2$

Infinite Current Sheet (cont.)

Part (a): Homogeneous plane wave

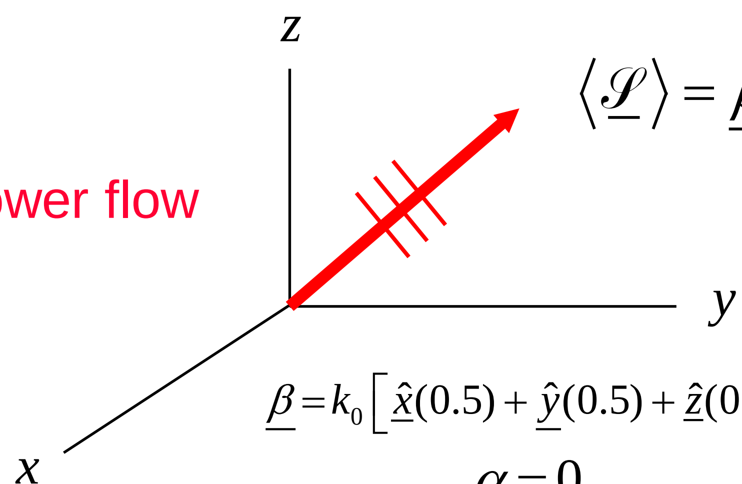
$$k_x = 0.5\,k_0 \qquad k_y = 0.5\,k_0$$

$$k_z = \sqrt{k_0^2 - 0.25\,k_0^2 - 0.25\,k_0^2} = \pm \frac{1}{\sqrt{2}}\,k_0$$

We must choose $k_z = +0.707\,k_0$ (outgoing wave)

Then $\underline{k} = k_0\left[\underline{\hat{x}}(0.5) + \underline{\hat{y}}(0.5) + \underline{\hat{z}}(0.707)\right]$

Power flow



$$\langle \underline{\mathscr{S}} \rangle = \underline{\beta} = \underline{k}$$

$$\underline{\beta} = k_0\left[\underline{\hat{x}}(0.5) + \underline{\hat{y}}(0.5) + \underline{\hat{z}}(0.707)\right]$$

$$\underline{\alpha} = \underline{0}$$

Infinite Current Sheet (cont.)

Part (b) Inhomogeneous plane wave

$$k_x = 2k_0 \quad k_y = 3k_0$$

$$\begin{aligned} k_z &= \sqrt{k_0^2 - 4k_0^2 - 9k_0^2} \\ &= \pm\, j k_0 \sqrt{12} \end{aligned}$$

We must choose $k_z = -\,j k_0 \sqrt{12}$

The wave is <u>evanescent</u> in the z direction.

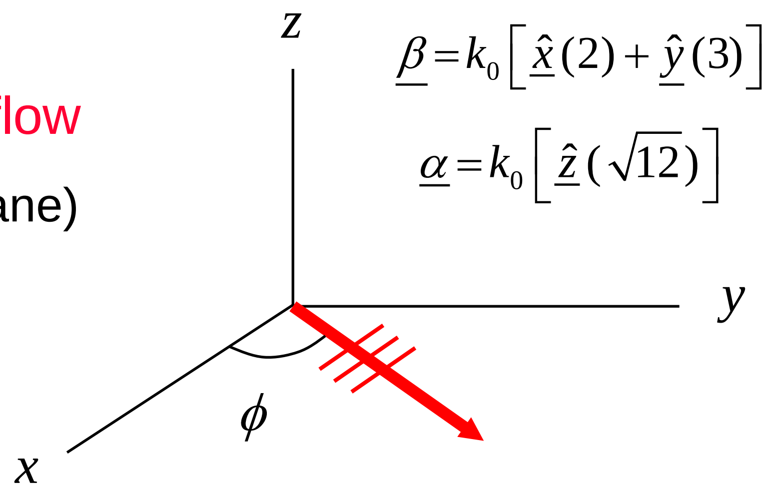
Then $\underline{k} = k_0 \left[\underline{\hat{x}}(2) + \underline{\hat{y}}(3) + \underline{\hat{z}}(-j\sqrt{12}) \right]$

$$\underline{\beta} = k_0 \left[\underline{\hat{x}}(2) + \underline{\hat{y}}(3) + \underline{\hat{z}}(0) \right] \qquad \underline{\alpha} = k_0 \left[\underline{\hat{z}}(\sqrt{12}) \right]$$

Infinite Current Sheet (cont.)

