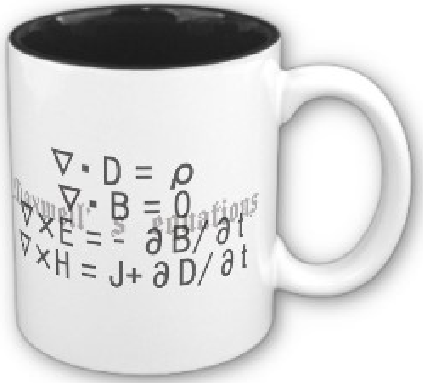
ECE 6340
Intermediate EM Waves

Fall 2016

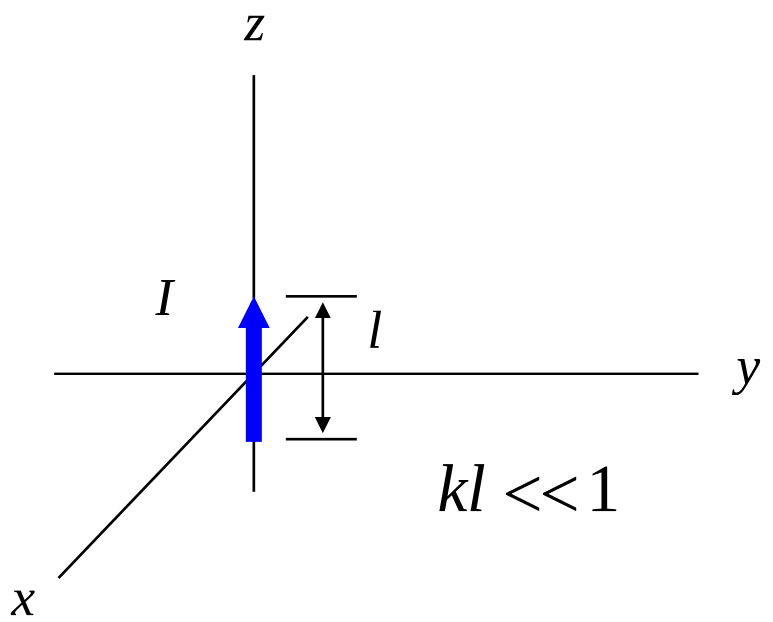
Prof. David R. Jackson
Dept. of ECE



Notes 22

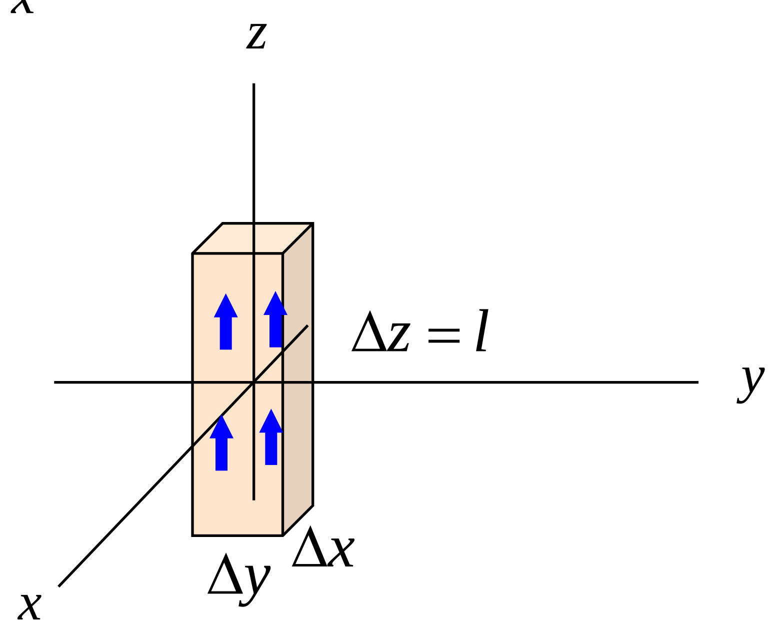
Radiation

Infinitesimal dipole:



Current model:

$$J_z^i = \frac{I}{\Delta x\,\Delta y} = \frac{Il}{\Delta x\,\Delta y\,\Delta z}$$



Radiation (cont.)

Define

$$P(x) = \begin{cases} \dfrac{1}{\Delta x} & x \in \left(-\dfrac{\Delta x}{2}, +\dfrac{\Delta x}{2} \right) \\ 0 & \text{otherwise} \end{cases}$$

Then

$$J_z^i(x, y, z) = (Il)\, P(x)\, P(y)\, P(z)$$

As $\Delta x \to 0, \quad P(x) \to \delta(x)$

Letting $\Delta x, \Delta y, \Delta z \to 0$

$$J_z^i(x, y, z) = (Il)\, \delta(x)\, \delta(y)\, \delta(z)$$

or

$$J_z^i(x, y, z) = (Il)\, \delta(\underline{r})$$

Radiation (cont.)

Maxwell's Equations:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \qquad (1)$$

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_c \underline{E} \qquad (2)$$

$$\nabla \cdot \underline{H} = 0 \qquad (3)$$

$$\nabla \cdot \underline{E} = \rho_v / \varepsilon \qquad (4)$$

Note: ε_c accounts for conductivity.

$$\underline{J}^{i}(x,y,z) = \underline{\hat{z}}(Il)\,\delta(\underline{r})$$

From (3):

$$\underline{H} = \frac{1}{\mu}\nabla \times \underline{A} \qquad (5)$$

Note:

The Harrington book uses

$$\underline{H} = \nabla \times \underline{A}$$

Most references use

$$\underline{B} = \nabla \times \underline{A}$$

Radiation (cont.)

From (1) and (5):

$$\nabla \times \underline{E} = -j\omega\mu \left[\frac{1}{\mu} (\nabla \times \underline{A}) \right]$$

or

$$\nabla \times (\underline{E} + j\omega \underline{A}) = \underline{0}$$

Hence

$$\underline{E} + j\omega \underline{A} = -\nabla \Phi$$

The function Φ is called the potential function. It is uniquely defined, even though voltage is not.

or

$$\underline{E} = -j\omega \underline{A} - \nabla \Phi \qquad (6)$$

This is the “mixed potential” form for $\underline{E}$.

Radiation (cont.)

Substitute (5) and (6) into (2): $\nabla \times \underline{H} = \underline{J}^{i} + j\omega\,\varepsilon_c\,\underline{E}$

$$\frac{1}{\mu}\nabla \times (\nabla \times \underline{A}) = \underline{J}^{i} + j\omega\varepsilon_c[-j\omega\underline{A} - \nabla\Phi]$$

$$\nabla \times (\nabla \times \underline{A}) - k^2\underline{A} = \mu\underline{J}^{i} - j\,\omega\,\mu\,\varepsilon_c\,\nabla\Phi$$

Use the vector Laplacian identity:

$$\nabla^2\underline{A} \equiv \nabla(\nabla \cdot \underline{A}) - \nabla \times (\nabla \times \underline{A})$$

The vector Laplacian has this nice property in <u>rectangular coordinates</u>:

$$\nabla^2\underline{A} = \underline{\hat{x}}\,\nabla^2 A_x + \underline{\hat{y}}\,\nabla^2 A_y + \underline{\hat{z}}\,\nabla^2 A_z$$

Radiation (cont.)

Hence

$$\nabla(\nabla\cdot\underline{A})-\nabla^2\underline{A}-k^2\underline{A}=\mu\underline{J}^i-j\omega\mu\varepsilon_c\nabla\Phi$$

or

$$\nabla^2\underline{A}+k^2\underline{A}=-\mu\underline{J}^i+\left[\nabla(\nabla\cdot\underline{A})+j\omega\mu\varepsilon_c\nabla\Phi\right]$$

Lorenz Gauge: **Note:** Lorenz, not Lorentz (as in Lorentz force law)!

Choose

$$\boxed{\nabla\cdot\underline{A}=-j\omega\mu\varepsilon_c\Phi}$$

Then we have

$$\boxed{\nabla^2\underline{A}+k^2\underline{A}=-\mu\underline{J}^i}$$

$$k^2=\omega^2\mu\varepsilon_c$$

Radiation (cont.)

