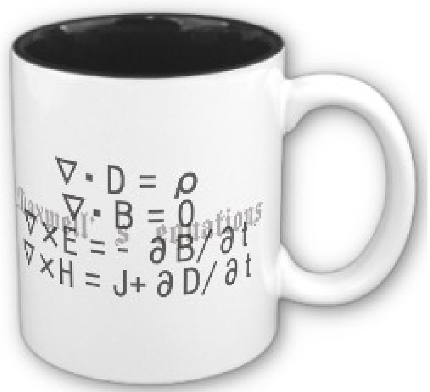
ECE 6340
Intermediate EM Waves

Fall 2016

Prof. David R. Jackson
Dept. of ECE

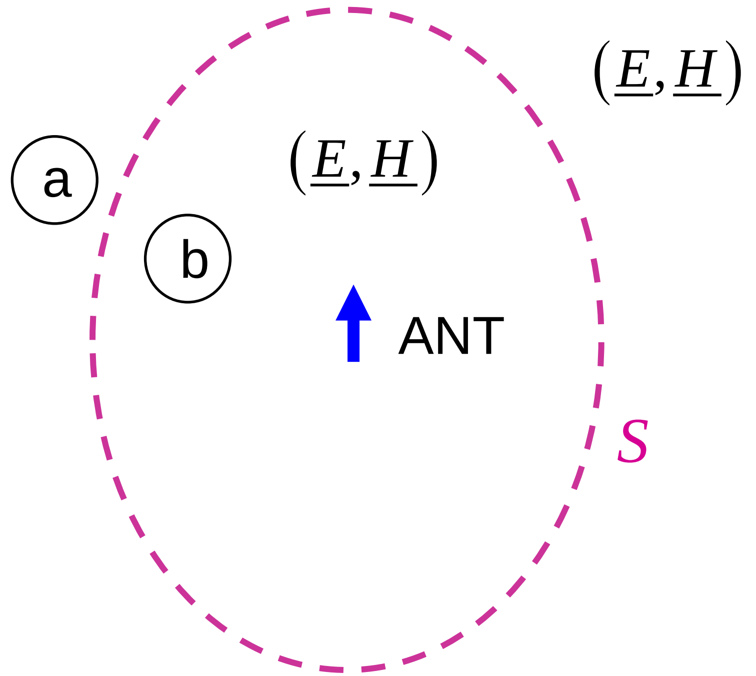


Notes 26

Equivalence Principle

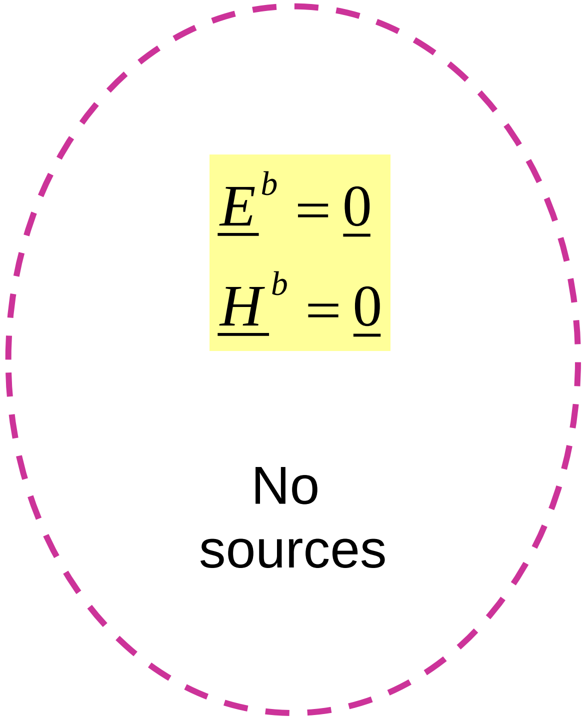
Basic idea:

We can replace the actual sources in a region by equivalent sources at the boundary of a closed surface.



- Keep original fields $\underline{E}$, $\underline{H}$ outside S.
- Put zero fields (and no sources) inside S.

Equivalence Principle (cont.)



