ECE 6340
Intermediate EM Waves

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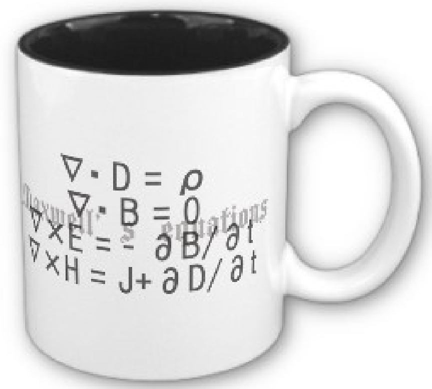
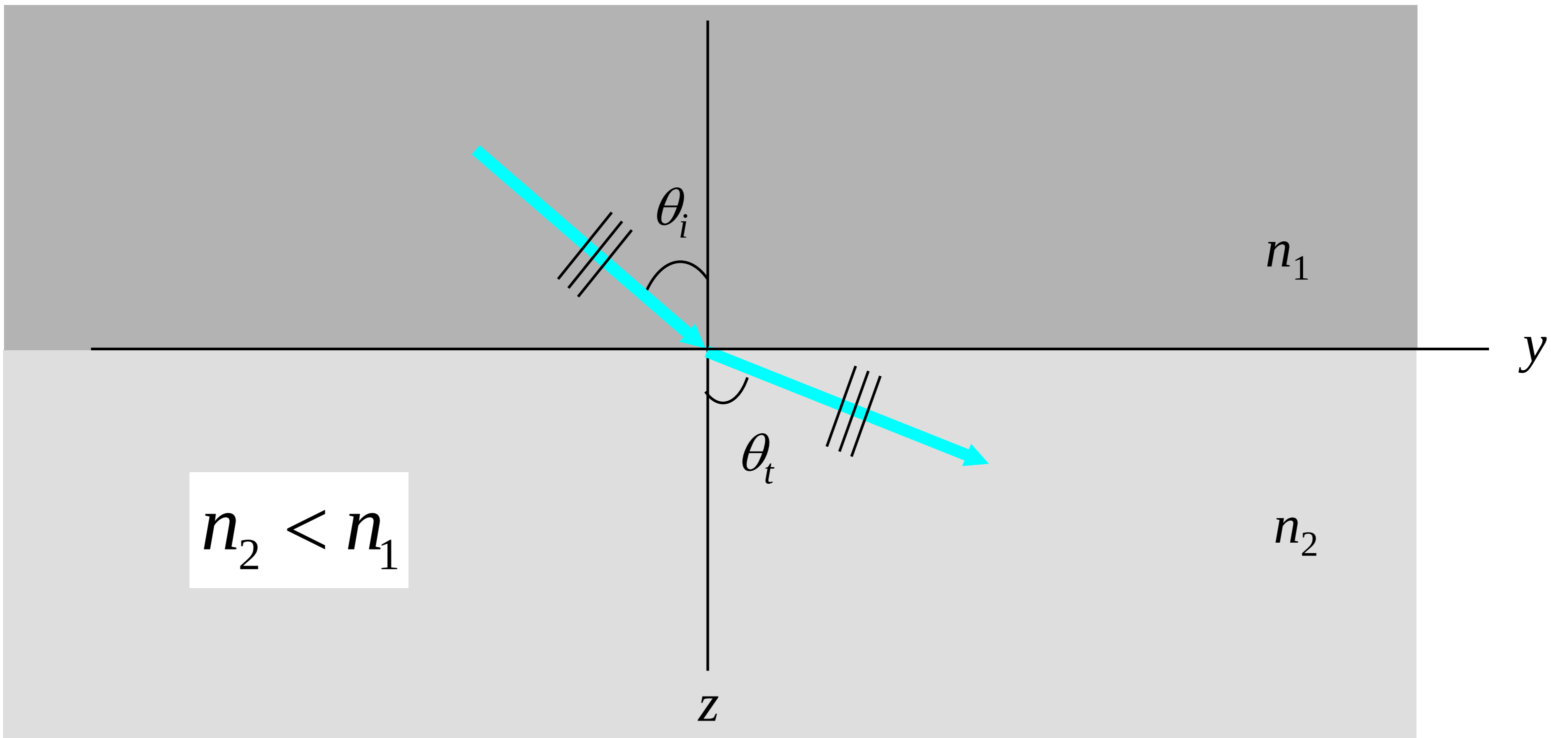