Take the z component:

$$\nabla^2 A_z + k^2 A_z = -\mu J_z^i$$

Hence we have

$$\boxed{\nabla^2 A_z + k^2 A_z = -\mu (Il)\,\delta(\underline{r})}$$

Note:

$$\nabla^2 A_x + k^2 A_x = 0$$
$$\nabla^2 A_y + k^2 A_y = 0$$

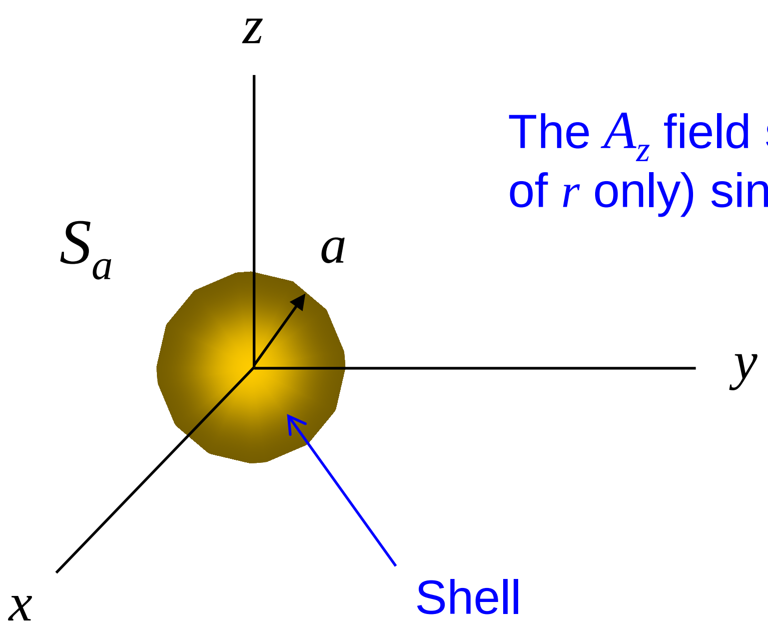
so

$$\boxed{\begin{aligned} A_x &= 0 \\ A_y &= 0 \end{aligned}}$$

Radiation (cont.)

$$\nabla^2 A_z + k^2 A_z = -\mu(Il)\delta(\underline{r})$$

Spherical shell model of 3D delta function:



The A_z field should be symmetric (a function of r only) since the source is.

In spherical coordinates:

$$\delta(\underline{r}) = \frac{1}{4\pi a^2}\delta(r-a)$$

Note : $\int_V \delta(\underline{r})\,dV = \int_{a^-}^{a^+} \delta(\underline{r})\,4\pi r^2 dr = 1$

Radiation (cont.)

Assume $A_z(\underline{r}) = R(r)$

For $r \neq 0$ we have that $\nabla^2 A_z + k^2 A_z = 0$

so that

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{\nabla^2 R(r)} + k^2 R = 0$$

Next, let

$$R(r) = \frac{1}{r} h(r) \quad \Rightarrow \quad r^2 \frac{dR(r)}{dr} = r^2\left[\frac{1}{r} h'(r) - \frac{1}{r^2} h(r)\right]$$

$$= r h'(r) - h(r)$$

Radiation (cont.)

We then have:

$$\frac{1}{r^2}\frac{d}{dr}\left(rh' - h\right) + k^2\frac{h}{r} = 0$$

$$\Rightarrow \frac{1}{r^2}\left(h' + rh'' - h'\right) + k^2\frac{h}{r} = 0$$

$$\Rightarrow \frac{1}{r}\left[h''(r) + k^2 h(r)\right] = 0$$

Solution: $h(r) = Ae^{-jkr} + Be^{+jkr}$

We choose $\exp(-jkr)$ for $r > a$ to satisfy the radiation condition at infinity.

We choose $\sin(kr)$ for $r < a$ to ensure that the potential is <u>finite</u> at the origin.

$\left(\text{Recall that } R\left(r\right) = h(r)/r.\right)$

Radiation (cont.)

Hence

$$h(r)=\begin{cases} Ae^{-jkr}, & r>a \\ B\sin(kr), & r<a \end{cases}$$

The R function satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2R=-\mu(Il)\delta(\underline{r})=-\mu(Il)\frac{1}{4\pi a^2}\delta(r-a)$$

$$\Downarrow$$

$$\frac{1}{r}\left[h''(r)+k^2h(r)\right]=-\mu(Il)\frac{1}{4\pi a^2}\delta(r-a)$$

$$\Downarrow$$

$$h''(r)+k^2h(r)=-\mu(Il)\frac{1}{4\pi a}\delta(r-a)$$

Radiation (cont.)

We require

$$h\left(a^{+}\right) = h\left(a^{-}\right) \qquad \text{(BC \#1)}$$

Also, from the differential equation we have

$$\int_{a^-}^{a^+} h''(r)\,dr + k^2 \int_{a^-}^{a^+} h(r)\,dr = -\mu\,(Il) \int_{a^-}^{a^+} \frac{1}{4\pi a}\,\delta\left(r-a\right)dr$$

$$\Downarrow$$

$$h'\left(a^{+}\right) - h'\left(a^{-}\right) = -\mu\,(Il)\,\frac{1}{4\pi a} \qquad \text{(BC \#2)}$$

Radiation (cont.)

Recall

$$h(r)=\begin{cases} Ae^{-jkr}, & r>a \\ B\sin(kr), & r<a \end{cases}$$

Hence

$$Ae^{-jka}=B\sin(ka) \qquad \text{(BC \#1)}$$

$$A(-jk)e^{-jka}-Bk\cos(ka)=-\mu(Il)\frac{1}{4\pi a} \qquad \text{(BC \#2)}$$

Radiation (cont.)

Substituting Ae^{-jka} from the first equation into the second, we have

$$(-jk)\left[B\sin(ka)\right]-Bk\cos(ka)=-\mu(Il)\frac{1}{4\pi a}$$

Hence, we have

$$B=\mu(Il)\frac{1}{4\pi a}\left[\frac{1}{k\left(j\sin(ka)+\cos(ka)\right)}\right]$$

From BC #1 we then have

$$A=\mu(Il)\frac{1}{4\pi a}\left[\frac{e^{jka}\sin(ka)}{k\left(j\sin(ka)+\cos(ka)\right)}\right]$$

Radiation (cont.)

Letting $a \to 0$, we have

$$A = \mu(Il)\frac{1}{4\pi a}\left[\frac{e^{jka}\sin(ka)}{k\left(j\sin(ka)+\cos(ka)\right)}\right]$$

$$\to \mu(Il)\frac{1}{4\pi}$$

so

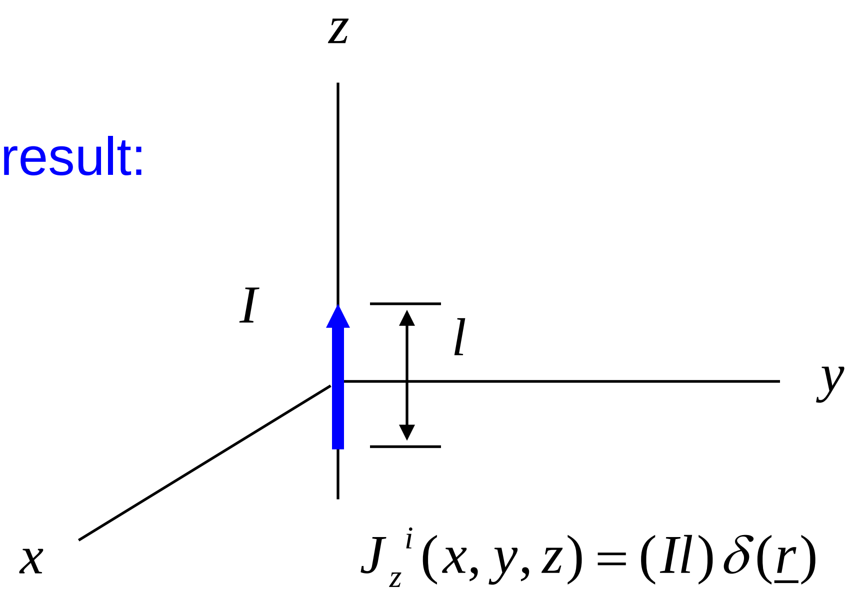
$$\boxed{A = \mu(Il)\frac{1}{4\pi}}$$

Hence

$$\boxed{A_z(r) = R(r) = \frac{1}{r}h(r) = \frac{1}{r}Ae^{-jkr} = \mu(Il)\frac{1}{4\pi r}e^{-jkr} \quad (r > a)}$$

Radiation (cont.)

Final result:



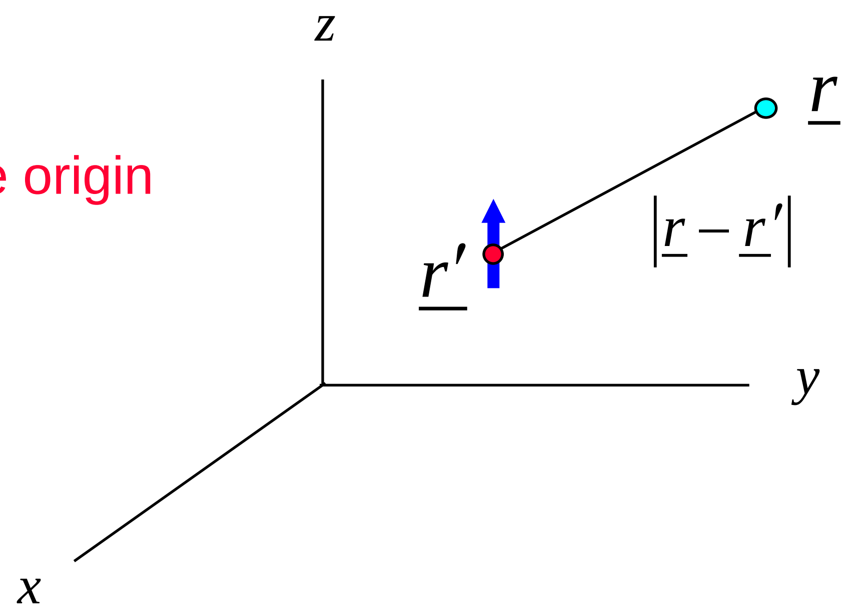
$$J_z^{\,i}(x,y,z) = (Il)\,\delta(\underline{r})$$

$$\begin{aligned}\underline{A} &= \underline{\hat{z}}\frac{\mu}{4\pi}(Il)\frac{e^{-jkr}}{r} \\ &= \left(\underline{\hat{r}}\cos\theta - \underline{\hat{\theta}}\sin\theta\right)\frac{\mu}{4\pi}(Il)\frac{e^{-jkr}}{r}\end{aligned}$$

Note: It is the moment *Il* that is important.

