ECE 6340
Intermediate EM Waves

Fall 2016

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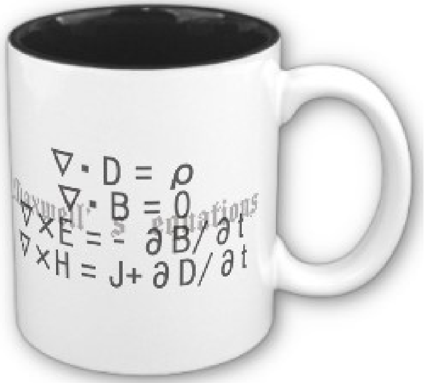


Notes 23

Duality

Assume a region of space with sources,
and allow for all types of sources and charges:

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu_c \underline{H}$$

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_c \underline{E}$$

$$\nabla \cdot (\varepsilon \underline{E}) = \rho_v$$

$$\nabla \cdot (\mu \underline{H}) = \rho_v^m$$

Maxwell's equations now have a completely symmetric form.

$$\varepsilon_c = \varepsilon - j\sigma / \omega$$

$$\mu_c = \mu - j\sigma_m / \omega$$

Note:
A magnetic conductivity σ_m is introduced for generality, though it is always zero in practice.

Duality (cont.)

The substitutions shown below leave Maxwell's equations unaffected.

$$\underline{E} \rightarrow \underline{H}$$

$$\underline{H} \rightarrow -\underline{E}$$

$$\varepsilon, \varepsilon_c \rightarrow \mu, \mu_c$$

$$\mu, \mu_c \rightarrow \varepsilon, \varepsilon_c$$

$$\underline{J}^i \rightarrow \underline{M}^i$$

$$\underline{M}^i \rightarrow -\underline{J}^i$$

$$\rho_v \rightarrow \rho_v^m$$

$$\rho_v^m \rightarrow -\rho_v$$

Duality (cont.)

Example:

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu_c \underline{H}$$

↓

$$\nabla \times \underline{H} = -\left(-\underline{J}^{i}\right) - j\omega\varepsilon_c \left(-\underline{E}\right)$$

↓

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_c \underline{E}$$

$$\nabla \times \underline{H} = \underline{J}^{i} + j\omega\varepsilon_c \underline{E}$$

↓

$$\nabla \times \left(-\underline{E}\right) = \underline{M}^{i} + j\omega\mu_c \underline{H}$$

↓

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu_c \underline{H}$$

Duality (cont.)

Duality allows us to find the radiation from a magnetic current.

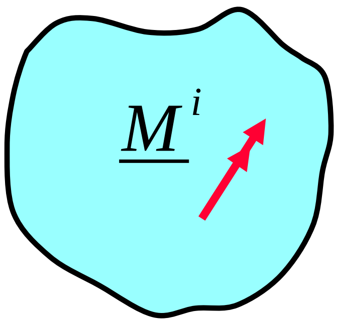
Steps:

1) Start with the given magnetic current source of interest.
2) Consider the dual problem that has an electric current source with the same shape or form (Case A).
3) Find the radiation from the electric current source.
4) Apply the dual substitutions to find the radiation from the original magnetic current (Case B).

Note:
When we apply the duality substitution, the numerical values of the permittivity and permeability switch. However, once we have the formula for radiation from the magnetic current, we can let the values be replaced by their conventional ones.

Radiation From Magnetic Current (cont.)

Problem of interest:



$$\underline{M}^i(x,y,z) = \underline{f}(x,y,z)$$

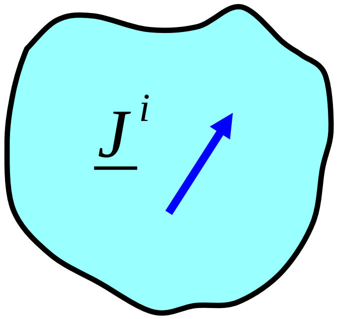
A magnetic current is radiating in free space.