Notes 19

Critical Angle

Assume lossless materials

$$n_i = \sqrt{\mu_{ri}\varepsilon_{ri}}, \quad i = 1, 2$$



Snell's law:

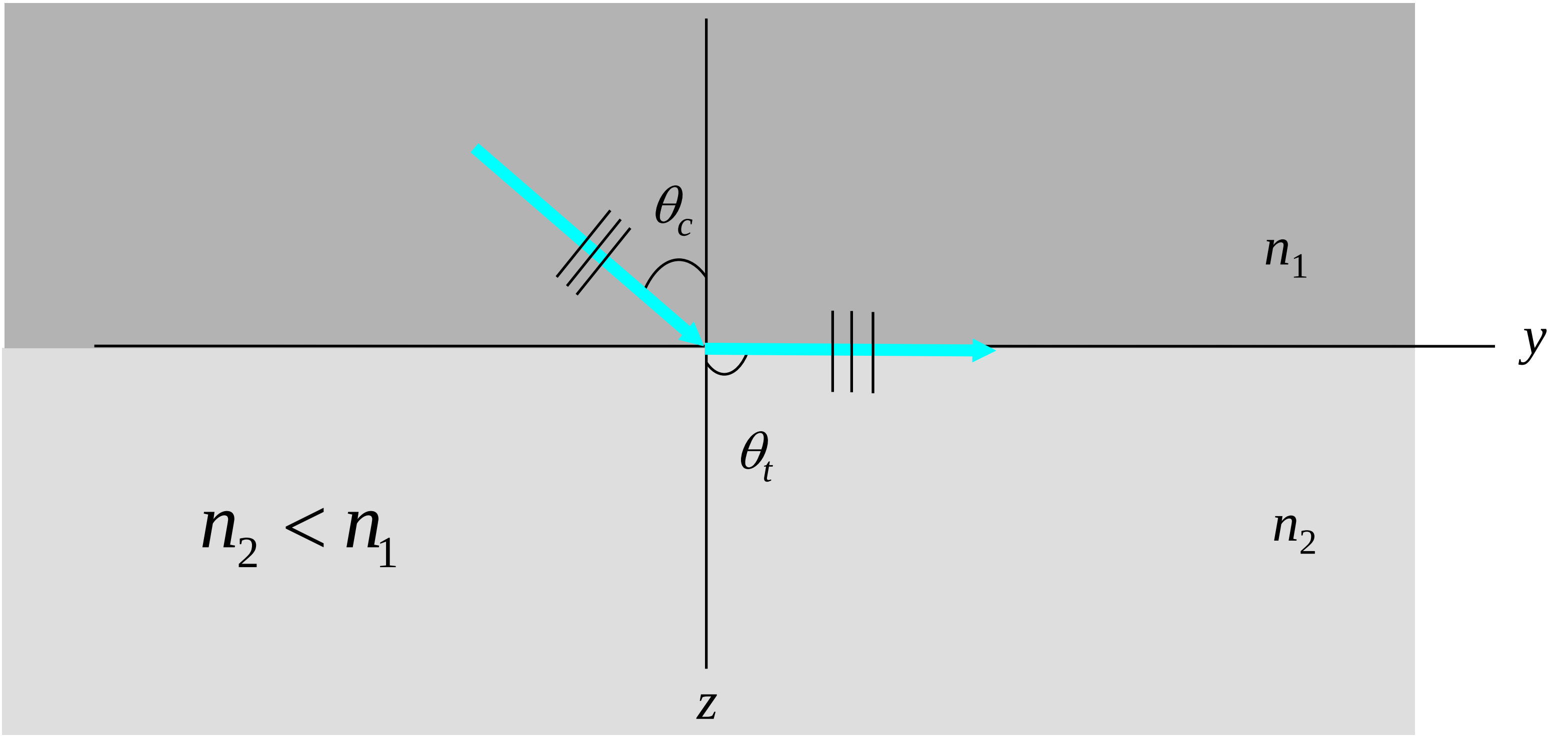
$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Critical Angle

At the critical angle ($\theta_i = \theta_c$): $\theta_t = 90^o$

$$\Rightarrow \quad \frac{n_1}{n_2}\sin\theta_i = \frac{n_1}{n_2}\sin\theta_c = 1 \quad \text{so} \quad \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



Critical Angle

Notes:

- The critical angle applies to any polarization.
- The critical angle is only defined for lossless materials.
- A critical angle exists only when going from a higher to a lower density medium ($n_1 > n_2$).