Power flow

(in xy plane)



$$\underline{\beta} = k_0 \left[\underline{\hat{x}}(2) + \underline{\hat{y}}(3) \right]$$

$$\underline{\alpha} = k_0 \left[\underline{\hat{z}}(\sqrt{12}) \right]$$

$$\langle \underline{\mathscr{S}} \rangle = \underline{\beta} \neq \underline{k}$$

$$\cos\theta = \frac{k_z}{k_0} = -j\sqrt{12}$$

so

$$\theta = \frac{\pi}{2} + j(1.956) \;\; [\text{rad}]$$

$$\tan\phi = \frac{\beta_y}{\beta_x} = \frac{3}{2}$$

so

$$\phi = 56.31^\circ$$

Note:
Another possible solution is the negative of the above angle. The inverse cosine should be chosen so that $\sin\theta$ is correct (to give the correct k_x and k_y): $\sin\theta > 0$.

Propagation Circle

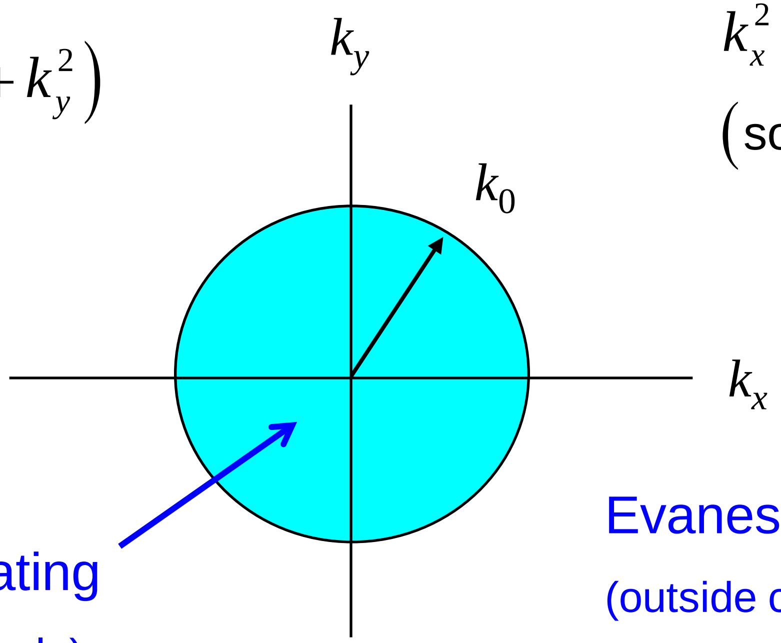
$$k_x^2 + k_y^2 + k_z^2 = k_0^2$$

$$\Rightarrow \; k_z^2 = k_0^2 - \left(k_x^2 + k_y^2\right)$$

Propagating waves:

$$k_x^2 + k_y^2 < k_0^2$$

$$\left(\text{so } k_z = \text{real}\right)$$



Propagating

(inside circle)

$$k_z = \sqrt{k_0^{\,2} - k_x^{\,2} - k_y^{\,2}}$$

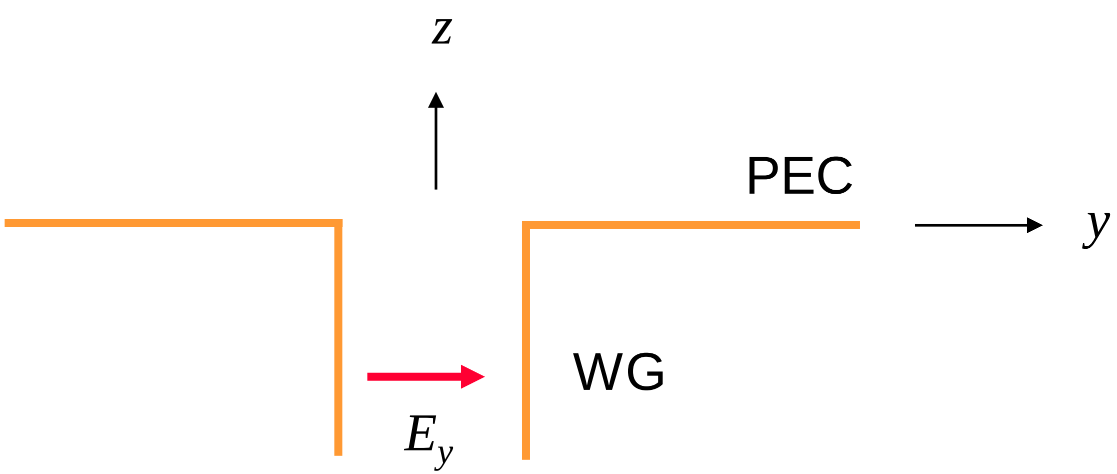
Evanescent

(outside circle)