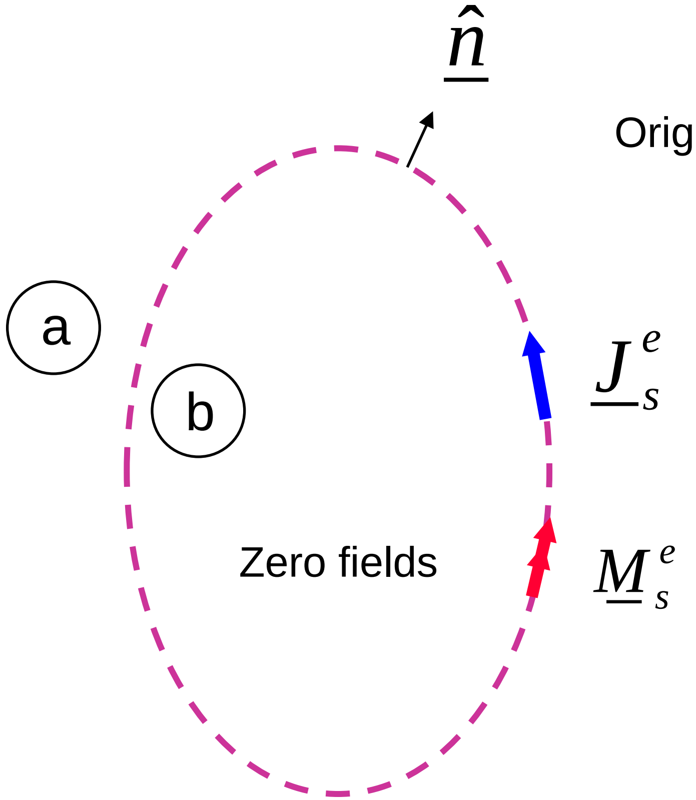
$$\underline{E}^b = \underline{0}$$
$$\underline{H}^b = \underline{0}$$

$$\underline{E}^a = \underline{E}$$
$$\underline{H}^a = \underline{H}$$

Note: ($\underline{E}^a$, $\underline{H}^a$) and ($\underline{E}^b$, $\underline{H}^b$) both satisfy Maxwell's equations.

Equivalence Principle (cont.)

- The B.C.'s on S are violated.
- Introduce equivalent sources on the boundary to make B.C.'s valid.



$$\underline{J}_s^e = \underline{\hat{n}} \times \left(\underline{H}^a - \underline{H}^b \right)$$

$$\underline{M}_s^e = -\underline{\hat{n}} \times \left(\underline{E}^a - \underline{E}^b \right)$$

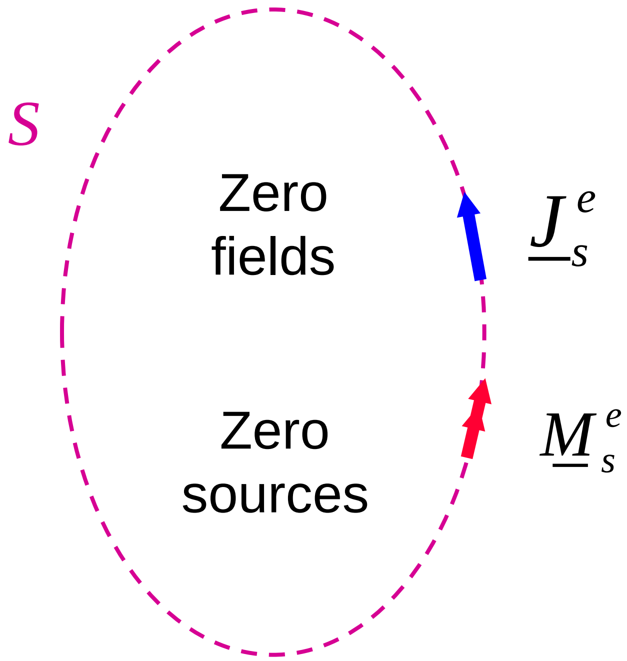
Equivalence Principle (cont.)