Extensions

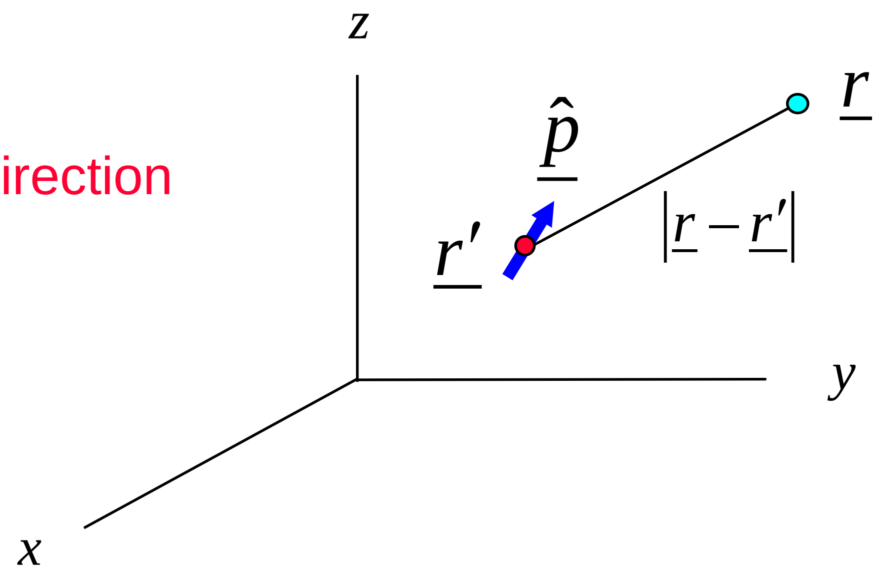
(a) Dipole not at the origin



$$\underline{A}=\underline{\hat{z}}\frac{\mu}{4\pi}(Il)\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$$

Extensions (cont.)

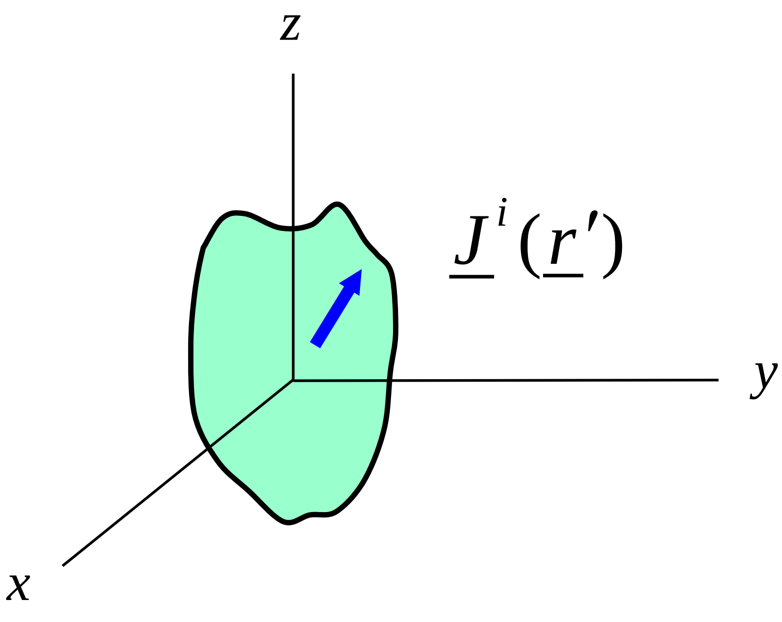
(b) Dipole not in the z direction



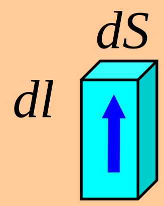
$$\underline{A} = \underline{\hat{p}}\,\frac{\mu}{4\pi}(Il)\,\frac{e^{-jk\left|\underline{r}-\underline{r}'\right|}}{\left|\underline{r}-\underline{r}'\right|}$$

Extensions (cont.)

(c) Volume current



Consider the z component of the current:



$$\underline{J}^i dV = \underline{\hat{z}}\, J_z^i \, dS\, dl$$
$$= \underline{\hat{z}} I \, dl$$

$$d\underline{A} = \underline{\hat{z}}\,(Idl)\frac{\mu}{4\pi}\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} = \underline{J}^i dV \frac{\mu}{4\pi}\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$$

Hence

$$\underline{A} = \frac{\mu}{4\pi}\int_V \underline{J}^i(\underline{r}')\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}dV'$$

The same result is obtained for the current components in the x and y directions, so this is a general result.

Extensions (cont.)

(d) Volume, surface, and filamental currents

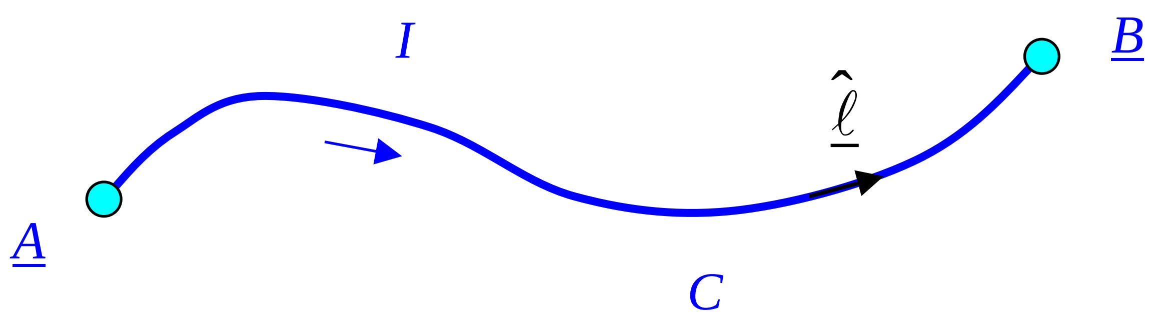
$$\underline{J}^i dV \to \begin{cases} \underline{J}^i dV & \text{(volume current density)} \\ \underline{J}_s^i \, dS & \text{(surface current density)} \\ I \underline{\hat{\ell}} \, d\ell & \text{(filamentary (wire) current)} \end{cases}$$

$$\underline{A}(\underline{r}) = \begin{cases} \dfrac{\mu}{4\pi} \displaystyle\int_V \underline{J}^i(\underline{r}') \dfrac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} dV' & \text{(volume current density)} \\ \dfrac{\mu}{4\pi} \displaystyle\int_S \underline{J}_s^i(\underline{r}') \dfrac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} dS' & \text{(surface current density)} \\ \dfrac{\mu}{4\pi} \displaystyle\int_C \underline{\hat{\ell}}(\underline{r}') I^i(\underline{r}') \dfrac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} d\ell' & \text{(filament of current)} \end{cases}$$

Extensions (cont.)

Filament of current (useful for wire antennas):

$$\underline{A}(\underline{r}) = \frac{\mu}{4\pi}\int_C \underline{\hat{\ell}}(\underline{r}')\, I^i(\underline{r}')\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\,d\ell'$$



Note: The current I^i is the current that flows in the direction of the unit tangent vector along the contour.

Fields

To find the fields:

$$\underline{H} = \frac{1}{\mu} \nabla \times \underline{A}$$

$$\underline{E} = \frac{1}{j\omega\varepsilon_c} \nabla \times \underline{H} \qquad \text{(valid for } r \neq 0\text{).}$$

Alternative way to find the electric field:

$$\underline{E} = -j\omega \underline{A} - \nabla \Phi$$

$$= -j\omega \underline{A} + \frac{1}{j\omega\mu\varepsilon_c} \nabla \left(\nabla \cdot \underline{A} \right)$$

Recall: $\nabla \cdot \underline{A} = -j\omega\mu\varepsilon_c \Phi$

Fields (cont.)