Free-space is assumed for simplicity. To be more general, we can replace

$$\varepsilon_0 \to \varepsilon_c \qquad \mu_0 \to \mu_c$$

Radiation From Magnetic Current

Case (A): $\underline{J}^{\,i}$ only



The shape of the electric current is the same as that of the original magnetic current that was given.

$$\underline{J}^{\,i}(x,y,z)=\underline{f}(x,y,z)$$

$$\underline{A}=\mu_0\int_V \underline{J}^{\,i}(\underline{r}')\frac{e^{-jk_0|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|}dV'$$

$$\underline{H}=\frac{1}{\mu_0}\nabla\times\underline{A}$$

Radiation From Magnetic Current (cont.)

Also

$$\underline{E} = -j\omega\underline{A} - \nabla\Phi$$

and

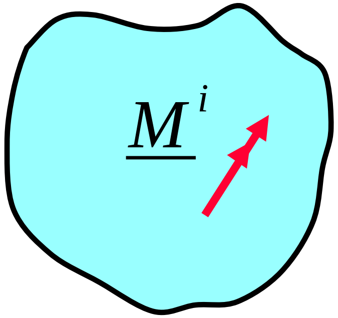
$$\nabla \cdot \underline{A} = -j\omega\mu_0\varepsilon_0\Phi$$

so

$$\underline{E} = -j\omega\underline{A} + \frac{1}{j\omega\mu_0\varepsilon_0}\nabla(\nabla \cdot \underline{A})$$

Radiation From Magnetic Current (cont.)

Case (B): $\underline{M}^{i}$ only



$$\underline{M}^{i}(x,y,z) = \underline{f}(x,y,z)$$

From duality:

Case A

$$\underline{H} = \frac{1}{\mu_0} \nabla \times \underline{A}$$

where

$$\underline{A} = \mu_0 \int_V \underline{J}^{i}(\underline{r}') \frac{e^{-jk_0|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} dV'$$

Case B

$$-\underline{E} = \frac{1}{\varepsilon_0} \nabla \times \underline{F}$$

where

$$\underline{F} = \varepsilon_0 \int_V \underline{M}^{i}(\underline{r}') \frac{e^{-jk_0|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} dV'$$

Radiation From Magnetic Current (cont.)

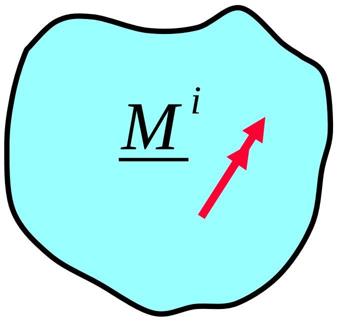
Also,

$$\underline{E} = -j\omega\underline{A} + \frac{1}{j\omega\mu_0\varepsilon_0}\nabla(\nabla\cdot\underline{A})$$ Case A

$$\Downarrow$$

$$\underline{H} = -j\omega\underline{F} + \frac{1}{j\omega\varepsilon_0\mu_0}\nabla(\nabla\cdot\underline{F})$$ Case B

Radiation from Magnetic Current Summary



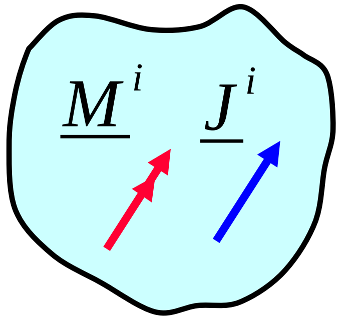
$$\underline{F} = \varepsilon_0 \int_V \underline{M}^{i}(\underline{r}') \frac{e^{-jk_0|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} dV'$$

$$\underline{E} = -\frac{1}{\varepsilon_0} \nabla \times \underline{F}$$

$$\underline{H} = -j\omega \underline{F} + \frac{1}{j\omega\varepsilon_0\mu_0} \nabla(\nabla \cdot \underline{F})$$

