ECE 6340
Intermediate EM Waves

Fall 2016

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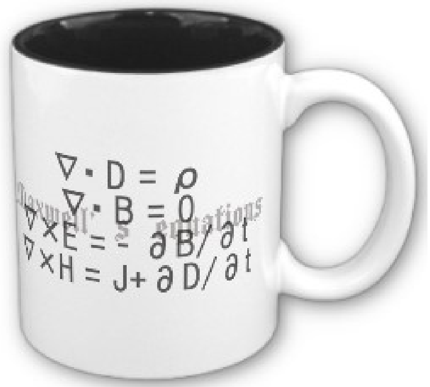


Notes 24

Uniqueness Theorem

- Shows what B.C.'s are necessary to uniquely determine fields.
- Justifies image theory and the equivalence principle.

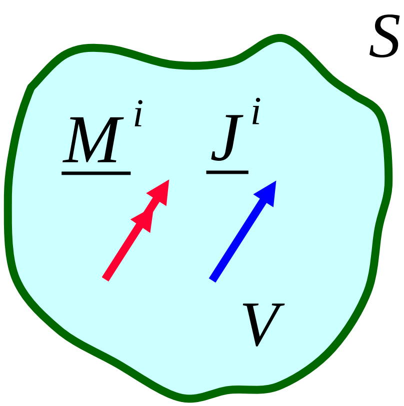
Theorem

Assume:

(1) Sources ($\underline{J}^i$, $\underline{M}^i$) are specified in V.

(2) $\underline{E}_t$ or $\underline{H}_t$ is specified on S†.

(3) Region V is slightly lossy.



Then ($\underline{E}$, $\underline{H}$) is unique inside V.

†It is allowed to specify $\underline{E}_t$ on one part and $\underline{H}_t$ on the other part.

Uniqueness Theorem (cont.)

Proof

Assume different two solutions that have the same sources and tangential field ($\underline{E}_t$ or $\underline{H}_t$) on S:

$$(\underline{E}^a, \underline{H}^a) \text{ and } (\underline{E}^b, \underline{H}^b)$$

$$\nabla \times \underline{E}^a = -j\omega\mu \underline{H}^a - \underline{M}^i$$

$$\nabla \times \underline{E}^b = -j\omega\mu \underline{H}^b - \underline{M}^i$$

Subtract:

$$\nabla \times \left(\underline{E}^a - \underline{E}^b \right) = -j\omega\mu \left(\underline{H}^a - \underline{H}^b \right)$$

Uniqueness Theorem (cont.)

Let

$$\Delta \underline{E} \equiv \underline{E}^{a} - \underline{E}^{b}$$

$$\Delta \underline{H} \equiv \underline{H}^{a} - \underline{H}^{b}$$

Then

$$\nabla \times \left(\Delta \underline{E} \right) = -j\omega\mu \left(\Delta \underline{H} \right)$$

Similarly (from Ampere's law),

$$\nabla \times \left(\Delta \underline{H} \right) = j\omega\varepsilon_{c} \left(\Delta \underline{E} \right)$$

Note that these are <u>source-free</u> equations.

Uniqueness Theorem (cont.)

Now use the complex Poynting theorem:

$$\frac{1}{2}\oint\left(\underline{E}\times\underline{H}^{*}\right)\cdot\underline{\hat{n}}\,dS$$

$$+2\omega\int_{V}\left(\frac{1}{4}\mu''\left|\underline{H}\right|^{2}+\frac{1}{4}\varepsilon''\left|\underline{E}\right|^{2}\right)dV$$

$$+2j\omega\int_{V}\left(\frac{1}{4}\mu\left|\underline{H}\right|^{2}-\frac{1}{4}\varepsilon'\left|\underline{E}\right|^{2}\right)dV=\int_{V}-\frac{1}{2}\left(\underline{E}\cdot\underline{J}^{i*}+\underline{H}^{*}\cdot\underline{M}^{i}\right)dV$$

with

$$\underline{E}\rightarrow\Delta\underline{E}\qquad\underline{H}\rightarrow\Delta\underline{H}$$

$$\underline{J}^{i}\rightarrow\Delta\underline{J}^{i}=\underline{0}\qquad\underline{M}^{i}\rightarrow\Delta\underline{M}^{i}=\underline{0}$$

Uniqueness Theorem (cont.)