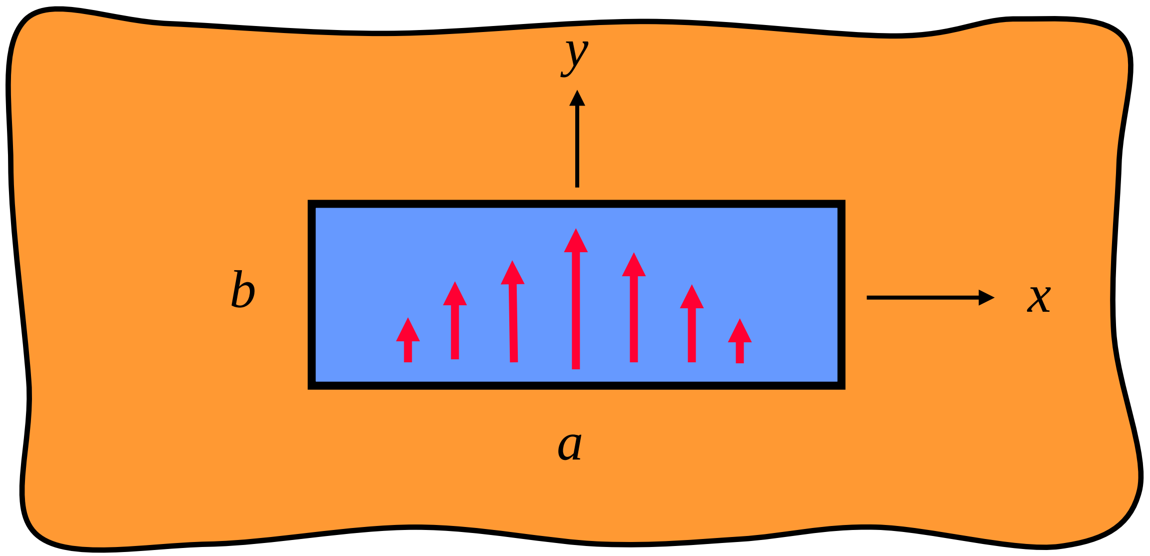
$$k_z = -j\sqrt{k_x^2 + k_y^2 - k_0^2}$$

Free space acts as a “low-pass filter.”

Radiation from Waveguide



$$E_y(x,y,0)=\cos\left(\frac{\pi x}{a}\right)$$



Radiation from Waveguide (cont.)

Fourier transform pair:

$$\tilde{E}_y(k_x, k_y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(x, y, z)\, e^{+j(k_x x + k_y y)}\, dx\, dy$$

$$E_y(x, y, z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}_y(k_x, k_y, z)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

Radiation from Waveguide (cont.)

$$E_y(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}_y(k_x,k_y,z)\,e^{-j(k_x x+k_y y)}dk_x dk_y$$

$$\nabla^2 E_y + k^2 E_y = 0$$

$$\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}+k^2 E_y=0$$

Hence

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-k_x^2\tilde{E}_y-k_y^2\tilde{E}_y+\frac{\partial^2\tilde{E}_y}{\partial z^2}+k^2\tilde{E}_y\right)e^{-j(k_x x+k_y y)}dk_x dk_y=0$$

Radiation from Waveguide (cont.)

Hence

$$-k_x^2 \tilde{E}_y - k_y^2 \tilde{E}_y + \frac{\partial^2 \tilde{E}_y}{\partial z^2} + k^2 \tilde{E}_y = 0$$

Next, define

$$k_z^2 \equiv k^2 - k_x^2 - k_y^2$$

We then have

$$\frac{\partial^2 \tilde{E}_y}{\partial z^2} + k_z^2 \tilde{E}_y = 0$$

Radiation from Waveguide (cont.)