General Radiation Formula

Both $\underline{J}^{\,i}$ and $\underline{M}^{\,i}$:



$$\underline{J}^{\,i} \rightarrow \underline{A}$$

$$\underline{M}^{\,i} \rightarrow \underline{F}$$

Use superposition:

$$\underline{E} = -\frac{1}{\varepsilon_0}\nabla \times \underline{F} - j\omega \underline{A} + \frac{1}{j\omega\mu_0\varepsilon_0}\nabla(\nabla \cdot \underline{A})$$

$$\underline{H} = \frac{1}{\mu_0}\nabla \times \underline{A} - j\omega \underline{F} + \frac{1}{j\omega\varepsilon_0\mu_0}\nabla(\nabla \cdot \underline{F})$$

Note: Duality also holds with the vector potentials, If we include these:

$$\underline{A} \rightarrow \underline{F}$$

$$\underline{F} \rightarrow -\underline{A}$$

Far-Field

Case (A) (electric current only):

$$\underline{E} \sim - j\omega \underline{A}_t(r,\theta,\phi)$$

$$\underline{A} \sim \left(\frac{\mu_0}{4\pi}\right)\psi(r)\,\underline{a}(\theta,\phi)$$

$$\underline{H} \sim \frac{1}{\eta_0}(\underline{\hat{r}} \times \underline{E})$$

where

$$\underline{a}(\theta,\phi) = \int_V \underline{J}^i(\underline{r}')\, e^{+j\underline{k}\cdot\underline{r}'} dV'$$

$$\psi(r) = \frac{e^{-jk_0 r}}{r}$$

Far-Field (cont.)

Case (B) (magnetic current only):

$$\underline{H} \sim -j\omega \underline{F}_t(r,\theta,\phi)$$

$$\underline{F} \sim \left(\frac{\varepsilon_0}{4\pi}\right)\psi(r)\,\underline{f}(\theta,\phi)$$

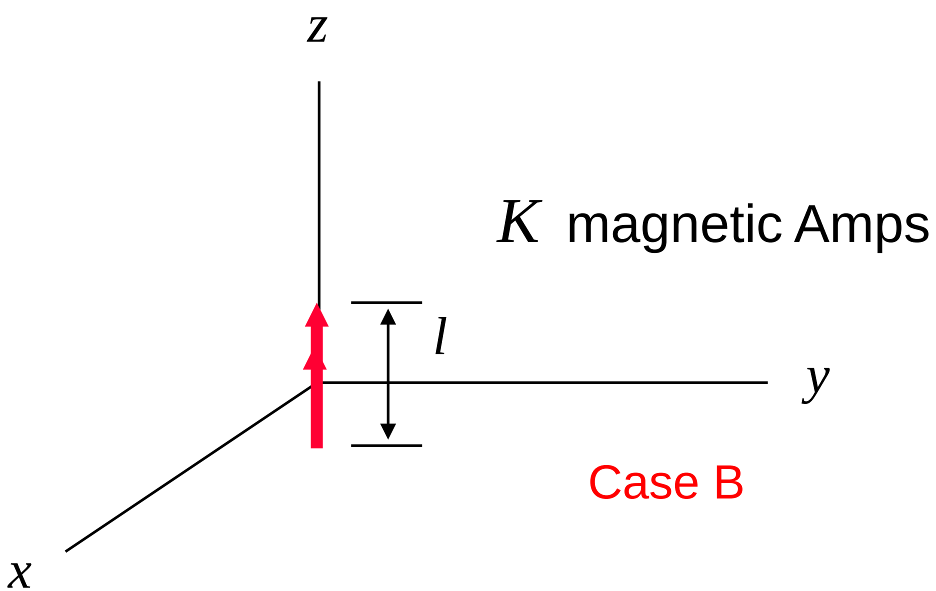
$$\underline{E} \sim -\eta_0\,(\hat{\underline{r}} \times \underline{H})$$

where

$$\underline{f}(\theta,\phi) \equiv \int_V \underline{M}^i(\underline{r}')\,e^{+j\underline{k}\cdot\underline{r}'}\,dV'$$