Hence, we have

$$\frac{1}{2}\oint\left(\Delta\underline{E}\times\Delta\underline{H}^{*}\right)\cdot\underline{\hat{n}}\,dS$$

$$+2\omega\int_V\left(\frac{1}{4}\mu''\left|\Delta\underline{H}\right|^2+\frac{1}{4}\varepsilon''\left|\Delta\underline{E}\right|^2\right)dV$$

$$+2\,j\omega\int_V\left(\frac{1}{4}\mu'\left|\Delta\underline{H}\right|^2-\frac{1}{4}\varepsilon'\left|\Delta\underline{E}\right|^2\right)dV\;=0$$

Next, examine the first term.

Uniqueness Theorem (cont.)

On S,

$$(\Delta \underline{E}) \times (\Delta \underline{H})^{*} \cdot \underline{\hat{n}} = (\Delta \underline{E}_t) \times (\Delta \underline{H}_t)^{*} \cdot \underline{\hat{n}} = 0$$

This follows since

$$\Delta \underline{E}_t = 0 \quad \text{or} \quad \Delta \underline{H}_t = 0 \quad \text{on } S.$$

Uniqueness Theorem (cont.)

Hence

$$+2\omega\int_V\left(\frac{1}{4}\mu''\left|\Delta\underline{H}\right|^2+\frac{1}{4}\varepsilon''\left|\Delta\underline{E}\right|^2\right)dV$$

$$+2j\omega\int_V\left(\frac{1}{4}\mu'\left|\Delta\underline{H}\right|^2-\frac{1}{4}\varepsilon'\left|\Delta\underline{E}\right|^2\right)dV=0$$

Set the real and imaginary parts to zero:

$$\int_V\left(\frac{1}{4}\mu''\left|\Delta\underline{H}\right|^2+\frac{1}{4}\varepsilon''\left|\Delta\underline{E}\right|^2\right)dV=0$$

and

$$\int_V\left(\frac{1}{4}\mu'\left|\Delta\underline{H}\right|^2-\frac{1}{4}\varepsilon'\left|\Delta\underline{E}\right|^2\right)dV=0$$

Uniqueness Theorem (cont.)

Examine the real part:

$$\int_V \left(\mu'' \left| \Delta \underline{H} \right|^2 + \varepsilon'' \left| \Delta \underline{E} \right|^2 \right) dV = 0$$

Assume

$$\varepsilon'' > 0 \quad \text{in } V$$

or

$$\mu'' > 0 \quad \text{in } V$$

Then

$$\Delta \underline{E} = \underline{0} \quad \text{in } V$$

or

$$\Delta \underline{H} = \underline{0} \quad \text{in } V$$

Uniqueness Theorem (cont.)

In either case, from Maxwell's equations we have that both must be zero:

$$\Delta \underline{E} = \underline{0} \quad \text{in } V$$

and

$$\Delta \underline{H} = \underline{0} \quad \text{in } V$$

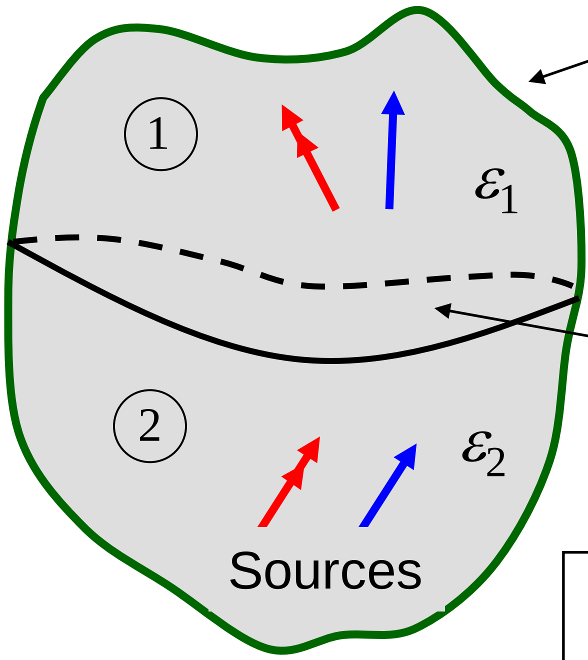
Hence

$$\underline{E}^{a} = \underline{E}^{b}$$

$$\underline{H}^{a} = \underline{H}^{b}$$

Generalization

Two regions with an interface:



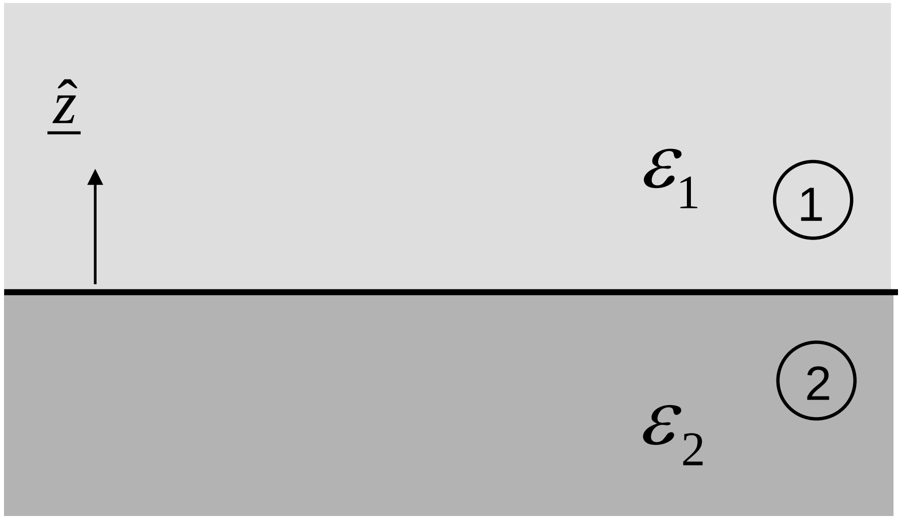
$\underline{E}_t$ or $\underline{H}_t$ is specified on S

$\underline{E}_t$ and $\underline{H}_t$ continuous on The interface

Note:
If there are surface currents on the interface, then we require the appropriate boundary conditions on the interface to be satisfied.

Generalization (cont.)

Note that D_n and B_n are <u>automatically continuous</u> at the boundary if the tangential fields are.



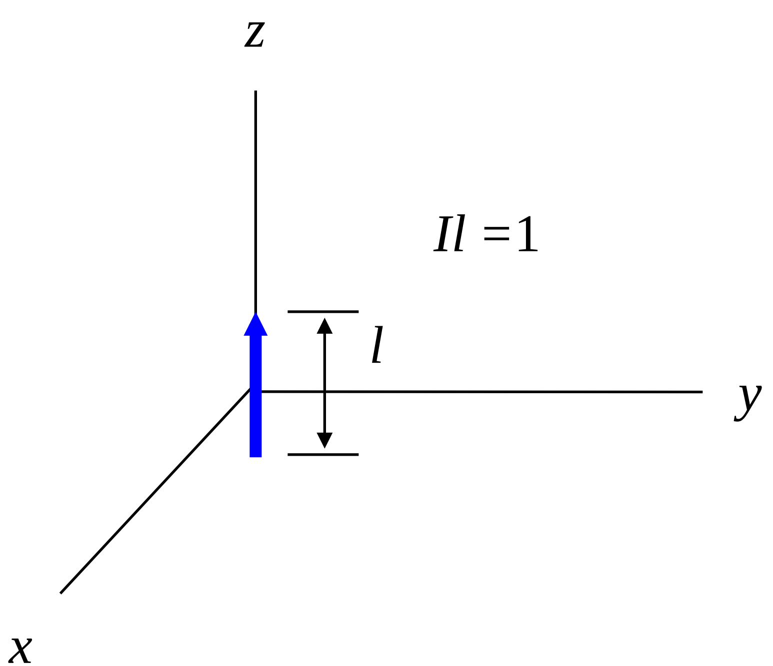
For example: $\nabla \times \underline{H} = j\omega \underline{D}$

$$D_z = \frac{1}{j\omega}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]$$

Example

Infinitesimal dipole:

$$\nabla^2 A_z + k^2 A_z = -\mu\,\delta(\underline{r})$$



Two possible solutions:

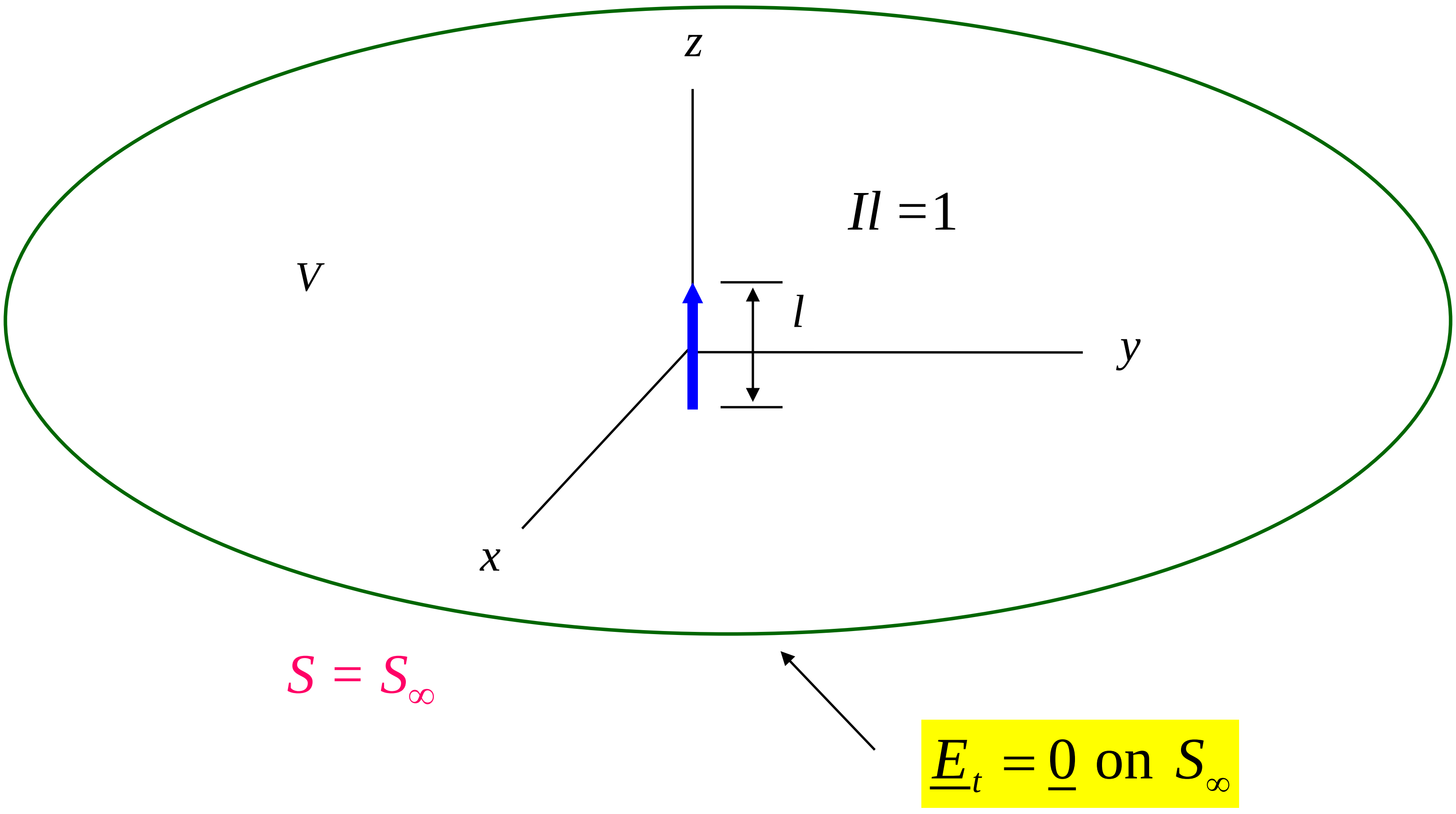
$$A_z^+ = \left(\frac{\mu}{4\pi}\right)\frac{e^{-jkr}}{r} \qquad A_z^- = \left(\frac{\mu}{4\pi}\right)\frac{e^{+jkr}}{r}$$

Both satisfy the Helmholtz equation for A_z for the given source.

Example (cont.)

We impose a B.C. at infinity for $\underline{E}_t$ or $\underline{H}_t$.

Assumption: The correct solution goes to zero at infinity.



Example (cont.)

Both solutions tend to zero at infinity if there is no loss
(we cannot tell which one is correct).

Assume $k = k' - jk'', \;\; k'' > 0$

The solution should now be unique.
(The requirements of the uniqueness theorem are satisfied, which includes having a small amount of loss.)