Results for infinitesimal, z-directed dipole at the origin:

$$E_r = \frac{Il}{2\pi}\eta e^{-jkr}\left(\frac{1}{r^2}\right)\left[1+\frac{1}{jkr}\right]\cos\theta$$

$$E_\theta = \frac{Il}{4\pi}(j\omega\mu)e^{-jkr}\left(\frac{1}{r}\right)\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\sin\theta$$

$$H_\phi = \frac{Il}{4\pi}(jk)e^{-jkr}\left(\frac{1}{r}\right)\left[1+\frac{1}{jkr}\right]\sin\theta$$

with $\eta = \sqrt{\dfrac{\mu}{\varepsilon_c}}$

Note on Dissipated Power

Note: If the medium is lossy, there will be an infinite amount of power dissipated by the infinitesimal dipole.

$$\langle P_d \rangle = \frac{1}{2}\omega\varepsilon'' \int_{V_\varepsilon} \left|\underline{E}\right|^2 dV$$

(power dissipation inside of a small spherical region V_ε of radius ε)

$$= \frac{1}{2}\omega\varepsilon'' \int_0^{\varepsilon}\int_0^{\pi}\int_0^{2\pi} \left|\underline{E}\right|^2 r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$\frac{1}{2}\omega\varepsilon'' \int_0^{\varepsilon}\int_0^{\pi}\int_0^{2\pi} \left(\left|E_\theta\right|^2 + \left|E_r\right|^2\right) r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \frac{1}{2}\omega\varepsilon''(2\pi)\left(\frac{4}{3}+\frac{2}{3}\right)\int_0^{\varepsilon}\left(\left|E_\theta^N\right|^2 + \left|E_r^N\right|^2\right) r^2 \, dr$$

The N superscript denotes that the θ dependence is suppressed in the fields.

$$\sim \frac{1}{2}\omega\varepsilon''(2\pi)\left(\frac{4}{3}+\frac{2}{3}\right)\left(\left|\frac{Il}{4\pi\omega\varepsilon_c}\right|\right)^2 \left(1+(2)^2\right)\int_0^{\varepsilon}\left(\frac{1}{r^3}\right)^2 r^2 \, dr$$

$$= \infty$$

Note:

$$\int_0^{\pi} \sin^2\theta \sin\theta \, d\theta = \frac{4}{3}$$

$$\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \frac{2}{3}$$

Far Field

Far-field of dipole: $\left(|kr| \gg 1\right)$

$$E_r \approx 0 \quad \left(\text{behaves as } O\left(\frac{1}{r^2}\right)\right)$$

$$E_\theta \sim (Il)\left(\frac{j\omega\mu}{4\pi r}\right)e^{-jkr}\sin\theta$$

$$H_\phi \sim (Il)\left(\frac{jk}{4\pi r}\right)e^{-jkr}\sin\theta$$

Note that $$\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{k} = \eta$$

The far-zone field acts like a homogeneous plane wave.

$$\underline{k} = k\underline{\hat{r}}, \quad \underline{r} = \underline{\hat{r}}r, \quad \underline{k} \cdot \underline{r} = kr$$

Radiated Power (lossless case)

$$P_{rad} = \frac{1}{2}\mathrm{Re}\int_{S_\infty} (\underline{E} \times \underline{H}^*) \cdot \underline{\hat{r}}\, dS$$

$$= \frac{1}{2}\mathrm{Re}\int_{S_\infty} E_\theta H_\phi^*\, dS$$

$$= \frac{1}{2}\mathrm{Re}\int_{S_\infty} \frac{|E_\theta|^2}{\eta}\, dS$$

$$= \frac{1}{2\eta}\int_0^{2\pi}\int_0^{\pi} |E_\theta|^2\, r^2 \sin\theta\, d\theta d\phi$$

$$= \frac{2\pi}{2\eta}\int_0^{\pi} |E_\theta|^2\, r^2 \sin\theta\, d\theta$$

The radius of the sphere is chosen as infinite in order to simplify the fields.

We assume lossless media here (η is real).

Radiated Power (cont.)

or

$$P_{rad} = \frac{\pi}{\eta}(Il)^2 \left(\frac{\omega\mu}{4\pi r}\right)^2 \int_0^{\pi} (\sin^2\theta)\, r^2 \sin\theta \, d\theta$$

$$= \frac{\pi}{\eta}(Il)^2 \left(\frac{\omega\mu}{4\pi}\right)^2 \int_0^{\pi} \sin^3\theta \, d\theta$$

Integral result: $\displaystyle\int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}$

Hence $\displaystyle P_{rad} = (Il)^2 \frac{\pi}{\eta}\left(\frac{\omega\mu}{4\pi}\right)^2 \left(\frac{4}{3}\right)$

Radiated Power (cont.)

Simplify using $\omega\mu = k\eta$

$$P_{rad} = (Il)^2 \frac{\pi}{\eta}\left(\frac{k\eta}{4\pi}\right)^2\left(\frac{4}{3}\right)$$

or

$$P_{rad} = (Il)^2 \frac{k^2\eta}{12\pi}$$

Use $k = \dfrac{2\pi}{\lambda}$ Then $P_{rad} = (Il)^2 \dfrac{4\pi^2\eta}{12\pi\lambda^2}$

Radiated Power (cont.)

We then have

$$P_{rad} = I^2 \left(\frac{l}{\lambda} \right)^2 \left(\frac{\pi \eta}{3} \right) \qquad [\mathrm{W}]$$

Note: The dipole radiation becomes significant when the dipole length is significant relative to a wavelength.

Potential Function

We do not <u>need</u> the potential function in order to calculate the fields, but we can obtain the potential function if we wish.

$$\underline{E} = -j\omega\underline{A} - \nabla\Phi$$ Needs Φ

Recall: $\nabla \cdot \underline{A} = -j\omega\mu\varepsilon_c\Phi$

$$\underline{E} = -j\omega\underline{A} + \frac{1}{j\omega\mu\varepsilon_c}\nabla\left(\nabla\cdot\underline{A}\right)$$ Does not need Φ

Another form:

$\underline{J}^i = 0$ (away from the dipole)

$$\underline{E} = \frac{1}{j\omega\varepsilon_c}\nabla\times\underline{H} - \frac{1}{j\omega\varepsilon_c}\cancel{\underline{J}^i}$$ (from Ampere's law)

$$= \frac{1}{j\omega\mu\varepsilon_c}\nabla\times\left(\nabla\times\underline{A}\right) - \frac{1}{j\omega\varepsilon_c}\cancel{\underline{J}^i}$$ Does not need Φ

Potential Function (cont.)

The potential function is given by:

$$\Phi(\underline{r}) = \int_V \frac{\rho_v^i(\underline{r}')}{4\pi\varepsilon_c \left|\underline{r} - \underline{r}'\right|} e^{-jk\left|\underline{r}-\underline{r}'\right|} dV'$$

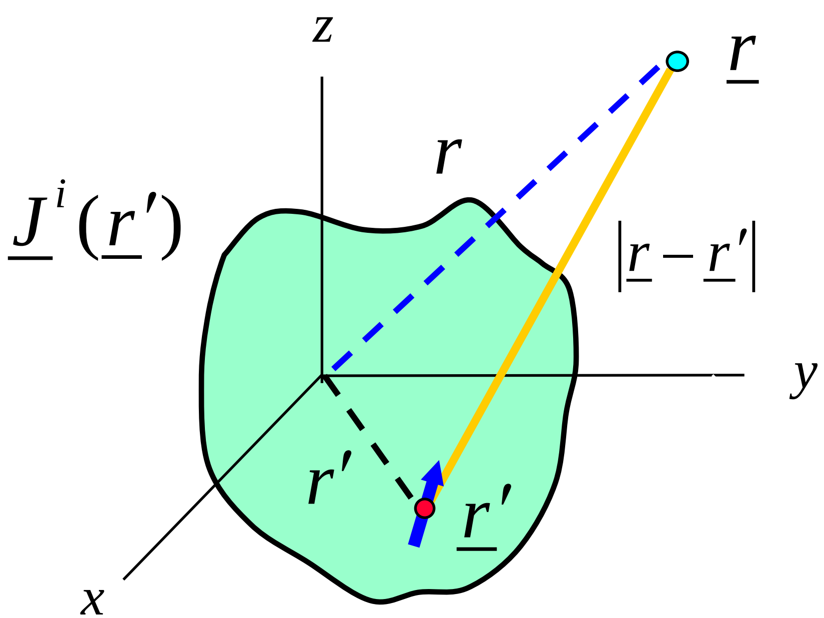
where

$$\nabla \cdot \underline{J}^i = -j\omega\rho_v^i(\underline{r}')$$

(This is left as a HW problem.)

Note: The potential function is uniquely defined, even at high frequency, but <u>voltage</u> is not. In statics, potential = voltage.