Beyond Critical Angle

Let's examine the transmitted angle:

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Assume $\theta_i > \theta_c$ $\Rightarrow$ $\frac{n_1}{n_2}\sin\theta_i > 1$ $\Rightarrow$ $\theta_t = \text{complex}$

$$k_y = k_1 \sin\theta_i > k_1 \sin\theta_c = k_2 \sin\theta_{tc} = k_2 \sin\left(90^{\text{o}}\right) = k_2$$

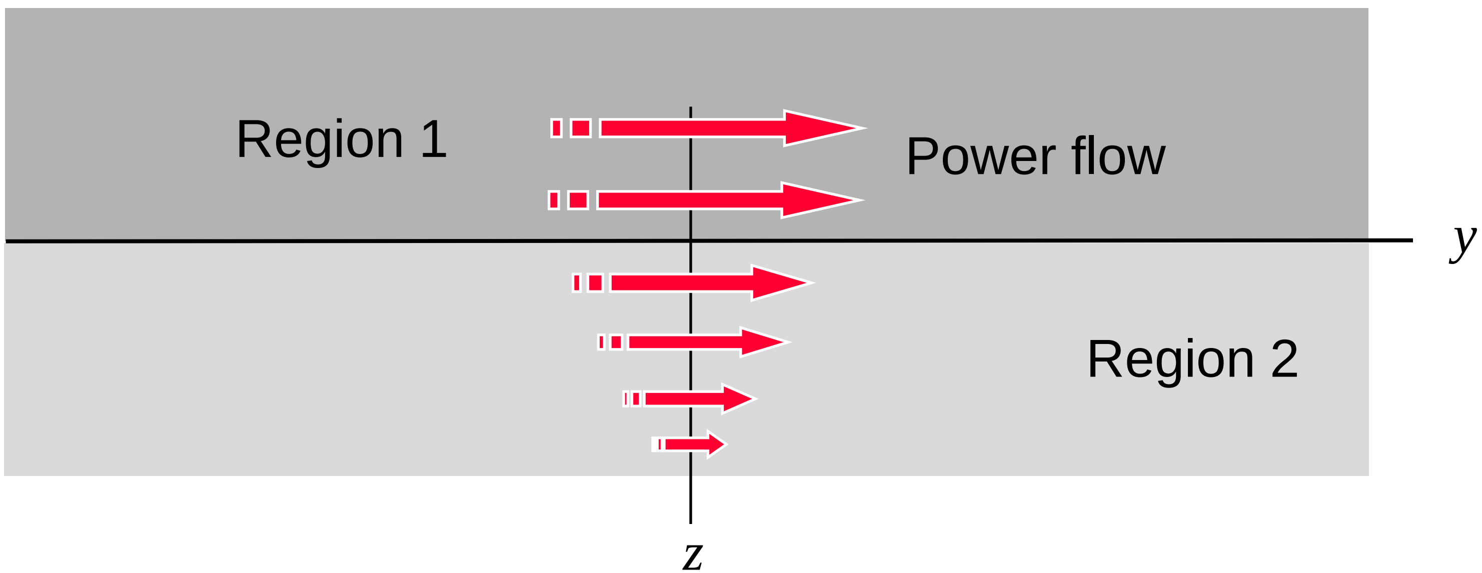
so $k_y > k_2$

Beyond Critical Angle (cont.)

$$k_y > k_2$$

Note: The power flow in the upper region is horizontal. No power crosses the boundary