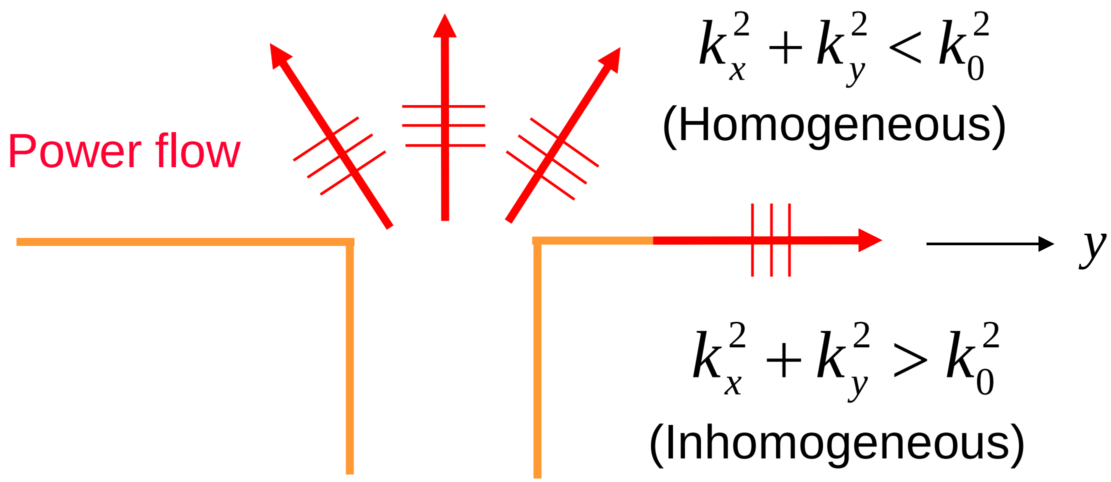
Solution:

$$\tilde{E}_y(k_x, k_y, z) = \tilde{E}_y(k_x, k_y, 0)\, e^{-jk_z z}$$

Note: We want outgoing waves only.

Hence

$$E_y(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}_y(k_x, k_y, 0)\; e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y$$



$$k_x^2 + k_y^2 < k_0^2$$

(Homogeneous)

$$k_x^2 + k_y^2 > k_0^2$$

(Inhomogeneous)

Radiation from Waveguide (cont.)

Fourier transform of aperture field:

$$\tilde{E}_y(k_x, k_y, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(x, y, 0)\, e^{+j(k_x x + k_y y)} dx\, dy$$

$$= \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) e^{+j(k_x x + k_y y)} dx\, dy$$

$$= \int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) e^{+j(k_x x)} dx \int_{-b/2}^{b/2} e^{+j(k_y y)} dy$$

$$\tilde{E}_y(k_x, k_y, 0) = \left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x \frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x a}{2}\right)^2} \right) \left(b \operatorname{sinc}\left(\frac{k_y b}{2}\right) \right)$$

Radiation from Waveguide (cont.)

Note:

$$\tilde{E}_x(k_x,k_y,z) = \tilde{E}_x(k_x,k_y,0)\,e^{-jk_z z} = 0$$

Hence

$$E_x(x,y,z) = 0$$

For E_z we have

$$\nabla \cdot \underline{E} = 0 \;\Rightarrow\; \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\Rightarrow\; \left(-jk_y\right)\tilde{E}_y + \left(-jk_z\right)\tilde{E}_z = 0$$

This follows from the mathematical form of E_y as an inverse transform.

Radiation from Waveguide (cont.)

Hence

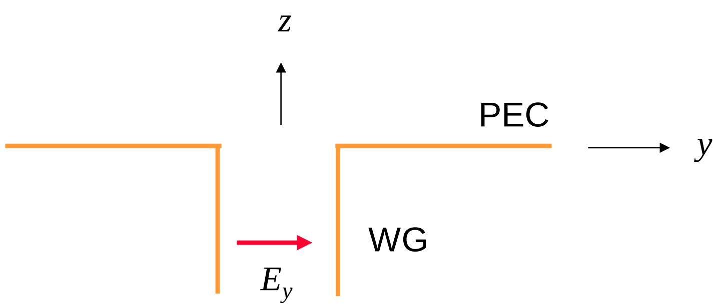
$$\tilde{E}_z = \tilde{E}_y \left(-\frac{k_y}{k_z} \right)$$

In the space domain, we have

$$E_z(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_y(k_x, k_y, 0)\, e^{-j(k_x x + k_y y + k_z z)} \left(-\frac{k_y}{k_z} \right) dk_x dk_y$$

Radiation from Waveguide (cont.)