Far-Field Radiation From Arbitrary Current



$$\underline{A} = \frac{\mu}{4\pi}\int_V \underline{J}^i(x', y', z')\frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}dV'$$

Far- Field (cont.)

$$\left|\underline{r}-\underline{r}'\right| = \left[\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right]^{1/2}$$

$$= \left[\left(x^2+y^2+z^2\right)+\left(x'^2+y'^2+z'^2\right)-2\left(xx'+yy'+zz'\right)\right]^{1/2}$$

$$= \sqrt{r^2+r'^2-2\underline{r}\cdot\underline{r}'} = \sqrt{r^2-2\underline{r}\cdot\underline{r}'+r'^2}$$

$$= r\sqrt{1\underbrace{-2\left(\frac{\underline{r}'\cdot\underline{\hat{r}}}{r}\right)+\left(\frac{r'}{r}\right)^2}_{x}}$$

Use Taylor series expansion: $\sqrt{1+x} \sim 1+\dfrac{x}{2}-\dfrac{1}{8}x^2+\ldots$

Far- Field (cont.)

$$\left|\underline{r}-\underline{r}'\right| = r\left[1-\left(\frac{\underline{r}'\cdot\underline{\hat{r}}}{r}\right)+\frac{1}{2}\left(\frac{r'}{r}\right)^2-\frac{1}{8}\left(\frac{-2\underline{r}'\cdot\underline{\hat{r}}}{r}\right)^2+O\left(\frac{1}{r^3}\right)\right]$$

$$\left|\underline{r}-\underline{r}'\right| = r-\underline{r}'\cdot\underline{\hat{r}}+\frac{1}{r}\left[\frac{1}{2}\left(r'\right)^2-\frac{1}{2}\left(\underline{r}'\cdot\underline{\hat{r}}\right)^2\right]+O\left(\frac{1}{r^2}\right)$$

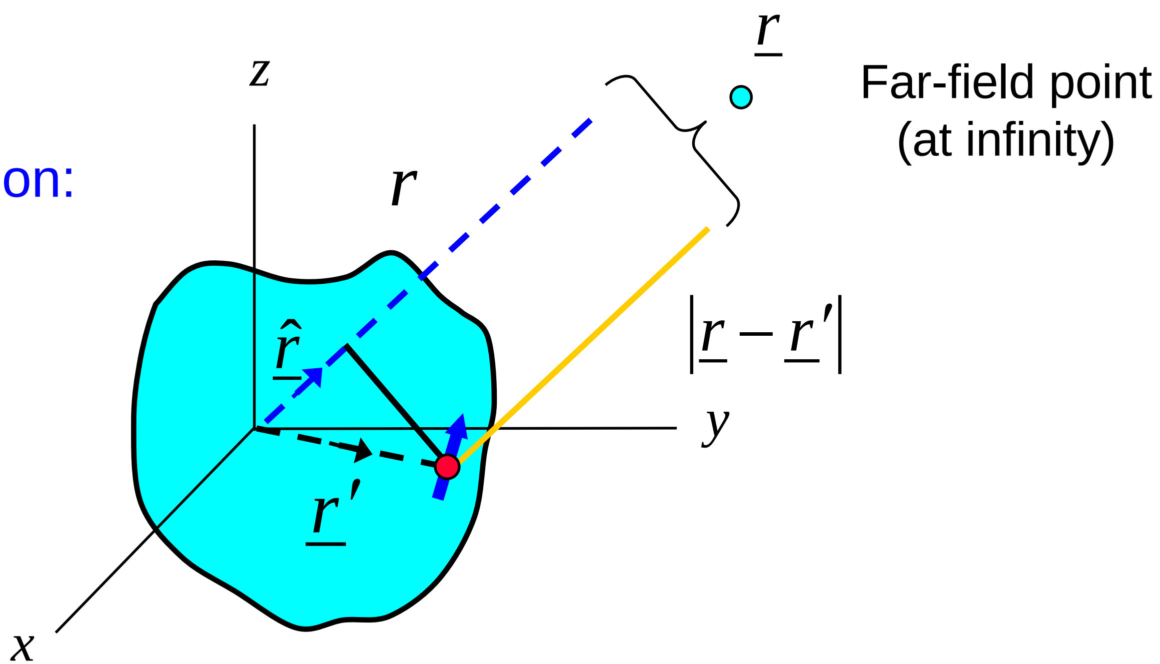
As $r \to \infty$

$$\left|\underline{r}-\underline{r}'\right| \sim r-\underline{r}'\cdot\underline{\hat{r}}$$

Far- Field (cont.)

$$\left|\underline{r}-\underline{r}'\right| \sim r-\underline{r}'\cdot\underline{\hat{r}}$$

Physical interpretation:



$$\underline{\hat{r}}=\underline{\hat{x}}\left(\sin\theta\cos\phi\right)+\underline{\hat{y}}\left(\sin\theta\sin\phi\right)+\underline{\hat{z}}\left(\cos\theta\right)$$

Far- Field (cont.)

Using this approximation, we have: $\left|\underline{r}-\underline{r}'\right| \sim r - \underline{r}'\cdot\underline{\hat{r}}$

$$\underline{A}(\underline{r}) = \frac{\mu}{4\pi}\int_V \underline{J}^i(x',y',z')\frac{e^{-jk\left|\underline{r}-\underline{r}'\right|}}{\left|\underline{r}-\underline{r}'\right|}dV' \sim \mu\frac{e^{-jkr}}{4\pi r}\int_V \underline{J}^i(\underline{r}')\, e^{jk\underline{r}'\cdot\underline{\hat{r}}}dV'$$

Define:

$$\psi(r) \equiv \frac{e^{-jkr}}{r}$$

$$\underline{k} \equiv k\underline{\hat{r}} = \underline{\hat{x}}\left(k\sin\theta\cos\phi\right) + \underline{\hat{y}}\left(k\sin\theta\sin\phi\right) + \underline{\hat{z}}\left(k\cos\theta\right)$$

(This is the $\underline{k}$ vector for a plane wave propagating towards the observation point.)

and

$$\underline{a}(\theta,\phi) \equiv \int_V \underline{J}^i(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'}dV'$$

Far- Field (cont.)

The we have

$$\underline{A}(\underline{r}) \sim \left(\frac{\mu}{4\pi}\right)\psi(r)\,\underline{a}(\theta,\phi)$$

where

$$\psi(r) = \frac{e^{-jkr}}{r}$$

$$\underline{a}(\theta,\phi) = \int_V \underline{J}^i(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'}\, dV'$$

$$\underline{k}\cdot\underline{r}' = x'\left(k\sin\theta\cos\phi\right) + y'\left(k\sin\theta\sin\phi\right) + z'\left(k\cos\theta\right)$$

Fourier Transform Interpretation

$$\underline{a}(\theta,\phi) = \int_V \underline{J}^i(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'}\, dV'$$

"vector array factor"

Denote

$$k_x = k \sin\theta \cos\phi$$
$$k_y = k \sin\theta \sin\phi$$
$$k_z = k \cos\theta$$

Then

$$\underline{a}(\theta,\phi) = \int_V \underline{J}^i(x', y', z')\; e^{+j\left(k_x x' + k_y y' + k_z z'\right)} dV'$$

The vector array factor is the 3D Fourier transform of the current.

Magnetic Field

We next calculate the fields far away from the source, starting with the magnetic field.

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$$

In the far field the spherical wave acts as a plane wave, and hence

$$\nabla \sim -j\underline{k} = -jk\underline{\hat{r}}$$

We then have

$$\underline{H} = \frac{1}{\mu}(-jk)\left(\underline{\hat{r}}\times\underline{A}\right)$$

Recall

$$\underline{A} \sim \left(\frac{\mu}{4\pi}\right)\psi(r)\,\underline{a}(\theta,\phi)$$

Hence

$$\underline{H} = \frac{1}{\mu}(-jk)\left(\frac{\mu}{4\pi}\right)\psi(r)\left(\underline{\hat{r}}\times\underline{a}\right)$$

Electric Field

We start with

$$\underline{H} \sim -\frac{jk}{4\pi}\left(\underline{\hat{r}} \times \underline{a}(\theta,\phi)\right)\psi(r)$$

The far-zone electric field is then given by

$$\begin{aligned}
\underline{E} &= \frac{1}{j\omega\varepsilon_c}\nabla \times \underline{H} \\
&\sim \frac{1}{j\omega\varepsilon_c}\left(-jk\underline{\hat{r}}\right) \times \underline{H} \\
&= \frac{1}{j\omega\varepsilon_c}\left(-jk\underline{\hat{r}}\right) \times \left(-\frac{jk}{4\pi}\left(\underline{\hat{r}} \times \underline{a}\right)\psi\right) \\
&= j\omega\left(\frac{\mu}{4\pi}\right)\psi\left(\underline{\hat{r}} \times \left(\underline{\hat{r}} \times \underline{a}\right)\right)
\end{aligned}$$

Electric Field (cont.)

We next simplify the expression for the far-zone electric field.

$$\underline{E} \sim j\omega\left(\frac{\mu}{4\pi}\right)\psi\left(\underline{\hat{r}}\times\left(\underline{\hat{r}}\times\underline{a}\left(\theta,\phi\right)\right)\right)$$

$$= -j\omega\left(\frac{\mu}{4\pi}\right)\psi\,\underline{a}_t\left(\theta,\phi\right)$$

Recall that

$$\underline{A}(\underline{r}) \sim \left(\frac{\mu}{4\pi}\right)\psi\left(r\right)\underline{a}\left(\theta,\phi\right)$$

Hence

$$\boxed{\underline{E} \sim -j\omega\underline{A}_t}$$

Electric Field (cont.)

Hence, we have

$$\underline{E} \sim -j\omega A_t\left(r,\theta,\phi\right)$$

Note: The exact field is

$$\underline{E} = -j\omega\underline{A} - \nabla\Phi$$

The exact electric field has all three components (r, θ, ϕ) in general, but in the far field there are only (θ, ϕ) components.