Hence

$$\underline{J}_s^e = \underline{\hat{n}} \times \underline{H}$$

$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

Equivalent sources:



Outside S, these sources radiate the same fields as the original antenna, and produce zero fields inside S.

This is justified by the uniqueness theorem:

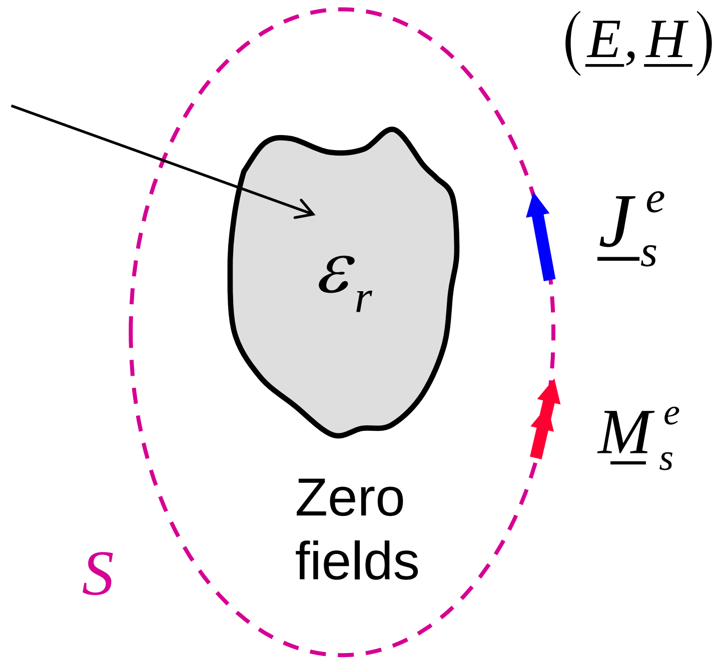
Maxwell's equations are satisfied along with boundary conditions at the interface.

Equivalence Principle (cont.)

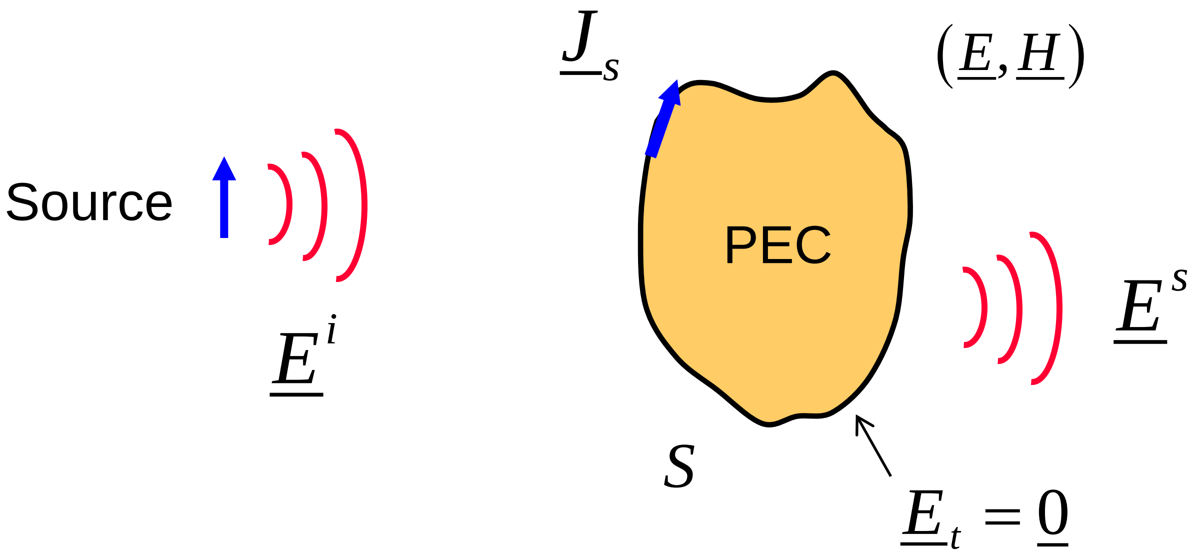
Note about materials:

If there are zero fields throughout a region, it doesn't matter what material is placed there (or removed).