Summary (for $z > 0$)



$$E_x(x,y,z) = 0$$

$$E_y(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_y(k_x,k_y,0)\; e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y$$

$$E_z(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_y(k_x,k_y,0)\, e^{-j(k_x x + k_y y + k_z z)} \left(-\frac{k_y}{k_z}\right) dk_x dk_y$$

$$\tilde{E}_y(k_x,k_y,0) = \left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x \frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x a}{2}\right)^2} \right) \left(b\,\mathrm{sinc}\left(\frac{k_y b}{2}\right) \right)$$

Some Plane-Wave "η" Theorems

Theorem #1

$$\underline{E} \cdot \underline{E} = \eta^2 \, \underline{H} \cdot \underline{H}$$ (always true)

Theorem #2

If PW is homogeneous:

$$\left|\underline{E}\right|^2 = \left|\eta\right|^2 \left|\underline{H}\right|^2$$ (lossy medium)

$$\left|\underline{E}\right|^2 = \eta^2 \left|\underline{H}\right|^2$$ (lossless medium)

Theorem #3

If medium is lossless:

$$\underline{\beta} \cdot \underline{\alpha} = 0$$

Example

Example

$$\underline{E} = \underline{\hat{y}}\, e^{-j(k_x x + k_y y + k_z z)} = \underline{\hat{y}}\, \psi(x, y, z)$$

Plane wave in free space

Given:

$$k_x = 2k_0$$

$$k_y = 0$$

$$k_z = -j\sqrt{3}\, k_0$$

Note: It can be seen that

$$k_x^2 + k_y^2 + k_z^2 = k_0^2$$

and

$$\underline{k} \cdot \underline{E} = 0$$

Find $\underline{H}$ and compare its magnitude with that of $\underline{E}$.

Verify theorems 1 and 3.

Example (cont.)

$$\nabla \times \underline{E} = -j\omega\mu_0 \underline{H}$$

so

$$-j\underline{k} \times \underline{E} = -j\omega\mu_0 \underline{H}$$

$$\underline{H} = \frac{1}{\omega\mu_0} \underline{k} \times \underline{E}$$

$$= \frac{k_0}{\omega\mu_0} (2, 0, -j\sqrt{3}) \times \left[(0,1,0)\psi(x,y,z) \right]$$

Example (cont.)

$$\underline{H} = \frac{1}{\eta_0}\left[\, \underline{\hat{z}}(2) + \underline{\hat{x}}(j\sqrt{3})\,\right]\psi$$

$$\left|\underline{H}\right| = \sqrt{\underline{H} \cdot \underline{H}^*} = \sqrt{H_x H_x^* + H_z H_z^*} = \sqrt{\left|H_x\right|^2 + \left|H_z\right|^2}$$

$$\left|\underline{H}\right| = \frac{1}{\eta_0}\sqrt{\left|2\right|^2 + \left|j\sqrt{3}\right|^2}\,\left|\psi\right| = \frac{\sqrt{7}}{\eta_0}\left|\psi\right|$$

Hence

$$\frac{\left|\underline{E}\right|}{\left|\underline{H}\right|} = \frac{\eta_0}{\sqrt{7}}$$

Note: The field magnitudes are not related by η_0 !

Example (cont.)

Verify: $\underline{E}\cdot\underline{E}=\eta_0^2\,\underline{H}\cdot\underline{H}$ (Theorem #1)

$$\underline{E}=\underline{\hat{y}}\,\psi(x,y,z)$$

$$\underline{H}=\frac{1}{\eta_0}\left[\,\underline{\hat{z}}(2)+\underline{\hat{x}}(j\sqrt{3})\,\right]\psi\left(x,y,z\right)$$

At the origin ($\psi = 1$) we have:

$$\underline{E}\cdot\underline{E}=1$$

$$\underline{H}\cdot\underline{H}=\frac{1}{\eta_0^2}\left[\,\underline{\hat{z}}(2)+\underline{\hat{x}}(j\sqrt{3})\,\right]\cdot\left[\,\underline{\hat{z}}(2)+\underline{\hat{x}}(j\sqrt{3})\,\right]=\frac{1}{\eta_0^2}\left(4-3\right)=\frac{1}{\eta_0^2}$$

Example (cont.)

Verify: $\underline{\beta} \cdot \underline{\alpha} = 0$ (Theorem #3)

$$\underline{\beta} = \mathrm{Re}\left(\underline{k}\right) = \underline{\hat{x}}\left(2k_0\right)$$

$$\underline{\alpha} = -\mathrm{Im}\left(\underline{k}\right) = \underline{\hat{z}}\left(\sqrt{3}k_0\right)$$

$$k_x = 2k_0$$
$$k_y = 0$$
$$k_z = -j\sqrt{3}\,k_0$$

Hence

$$\underline{\beta} \cdot \underline{\alpha} = 0$$