Relation Between $\underline{E}$ and $\underline{H}$

Start with

$$\underline{E} \sim -j\omega\left(\frac{\mu}{4\pi}\right)\psi\,\underline{a}_t(\theta,\phi)$$

$$\underline{H} \sim -\frac{jk}{4\pi}\left(\underline{\hat{r}}\times\underline{a}\right)\psi$$

$$= -\frac{jk}{4\pi}\left(\underline{\hat{r}}\times\underline{a}_t\right)\psi$$

$$= -\frac{jk}{4\pi}\left(\underline{\hat{r}}\times\left[\frac{\underline{E}}{-j\omega\mu\psi/(4\pi)}\right]\right)\psi$$

Simplify using

$$\frac{k}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon_c}}{\omega\mu} = \sqrt{\frac{\varepsilon_c}{\mu}} = \frac{1}{\eta}$$

Hence

$$\underline{H} \sim \frac{1}{\eta}\left(\underline{\hat{r}}\times\underline{E}\right)$$

or

$$\frac{E_\theta}{H_\phi} = \eta \quad , \quad \frac{E_\phi}{H_\theta} = -\eta$$

Summary of Far Field Recipe

$$\underline{E} \sim -j\omega A_t\left(r,\theta,\phi\right)$$

(keep only θ and ϕ components of $\underline{A}$)

$$\underline{A}(r,\theta,\phi) \sim \frac{\mu}{4\pi}\psi\left(r\right)\underline{a}\left(\theta,\phi\right)$$

$$\psi(r) = \frac{e^{-jkr}}{r} \qquad \underline{a}(\theta,\phi) = \int_V \underline{J}^i(\underline{r}')\,e^{+j\underline{k}\cdot\underline{r}'}dV'$$

$$\underline{H} \sim \frac{1}{\eta}\left(\underline{\hat{r}} \times \underline{E}\right)$$

$$\underline{k}\cdot\underline{r}' = k_x x' + k_y y' + k_z z' = \left(k\sin\theta\cos\phi\right)x' + \left(k\sin\theta\sin\phi\right)y' + \left(k\cos\theta\right)z'$$

Summary of Far Field Recipe

To specialize to any dimensionality, we have:

$$\underline{a}(\theta,\phi) = \int_V \underline{J}^{\,i}(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'} dV'$$

$$\underline{a}(\theta,\phi) = \int_S J_s^{\,i}(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'} dS'$$

$$\underline{a}(\theta,\phi) = \int_C \underline{\hat{\ell}}(\underline{r}')\, I(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'} d\ell'$$

Poynting Vector

$$
\begin{aligned}
\underline{S} &= \frac{1}{2}\underline{E} \times \underline{H}^{*} \\
&\sim \frac{1}{2}\underline{E}_t \times \underline{H}_t^{*} \\
&= \frac{1}{2}\left(\underline{\hat{\theta}}E_\theta + \underline{\hat{\phi}}E_\phi\right) \times \left(\underline{\hat{\theta}}H_\theta^{*} + \underline{\hat{\phi}}H_\phi^{*}\right) \\
&= \frac{1}{2}\underline{\hat{r}}\left(E_\theta H_\phi^{*} - E_\phi H_\theta^{*}\right) \\
&= \frac{1}{2}\underline{\hat{r}}\left(\frac{E_\theta E_\theta^{*}}{\eta^{*}} + \frac{E_\phi E_\phi^{*}}{\eta^{*}}\right)
\end{aligned}
$$

Poynting Vector (cont.)

$$\underline{S} = \underline{\hat{r}} \left(\frac{\left| E_\theta \right|^2}{2\eta^*} + \frac{\left| E_\phi \right|^2}{2\eta^*} \right) = \underline{\hat{r}} \left(\frac{\left| \underline{E} \right|^2}{2\eta^*} \right)$$

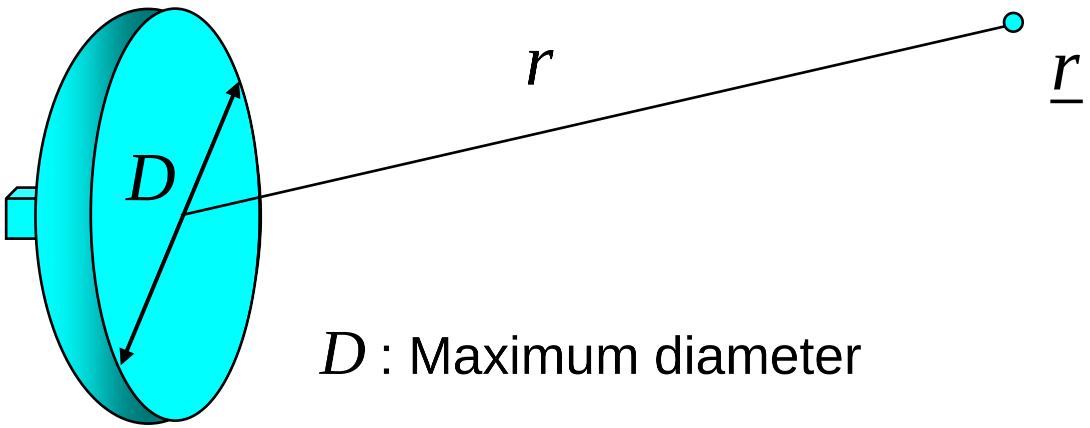
Assuming a <u>lossless</u> media,

$$\underline{S} = \underline{\hat{r}} \left(\frac{\left| E_\theta \right|^2}{2\eta} + \frac{\left| E_\phi \right|^2}{2\eta} \right) = \underline{\hat{r}} \left(\frac{\left| \underline{E} \right|^2}{2\eta} \right)$$

In a <u>lossless</u> medium, the Poynting vector is <u>purely real</u>
(no imaginary power flow in the far field).

Far-Field Criterion

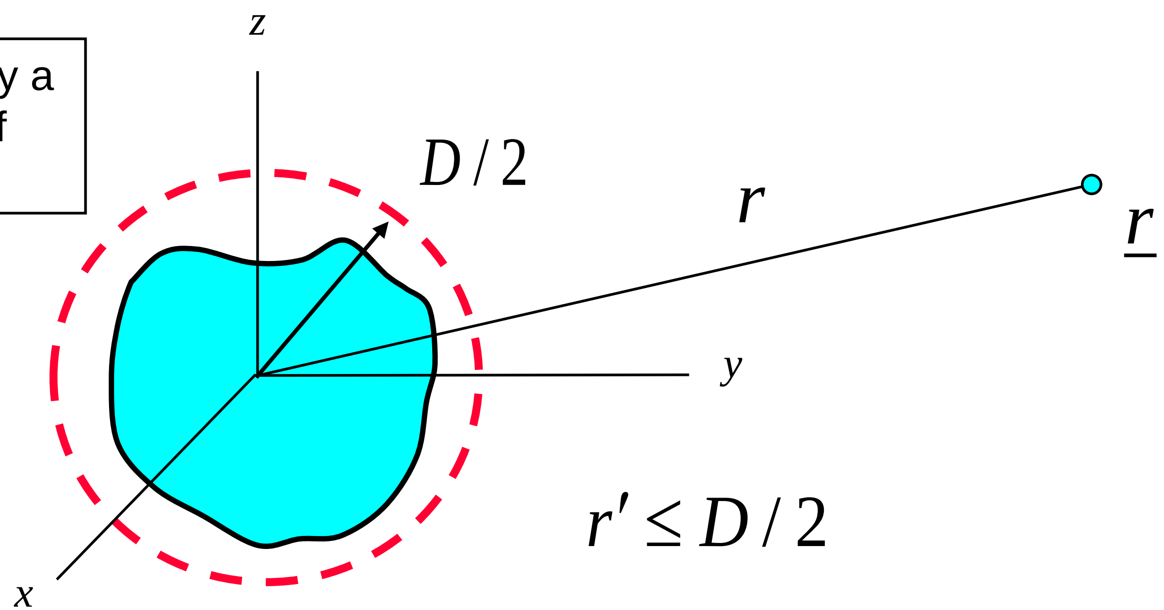
Antenna
(in free-space)



D : Maximum diameter

How large does r have to be to ensure an accurate far-field approximation?