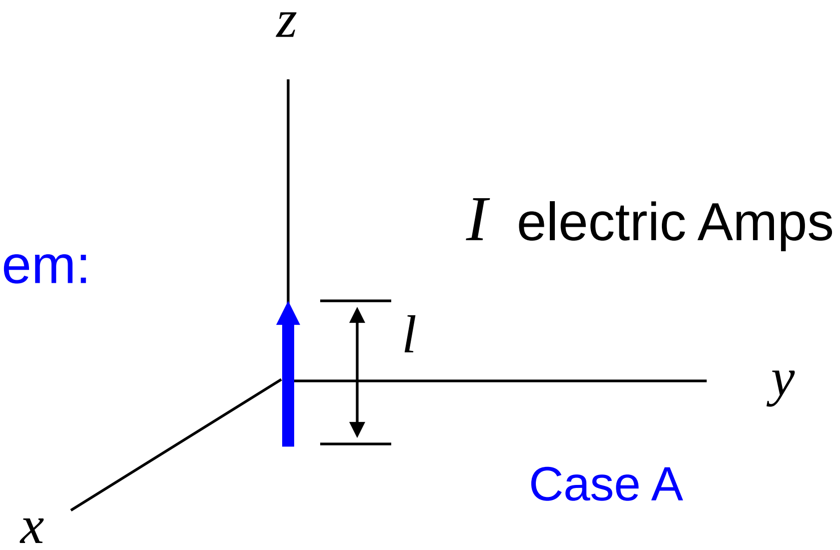
Example

Radiation from a magnetic dipole



Case B

Dual problem:



Case A

Example (cont.)

For the electric dipole:

$$\underline{H} = \underline{\hat{\phi}} \frac{(Il)}{4\pi} e^{-jk_0 r} \left(\frac{jk_0}{r} + \frac{1}{r^2} \right) \sin\theta$$

Duality:

$$I \to K$$

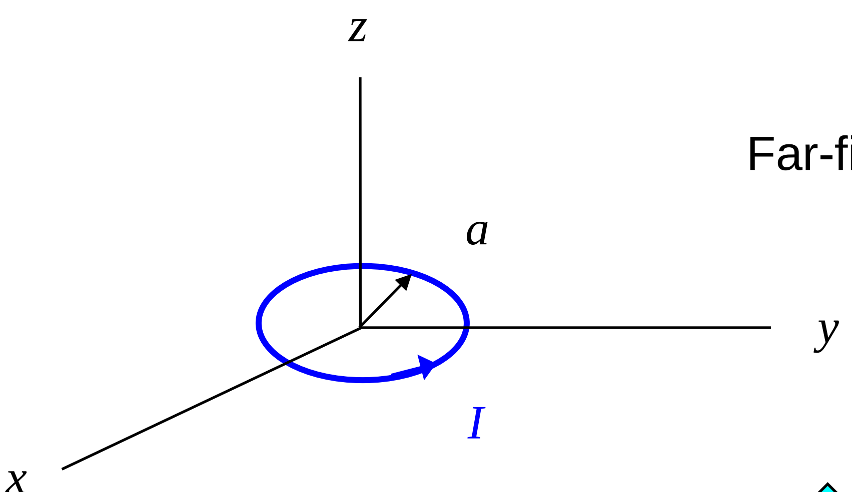
$$\underline{H} \to -\underline{E}$$

So, for the magnetic dipole:

$$\underline{E} = -\underline{\hat{\phi}} \frac{(Kl)}{4\pi} e^{-jk_0 r} \left(\frac{jk_0}{r} + \frac{1}{r^2} \right) \sin\theta$$

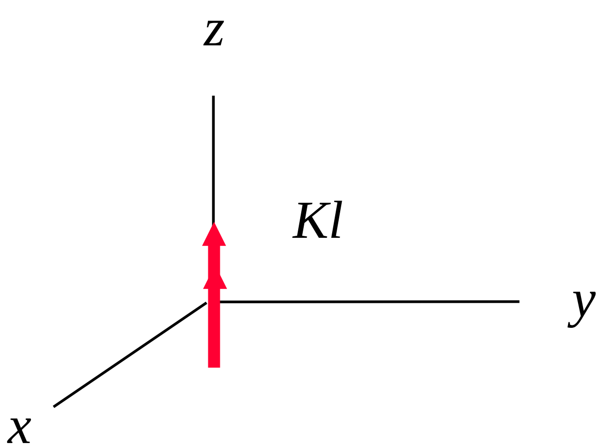
Example

A small loop antenna is equivalent to a magnetic dipole antenna,



Far-field: $$E_\phi \approx \left(\frac{\omega \mu_0 k_0 a^2 I_0}{4} \right) \left(\frac{e^{-jk_0 r}}{r} \right) \sin\theta$$