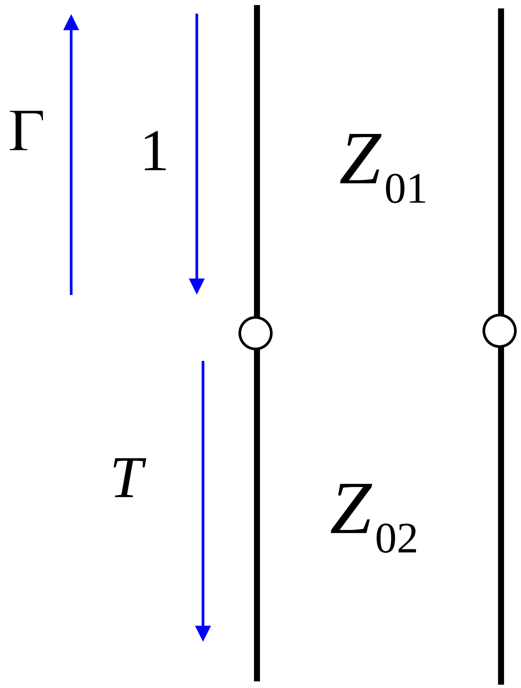
$$k_{z2} = \sqrt{k_2^{\,2} - k_y^{\,2}} = -j\sqrt{k_y^{\,2} - k_2^{\,2}} = -\,j\alpha_{z2}$$



$$\underline{k}_2 = \underline{\hat{y}}\,k_y + \underline{\hat{z}}(-j\alpha_{z2}) \qquad \underline{\beta}_2 = \underline{\hat{y}}\,k_y$$

Note: The power flow in the lower region is horizontal, and decays with z.

Beyond Critical Angle (cont.)



$$\theta_i > \theta_c$$

$$\Rightarrow k_{z2} = -j\alpha_{z2}$$

$$\Rightarrow Z_{02} \text{ is imaginary}$$

$$\Rightarrow |\Gamma| = 1$$

All of the incident power is reflected

Beyond Critical Angle (cont.)

Determine the transmitted angle θ_t : $\sin\theta_t = \left(\frac{n_1}{n_2}\sin\theta_i\right) > 1$

Let $\theta_t = x + jy$

Use

(real)

$$\sin\theta_t = \sin(x + jy) = \sin x \cosh y + j\cos x \sinh y = \frac{n_1}{n_2}\sin\theta_i > 1$$

$y = 0$ or $x = \pm\pi/2$

must use +sign

$x = \pi/2$

Not possible

$$\sin x = \frac{n_1}{n_2}\sin\theta_i > 1$$

$$\pm\cosh y = \frac{n_1}{n_2}\sin\theta_i > 1$$

Beyond Critical Angle (cont.)

$$\cosh y = \frac{n_1}{n_2}\sin\theta_i$$

$$\Rightarrow \quad y = \pm\cosh^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Hence $$\theta_t = \underbrace{\frac{\pi}{2}}_{x} \underbrace{\pm j\cosh^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)}_{jy}$$

Beyond Critical Angle (cont.)

$$\theta_t = \frac{\pi}{2} \pm j\cosh^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Choose + sign to obtain correct value for k_{z2} :

$$\begin{aligned} k_{z2} &= k_2\cos\theta_t = k_2\cos(x + jy) = k_2(\cos x\cosh y - j\sin x\sinh y) \\ &= -jk_2\sinh y \\ &= -j\,k_2\sinh\left[\pm\cosh^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)\right] \end{aligned}$$

The + sign is chosen to obtain a decaying wave.

Beyond Critical Angle (cont.)

Hence

$$\theta_t = \frac{\pi}{2} + j\cosh^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Practical note: When dealing with inhomogeneous plane waves (complex angles), it is usually easier to avoid working with angles and use the separation equation instead.