This object can be added



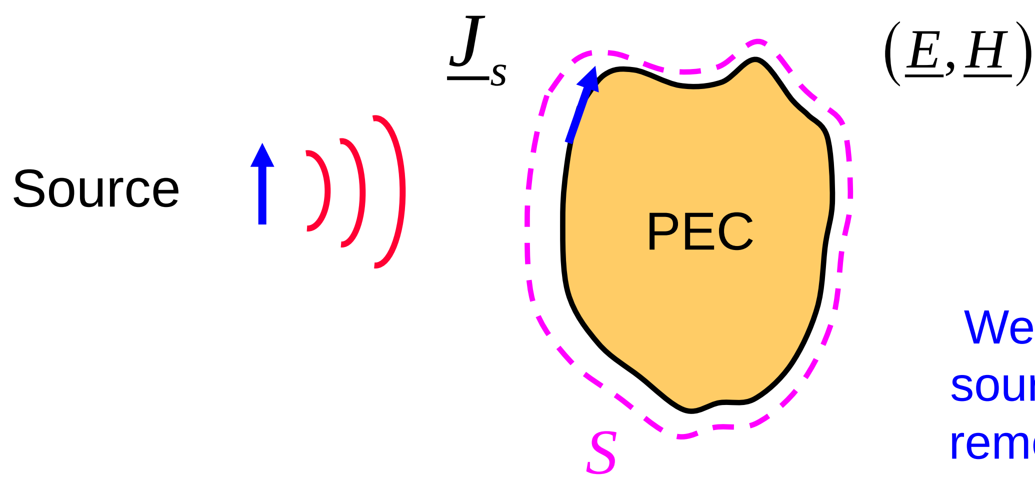
Scattering by a PEC



$$\underline{E} = \underline{E}^i + \underline{E}^s$$

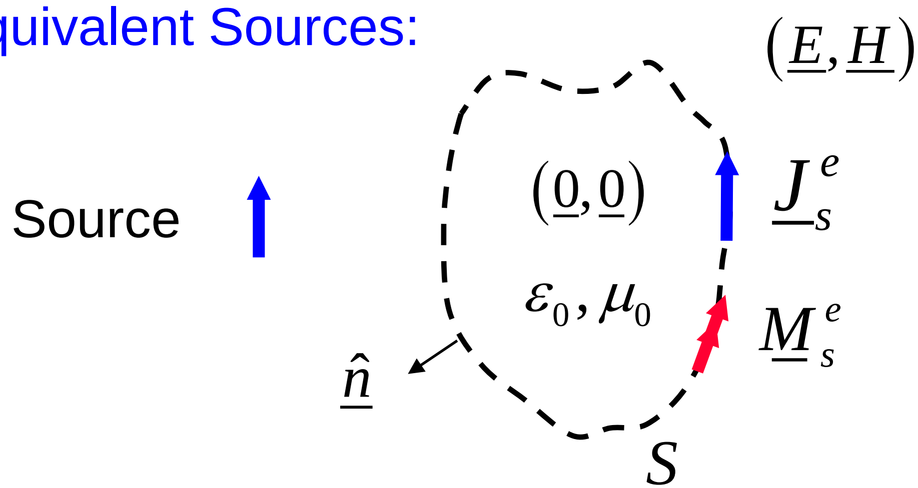
$$\underline{H} = \underline{H}^i + \underline{H}^s$$

Scattering by a PEC (cont.)



We put zero fields and sources inside of S, and remove the PEC object.