The antenna is enclosed by a *circumscribing sphere* of diameter D.



$$r' \leq D/2$$

Far-Field Criterion (cont.)

$$\left|\underline{r}-\underline{r}'\right| = r - \underline{r}'\cdot\underline{\hat{r}} + \underbrace{\frac{r'^2}{2r} - \frac{1}{2}\frac{\left(\underline{r}'\cdot\underline{\hat{r}}\right)^2}{r}}_{\text{Error}} + O\left(\frac{1}{r^2}\right)$$

Neglect

Let $\Delta\phi$ = maximum phase error in the exponential term inside the integrand.

$$\Delta\phi = k_0\left[\frac{r'^2}{2r} - \frac{1}{2}\frac{\left(\underline{r}'\cdot\underline{\hat{r}}\right)^2}{r}\right]_{\max}$$

Hence

$$\Delta\phi < \left.\frac{k_0 r'^2}{2r}\right|_{\max} = \frac{k_0 (D/2)^2}{2r}$$

Far-Field Criterion (cont.)

$$\Delta\phi_{\max} = \frac{k_0 D^2}{8r}$$

Set

$$\Delta\phi_{\max} = \frac{\pi}{8}\,[\mathrm{rad}] \quad (22.5^{\circ})$$

Then

$$\frac{k_0 D^2}{8r} = \frac{\pi}{8}$$

Far-Field Criterion (cont.)

$$r = \frac{k_0 D^2}{\pi} = \frac{2\pi D^2}{\lambda_0 \pi} = \frac{2D^2}{\lambda_0}$$

Hence, we have the restriction

$$\boxed{r \geq \frac{2D^2}{\lambda_0}}$$

"Fraunhofer criterion"

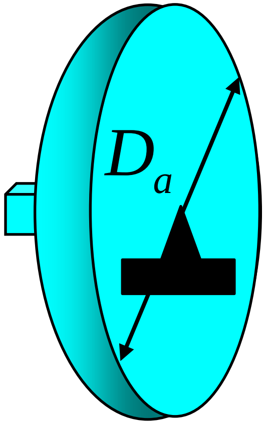
for $\Delta\phi_{\max} = \dfrac{\pi}{8}\ [\mathrm{rad}]$

Far-Field Criterion (cont.)

Note on choice of origin:

The diameter D can be minimized by a judicious choice of the origin. This corresponds to selecting the best possible mounting point when mounting an antenna on a rotating measurement platform.

$$D = D_a$$

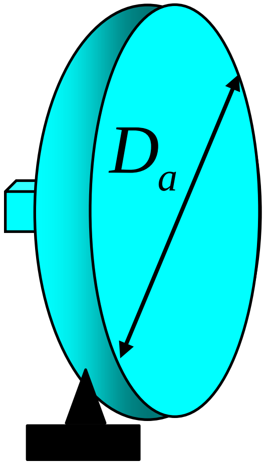


Mounted at center

$$r \geq \frac{2D^2}{\lambda_0}$$

D = diameter of circumscribing sphere
D_a = diameter of antenna

$$D = 2D_a$$



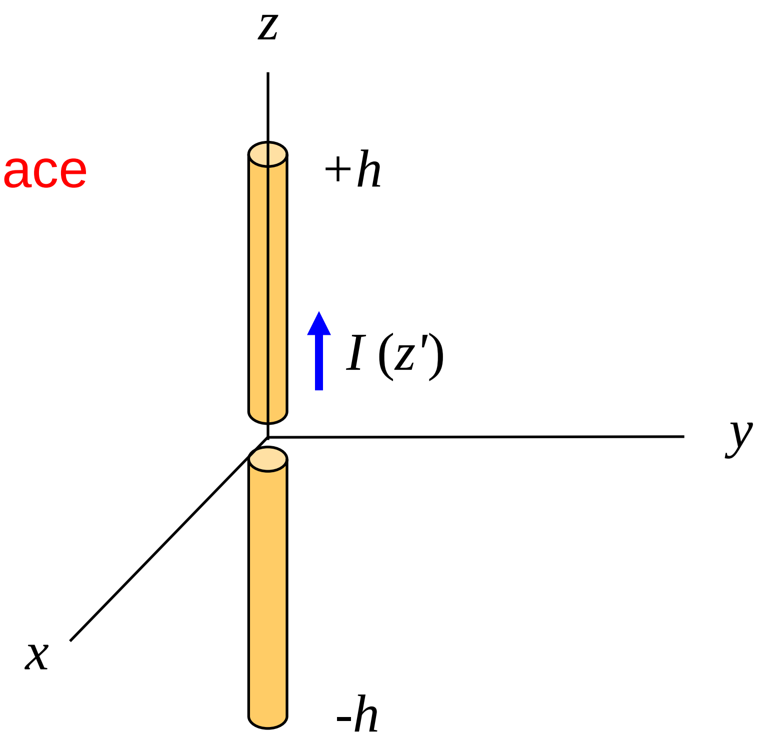
Mounted at edge

Far-Field Criterion (cont.)

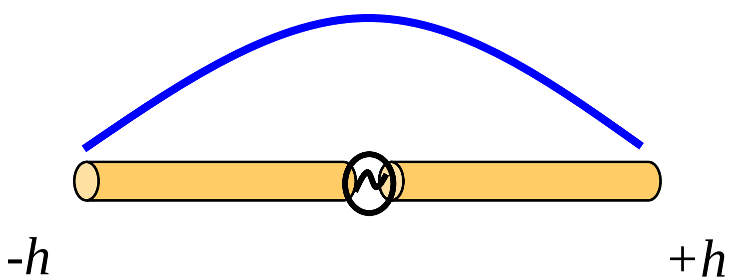
It is best to mount the antenna at the <u>center</u> of the rotating platform.

Wire Antenna

Wire antenna in free space



Assume $I(z') = I_0 \sin\left[k_0\left(h - |z'|\right)\right]$



Wire Antenna (cont.)

$$\underline{a}(\theta,\phi)=\int_V \underline{J}^{\,i}(\underline{r}')\,e^{+j\underline{k}\cdot\underline{r}'}dV'$$

$$\underline{J}^{\,i}dV' \rightarrow \underline{\hat{z}}\big(I(z')\,dz'\big)$$

Hence

$$\underline{a}(\theta,\phi)=\underline{\hat{z}}\int_{-h}^{+h} I(z')\,e^{+j\underline{k}\cdot\underline{r}'}dz'$$

$$=\underline{\hat{z}}\int_{-h}^{+h} I(z')\,e^{+j\left(\cancel{x'}k_0\sin\theta\cos\phi+\cancel{y'}k_0\sin\theta\sin\phi+z'k_0\cos\theta\right)}dz'$$

$$=\underline{\hat{z}}\int_{-h}^{+h} I(z')\,e^{+jz'\left(k_0\cos\theta\right)}dz'$$

Wire Antenna (cont.)

$$\underline{a}(\theta,\phi) = \underline{\hat{z}} \int_{-h}^{+h} I_0 \sin\left[k_0\left(h - |z'|\right)\right] e^{+jz'(k_0 \cos\theta)} dz'$$

The result is

$$\underline{a}(\theta,\phi) = \underline{\hat{z}}\,(2I_0)\left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{k_0 \sin^2\theta}\right]$$

Recall that

$$\underline{E} \sim -j\omega \underline{A}_t \sim -j\omega\left(\frac{\mu_0}{4\pi}\right)\psi(r)\,\underline{a}_t(\theta,\phi)$$

For the wire antenna we have

$$\underline{a}_t(\theta,\phi) = \underline{\hat{\theta}}\left(\underline{\hat{\theta}} \cdot \left(\underline{\hat{z}}\, a_z\right)\right) + \underline{\hat{\phi}}\left(\cancel{\underline{\hat{\phi}} \cdot}\left(\underline{\hat{z}} a_z\right)\right)$$

$$= \underline{\hat{\theta}}\left(-\sin\theta\right) a_z$$

Wire Antenna (cont.)

Hence

$$\underline{E} \sim -j\omega\left(\frac{\mu_0}{4\pi}\right)\left(\frac{e^{-jk_0 r}}{r}\right)\left[\underline{\hat{\theta}}\left(-\sin\theta\right)a_z(\theta,\phi)\right]$$

or

$$\underline{E} \sim \underline{\hat{\theta}}\left(j\omega\mu_0 \sin\theta\right)\left(\frac{e^{-jk_0 r}}{4\pi r}\right)a_z(\theta,\phi)$$

$$= \underline{\hat{\theta}}\left(j\omega\mu_0 \sin\theta\right)\left(\frac{e^{-jk_0 r}}{4\pi r}\right)(2I_0)\left[\frac{\cos(k_0 h\cos\theta)-\cos(k_0 h)}{k_0 \sin^2\theta}\right]$$

Next, simplify using $\quad \dfrac{\omega\mu_0}{k_0} = \eta_0$

Wire Antenna (cont.)

We then have

$$\underline{E} \sim \underline{\hat{\theta}}\left(j\eta_0\right) I_0 \left(\frac{e^{-jk_0 r}}{2\pi r}\right)\left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{\sin\theta}\right]$$

Note: The pattern goes to zero at $\theta = 0,\ \pi$.
(You can verify this by using L'Hôpital's rule.)

Wire Antenna: Radiated Power

$$P_{rad} = \frac{1}{2\eta_0} \int_0^{2\pi} \int_0^{\pi} \left|E_\theta\right|^2 r^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{2\eta_0} (2\pi) r^2 \int_0^{\pi} \left|E_\theta\right|^2 \sin\theta d\theta$$

$$= \left(\frac{\pi}{\eta_0}\right) r^2 \int_0^{\pi} \left|E_\theta\right|^2 \sin\theta d\theta$$

$$= \left(\frac{\pi}{\eta_0}\right) r^2 \eta_0^2 I_0^2 \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{r}\right)^2 \int_0^{\pi} \left[\frac{\cos(k_0 h \cos\theta) - \cos(k_0 h)}{\sin\theta}\right]^2 \sin\theta d\theta$$

Radiated Power (cont.)