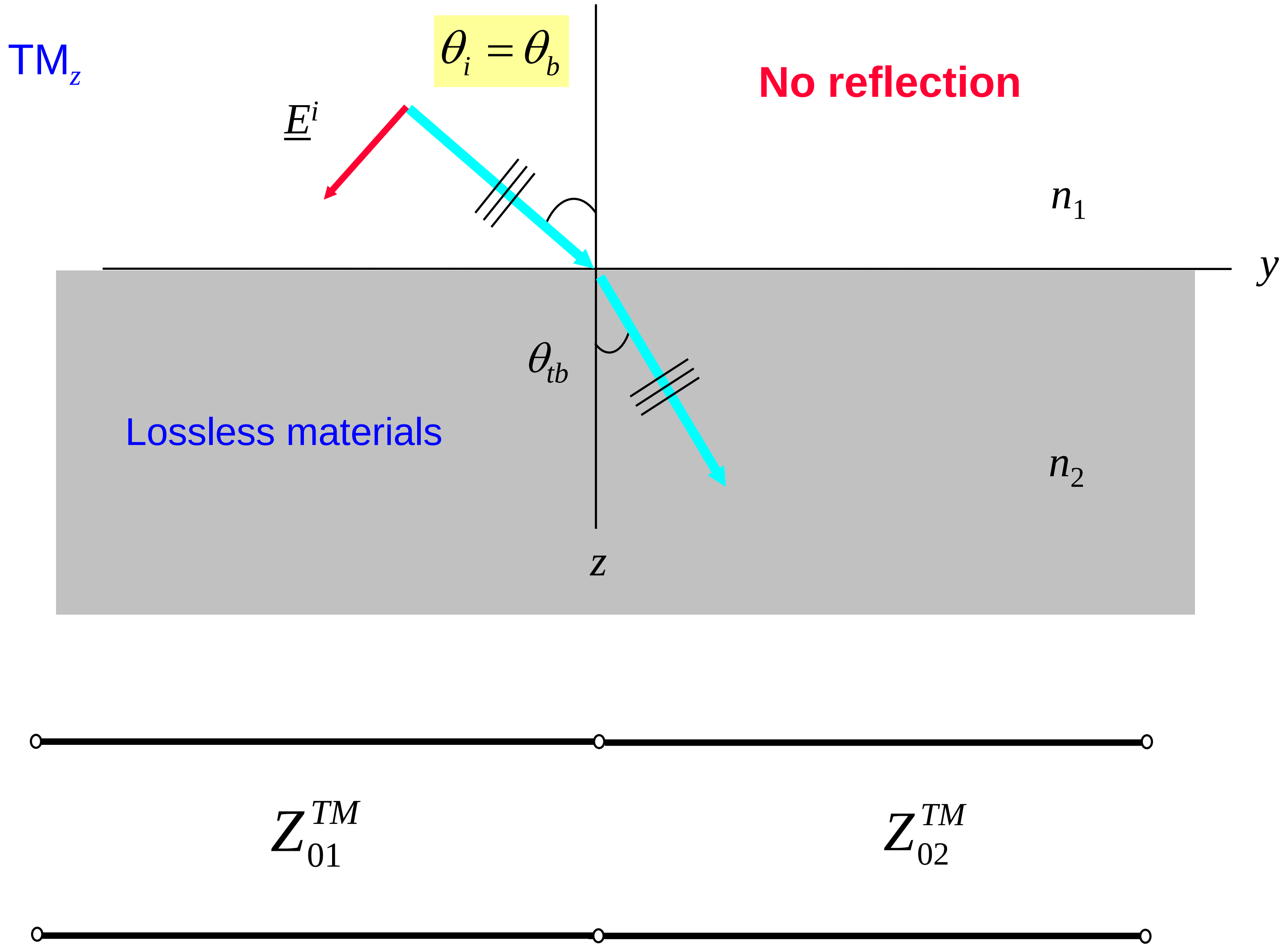
$$k_{z2} = k_2 \cos\theta_t \quad \Longleftrightarrow \quad k_{z2} = -j\sqrt{k_y^2 - k_2^{\,2}}$$

Requires complex angle

$$= -j\sqrt{k_1^2 \sin^2\theta_i - k_2^{\,2}}$$

Does not requires complex angle

Brewster Angle



Brewster Angle (cont.)

Perfect match: $Z_{01}^{TM} = Z_{02}^{TM}$

$$\frac{k_{z1}}{\omega \varepsilon_1} = \frac{k_{z2}}{\omega \varepsilon_2}$$

$$\frac{k_1 \cos\theta_b}{\varepsilon_1} = \frac{\sqrt{k_2^2 - k_1^2 \sin^2\theta_b}}{\varepsilon_2}$$

$$\frac{k_1^2 (1 - \sin^2\theta_b)}{\varepsilon_1^2} = \frac{k_2^2 - k_1^2 \sin^2\theta_b}{\varepsilon_2^2}$$

Brewster Angle (cont.)