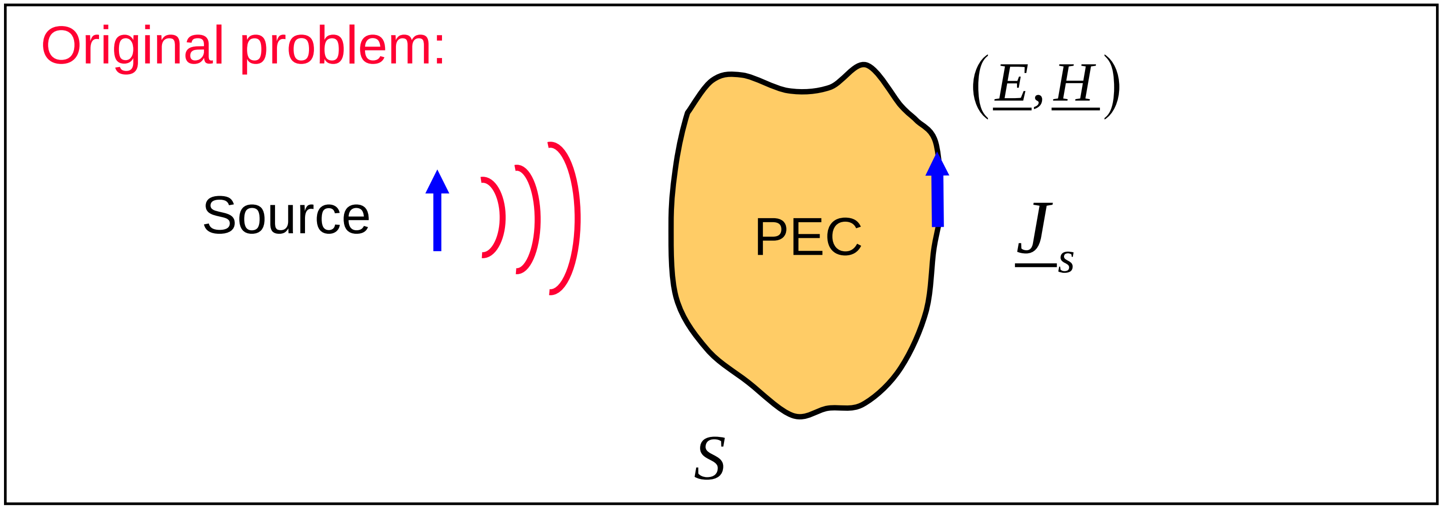
Equivalent Sources:

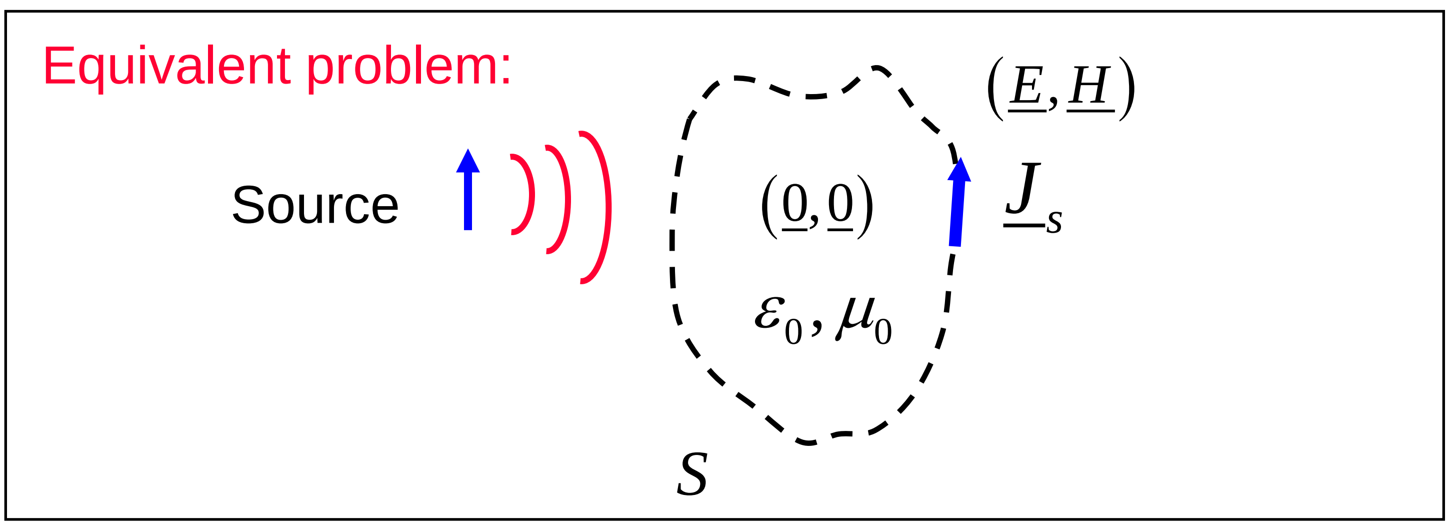


$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E} = -\underline{\hat{n}} \times \underline{E}_t = \underline{0}$$