$$Kl \equiv j\omega\mu_0 \left(A I_0 \right)$$
$$A = \pi a^2$$

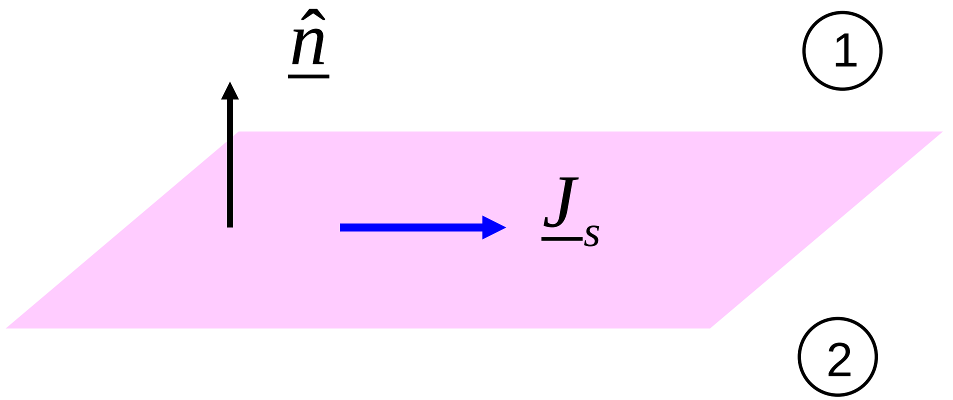


Far-field: $$E_\phi \approx -\frac{(Kl)}{4\pi} e^{-jk_0 r} \left(\frac{jk_0}{r} \right) \sin\theta$$

This is the correct far field: The loop can be modeled as a magnetic dipole.

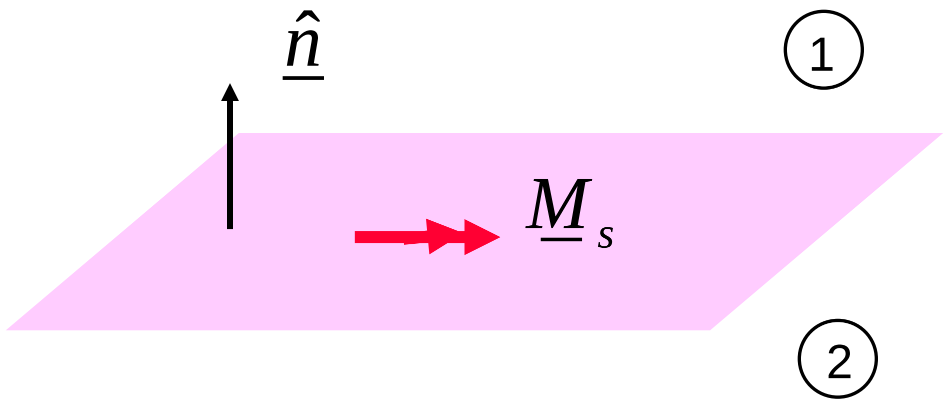
Boundary Conditions

Electric surface current:



$$\hat{\underline{n}} \times \left(\underline{H}^{(1)} - \underline{H}^{(2)} \right) = \underline{J}_s$$