Assume $\mu_1 = \mu_2 = \mu$:

$$\frac{\omega^2 \mu \varepsilon_1 (1 - \sin^2 \theta_b)}{\varepsilon_1^2} = \frac{\omega^2 \mu \varepsilon_2 - \omega^2 \mu \varepsilon_1 \sin^2 \theta_b}{\varepsilon_2^2}$$

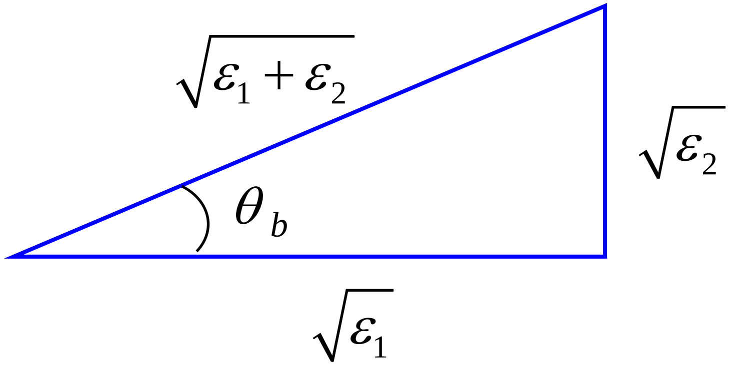
$$\frac{1 - \sin^2 \theta_b}{\varepsilon_1} = \frac{\varepsilon_2 - \varepsilon_1 \sin^2 \theta_b}{\varepsilon_2^2}$$

$$\varepsilon_2^2 - \varepsilon_2^2 \sin^2 \theta_b = \varepsilon_1 \varepsilon_2 - \varepsilon_1^2 \sin^2 \theta_b$$

$$\sin^2 \theta_b = \frac{\varepsilon_2^2 - \varepsilon_1 \varepsilon_2}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{\varepsilon_2 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_2^2 - \varepsilon_1^2} = \frac{\varepsilon_2 (\varepsilon_2 - \varepsilon_1)}{(\varepsilon_2 + \varepsilon_1)(\varepsilon_2 - \varepsilon_1)} = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

Brewster Angle (cont.)

$$\sin\theta_b = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$$



Hence

$$\theta_b = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Brewster Angle (cont.)

Notes:

- For non-magnetic media, only the TM_z polarization has a Brewster angle.
- A Brewster angle exists for any material contrast ratio.

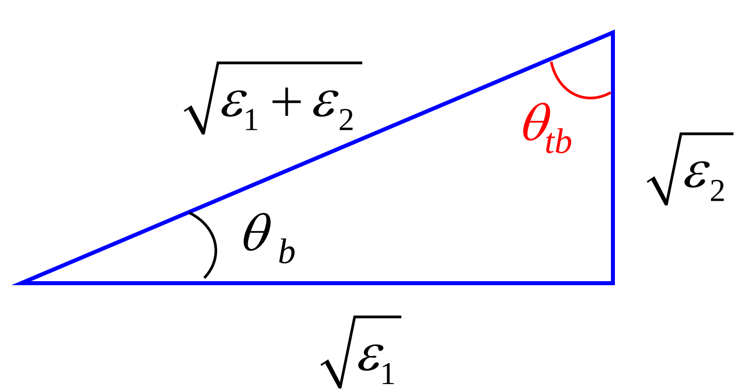
Brewster Angle (cont.)

From Snell's law:

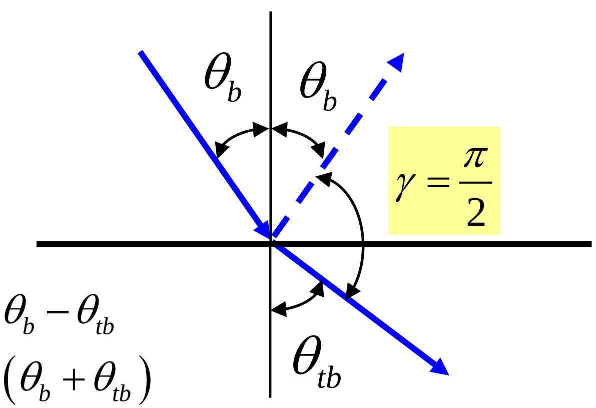
$$\sin\theta_{tb} = \frac{n_1}{n_2}\sin\theta_b = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_b$$

$$= \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}}$$

$$= \sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}}$$



This means that θ_{tb} is the angle shown in this figure.



$$\gamma = \pi - \theta_b - \theta_{tb}$$
$$= \pi - (\theta_b + \theta_{tb})$$
$$= \pi - (\pi/2)$$
$$= \pi/2$$

Hence

$$\theta_b + \theta_{tb} = \frac{\pi}{2}$$

The reflected and transmitted $\underline{k}$ vectors are perpendicular.