We then have

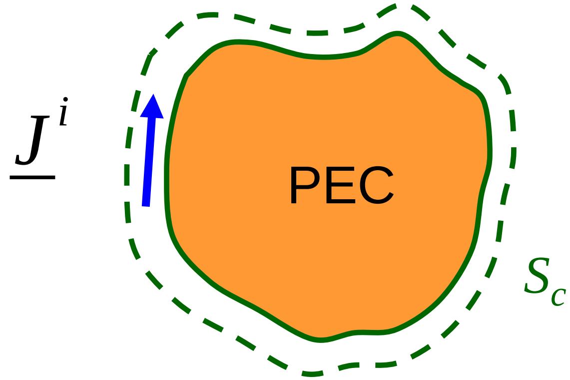
$$\left|A_z^{+}\right| = \left(\frac{\mu}{4\pi}\right)\frac{e^{-k''r}}{r} \rightarrow 0$$

$$\left|A_z^{-}\right| = \left(\frac{\mu}{4\pi}\right)\frac{e^{+k''r}}{r} \rightarrow \infty \quad \text{as} \quad r \rightarrow \infty$$

The correct choice is then $A_z^{+} = \left(\frac{\mu}{4\pi}\right)\frac{e^{-jkr}}{r}$

Example

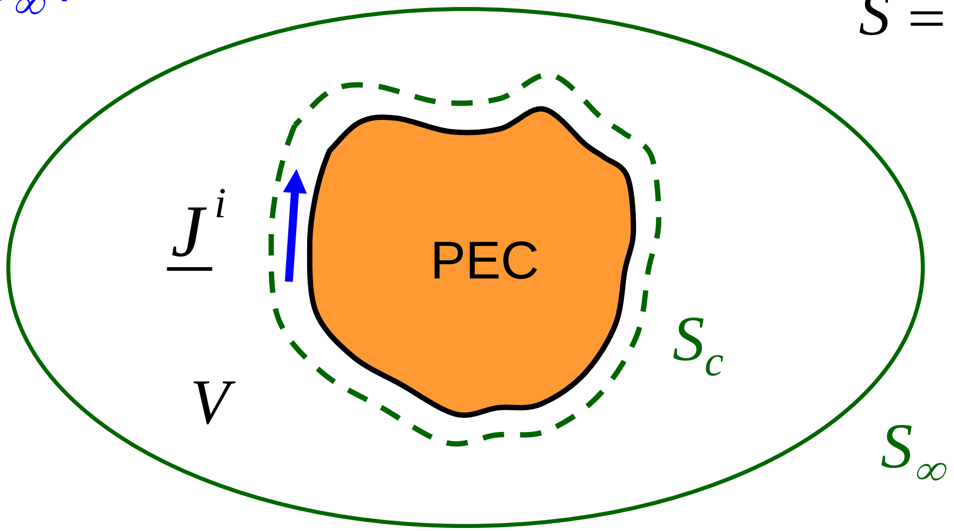
Current source tangent to (and just outside) PEC body.



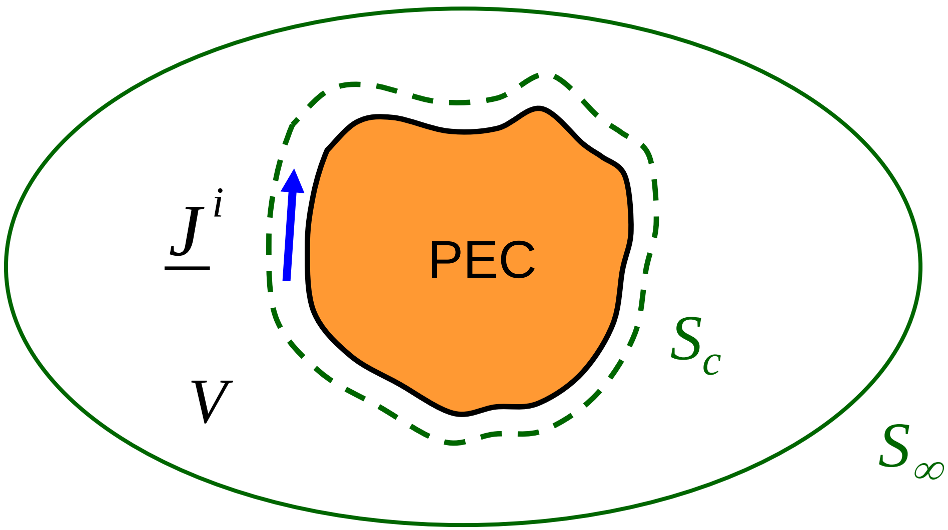
$$\underline{E}_t\left[\underline{J}^i\right]=\underline{0} \;\text{ on }\; S_c$$

Surround S_c by S_∞ :

$$S = S_c \cup S_\infty$$



Example (cont.)