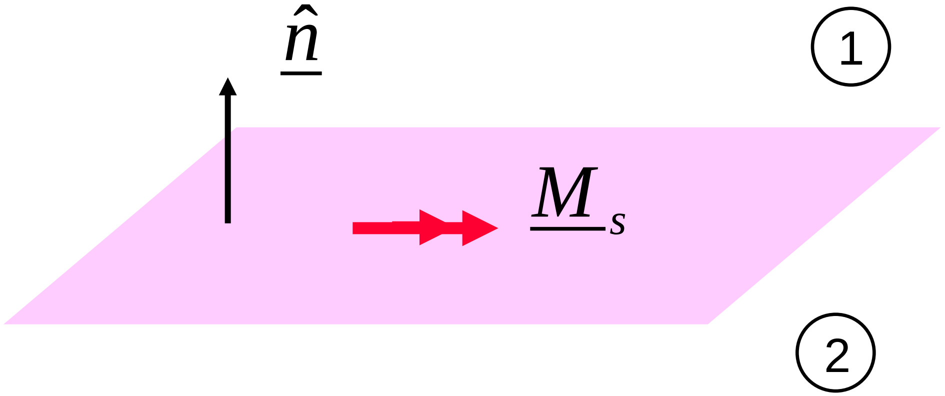
Duality:



$$-\hat{\underline{n}} \times \left(\underline{E}^{(1)} - \underline{E}^{(2)} \right) = \underline{M}_s$$

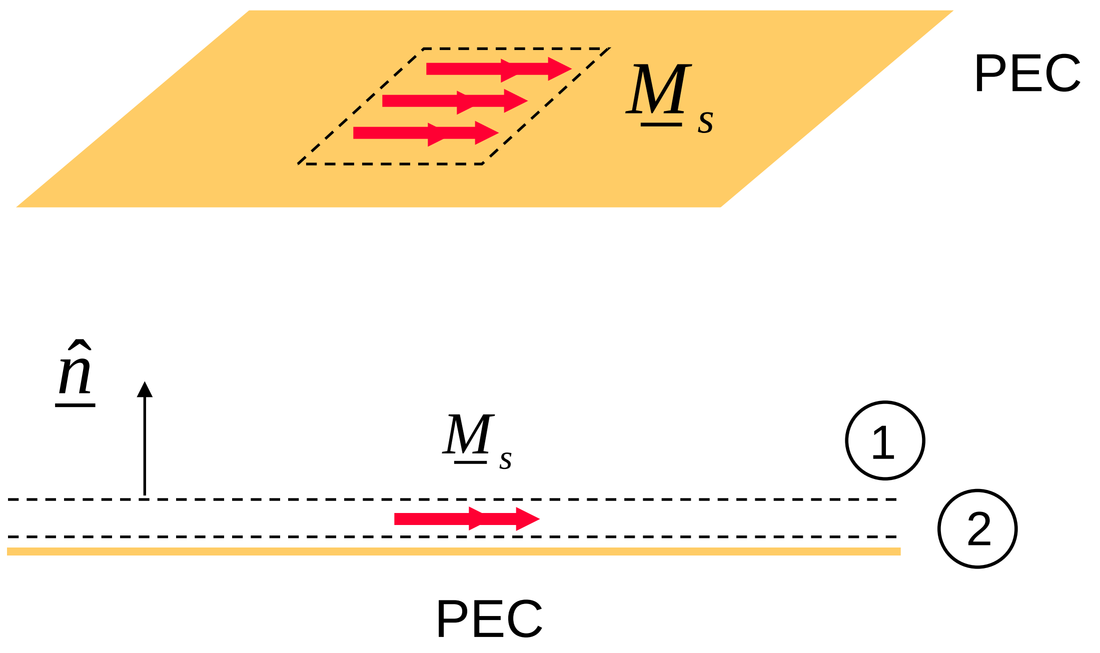
Boundary Conditions (cont.)

$$\underline{\hat{n}} \times \left(\underline{E}^{(1)} - \underline{E}^{(2)} \right) = -\underline{M}_s$$



Example: Magnetic Current on PEC

A magnetic surface current flows on top of a PEC.



$$\underline{\hat{n}} \times \left(\underline{E}^{(1)} - \underline{E}^{(2)} \right) = -\underline{M}_s$$

Example (cont.)