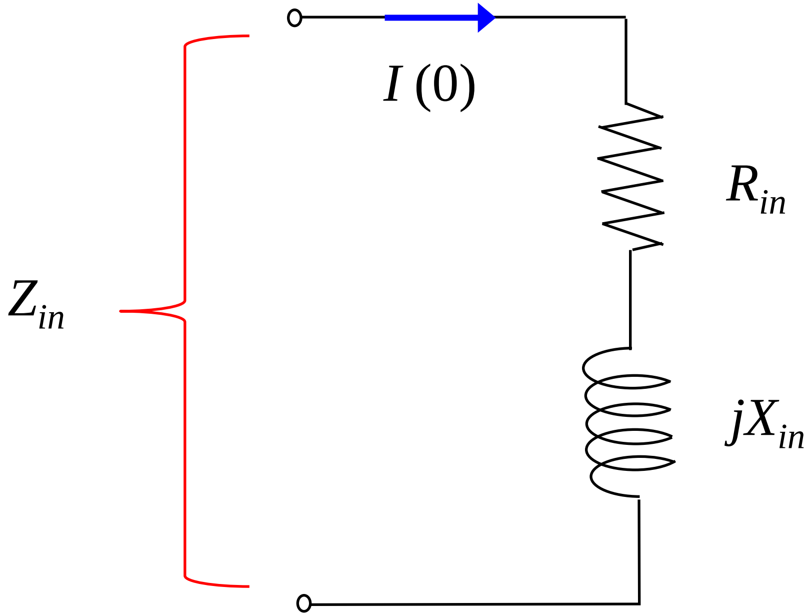
or

$$P_{rad} = I_0^{\,2}\left(\frac{\eta_0}{4\pi}\right)\int_0^{\pi} \frac{\left[\cos(k_0 h\cos\theta) - \cos(k_0 h)\right]^2}{\sin\theta}\,d\theta$$

Input Resistance

$$P_{rad} = I_0^{\,2}\left(\frac{\eta_0}{4\pi}\right)\int_0^{\pi}\frac{\left[\cos(k_0 h\cos\theta)-\cos(k_0 h)\right]^2}{\sin\theta}\,d\theta$$

Circuit model of antenna:



$$P_{rad} = \frac{1}{2}R_{in}\left|I(0)\right|^2$$

$$R_{in} = \frac{2P_{rad}}{\left|I(0)\right|^2}$$

Input Resistance (cont.)

$$I(0) = I_0 \sin\left[k_0\left(h - \left|\cancel{z'}\right|\right)\right]$$
$$= I_0 \sin(k_0 h)$$

$$R_{in} = \left(\frac{\eta_0}{2\pi}\right)\left(\frac{1}{\sin(k_0 h)}\right)^2 \int_0^{\pi} \frac{\left[\cos(k_0 h \cos\theta) - \cos(k_0 h)\right]^2}{\sin\theta}\, d\theta$$

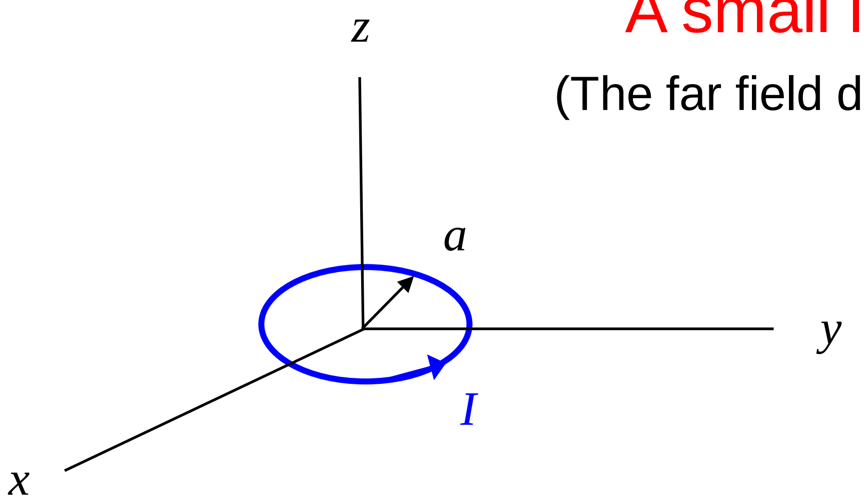
$\lambda/2$ Dipole: $\quad h = \dfrac{\lambda_0}{4}, \quad k_0 h = \dfrac{\pi}{2} \quad \Rightarrow \quad R_{in} \approx 73 \;\; [\Omega]$

(resonant)

Example

A small loop antenna

(The far field does not vary with ϕ.)



$$a << \lambda_0$$

(The current is approximately uniform).

$$\Rightarrow I(\phi) \approx I_0$$

$$\underline{a}(\theta,\phi) = \int_C \underline{\hat{\ell}}(\underline{r}')\, I(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'} d\ell'$$

Assume $\phi = 0$: $E_\phi = E_y$

$$k_x = k_0 \sin\theta$$

$$k_y = 0$$

so

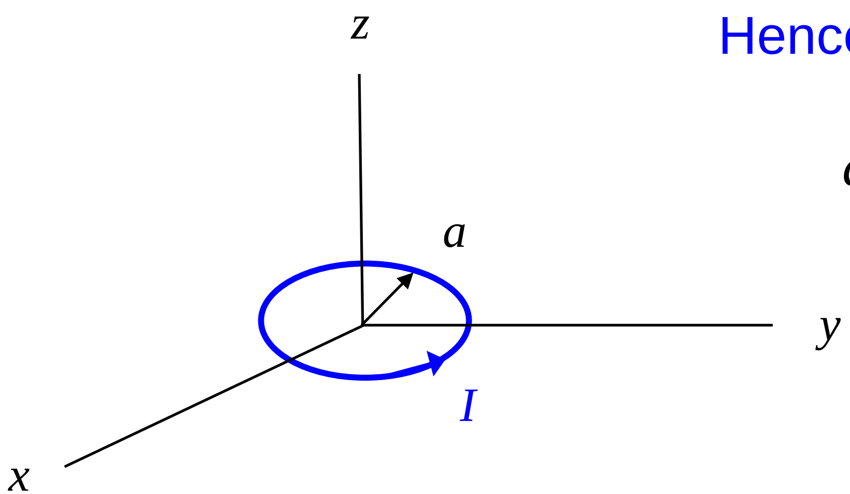
$$\underline{a}(\theta,\phi) = I_0 \int_C \underline{\hat{\phi}}'\; e^{+j\underline{k}\cdot\underline{r}'}\, a\, d\phi'$$

$$x' = a\cos\phi'$$

$$\underline{k}\cdot\underline{r}' = k_x x' + \cancel{k_y y'} + \cancel{k_z z'} = k_0 a \sin\theta \cos\phi'$$

$$\underline{\hat{\phi}}' \cdot \underline{\hat{y}} = \cos\phi'$$

Example (cont.)



Hence

$$a_\phi = a_y\Big|_{\phi=0} = I_0 \int_0^{2\pi} \cos\phi' \, e^{jk_0 a \sin\theta\cos\phi'} a\, d\phi'$$

Assume $\phi = 0$: By symmetry, we have $\boxed{a_x = 0}$

The x component of the array factor cancels for points ϕ' and $-\phi'$.

At $\phi = 0$:

$$a_\theta = \underline{\hat{\theta}} \cdot \left(\underline{\hat{x}} a_x + \underline{\hat{y}} a_y \right) = \cancel{a_x} \left(\underline{\hat{\theta}} \cdot \underline{\hat{x}} \right) + a_y \left(\cancel{\underline{\hat{\theta}} \cdot \underline{\hat{y}}} \right) = 0$$

Example (cont.)

Identity: $$\int_0^{2\pi} \cos\phi' \, e^{jx\cos\phi'} d\phi' = j2\pi J_1(x)$$

$$x \equiv k_0 a \sin\theta$$

Hence

$$\boxed{a_\phi = (j2\pi) a I_0 J_1(k_0 a \sin\theta)}$$

We then have

$$E_\phi = -j\omega\left(\frac{\mu_0}{4\pi}\right)\left(\frac{e^{-jk_0 r}}{r}\right) a_\phi$$

$$= -j\omega\left(\frac{\mu_0}{4\pi}\right)\left(\frac{e^{-jk_0 r}}{r}\right)(j2\pi) a I_0 J_1(k_0 a \sin\theta)$$

Example (cont.)

Hence

$$\underline{E} = \underline{\hat{\phi}}\left(\frac{\omega\mu_0 a I_0}{2}\right)\left(\frac{e^{-jk_0 r}}{r}\right)J_1\left(k_0 a \sin\theta\right)$$

Approximation of Bessel function:

$$J_1\left(x\right) \approx \frac{x}{2}, \quad x << 1$$

The field can be thought of as coming from a small vertical magnetic dipole, as we will discuss later (from duality).

The result is then

$$Kl = j\omega\mu_0\left(\pi a^2\right)I_0$$

$$\underline{E} \approx \underline{\hat{\phi}}\left(\frac{\omega\mu_0 k_0 a^2 I_0}{4}\right)\left(\frac{e^{-jk_0 r}}{r}\right)\sin\theta$$