√ $\underline{E}_t = \underline{0}$ on $S = S_c \cup S_\infty$

√ The sources inside V are specified (no sources in V).

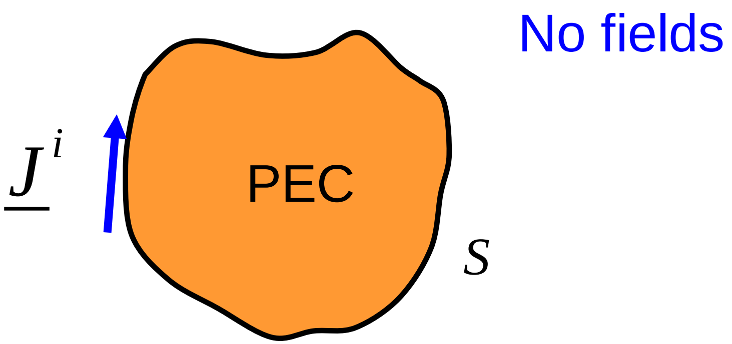
Hence, ($\underline{E}$, $\underline{H}$) are unique inside V.

$$\left(\underline{E}, \underline{H}\right) = \left(\underline{0}, \underline{0}\right) \text{ inside } V$$

This satisfies the source condition and the B.C.s, so it must be the correct solution.

Example (cont.)

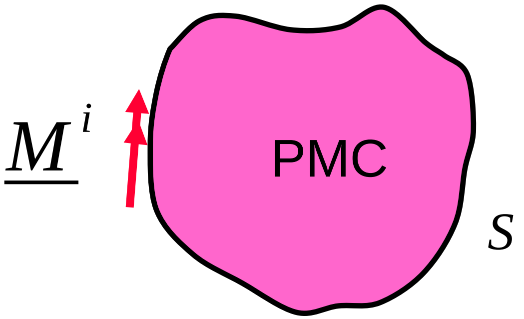
Conclusion:



An electric current tangent to a PEC body does not radiate.

Example (cont.)

Similarly, we can examine a tangential magnetic current on a PMC body.



A magnetic current tangent to a PMC body does not radiate.

Sommerfeld Radiation Condition

This is a more "powerful" boundary condition at <u>infinity</u> that does not require the medium to be lossy.

Let $\psi = A_x, A_y, A_z, E_x, E_y, E_z,$ etc.

Assume that $\nabla^2\psi + k^2\psi = S(\underline{r})$

Then ψ is <u>unique</u> if:

$$(1)\quad \lim_{r\to\infty} \psi(\underline{r}) = 0$$

$$(2)\quad \lim_{r\to\infty} r\left[\frac{\partial\psi}{\partial r} + jk\psi\right] = 0$$

Example

Use

$$\psi = \psi^{+} = \frac{e^{-jkr}}{r}$$

$$r\left[\frac{\partial \psi}{\partial r} + jk\psi\right] = r\left[\left(-jk - \frac{1}{r}\right)\frac{e^{-jkr}}{r} + jk\left(\frac{e^{-jkr}}{r}\right)\right] = r\left[-\frac{1}{r^2}e^{-jkr}\right]$$

$$= -\frac{1}{r}e^{-jkr}$$

$$\to 0 \quad \text{as} \quad r \to \infty$$

The function ψ+ satisfies the Sommerfeld radiation condition at infinity.

Example (cont.)

Now use

$$\psi = \psi^{-} = \frac{e^{+jkr}}{r}$$

$$r\left[\frac{\partial \psi}{\partial r} + jk\psi\right] = r\left[\left(+jk - \frac{1}{r}\right)\frac{e^{+jkr}}{r} + jk\left(\frac{e^{+jkr}}{r}\right)\right]$$

$$= 2jke^{+jkr} - \frac{1}{r}e^{+jkr}$$

$$\not\to 0 \quad \text{as} \quad r \to \infty$$

The function ψ- does <u>not satisfy</u> the Sommerfeld radiation condition at infinity.