$$\underline{\hat{n}} \times \underline{E}^{(1)} = -\underline{M}_s \quad \Rightarrow \quad \underline{\hat{n}} \times \left(\underline{\hat{n}} \times \underline{E}_t^{(1)} \right) = -\underline{\hat{n}} \times \underline{M}_s$$

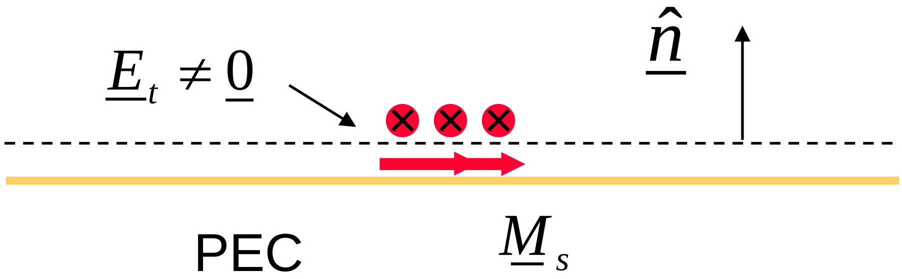
or

$$\underline{E}_t^{(1)} = \underline{\hat{n}} \times \underline{M}_s$$

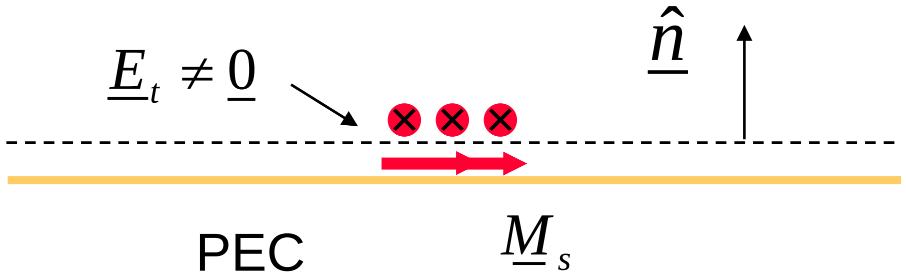
Dropping the superscript, we have

$$\underline{E}_t = \underline{\hat{n}} \times \underline{M}_s$$

Note:
The relation between the direction of the electric field and the direction of the magnetic current obeys a "left-hand rule".



Example (cont.)



$$\underline{E}_t = \hat{\underline{n}} \times \underline{M}_s$$

We then have

$$\hat{\underline{n}} \times \underline{E}_t = \hat{\underline{n}} \times \left(\hat{\underline{n}} \times \underline{M}_s \right)$$

$$\Rightarrow \quad \hat{\underline{n}} \times \underline{E}_t = -\underline{M}_s$$

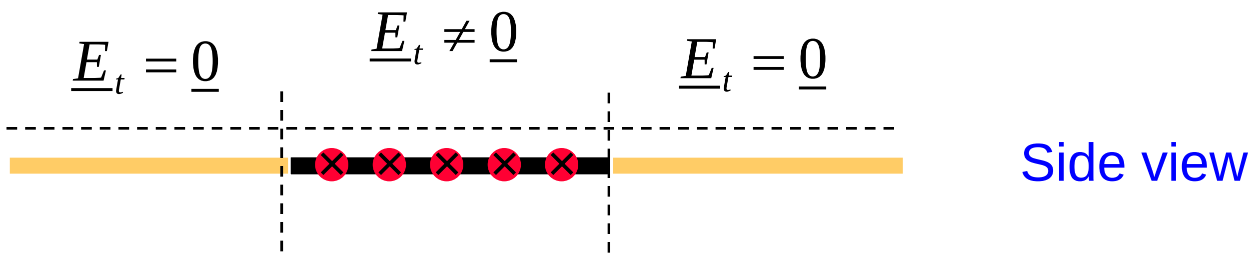
Note: The tangential subscript can be dropped.

Hence

$$\underline{M}_s = -\hat{\underline{n}} \times \underline{E}$$

This allows us to find the magnetic surface current necessary to produce any desired tangential electric field.

Modeling of Slot Antenna Problem



$$\underline{M}_s = -\underline{\hat{n}} \times \underline{E}$$