$$\underline{J}_s^e = \underline{\hat{n}} \times \underline{H} = \underline{J}_s$$

Scattering by a PEC (cont.)

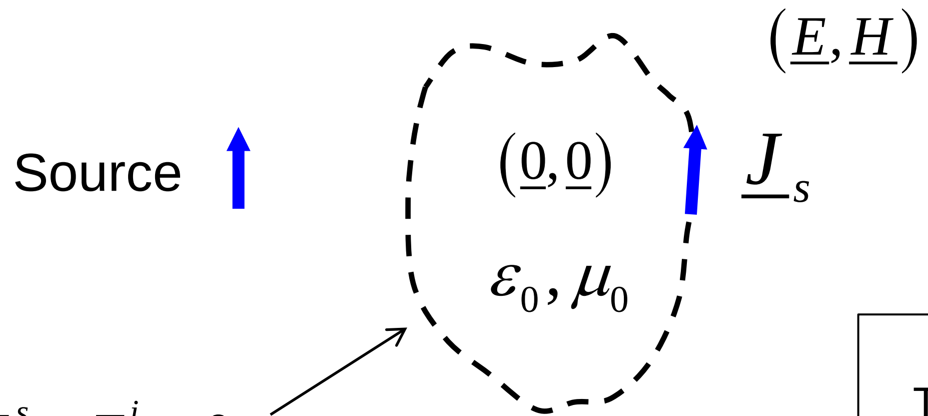




Conclusion: The conductor can be removed.

Scattering by a PEC (cont.)

Integral equation for the unknown current



$$\underline{E}_t^s + \underline{E}_t^i = \underline{0}$$

> **Note:**
> The bracket notation means "field due to a source".

$$\underline{E}_t\left[\,\underline{J}_s\,\right] + \underline{E}_t^i = \underline{0}, \quad \underline{r} \in S$$

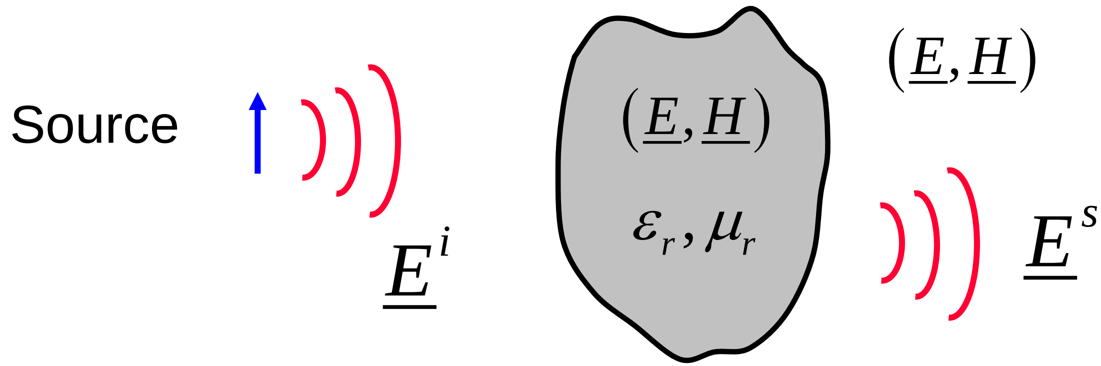
so

$$\underline{E}_t\left[\,\underline{J}_s\,\right] = -\underline{E}_t^i$$

"Electric Field Integral Equation" (EFIE)

This integral equation has to be solved numerically.

Scattering by Dielectric Body



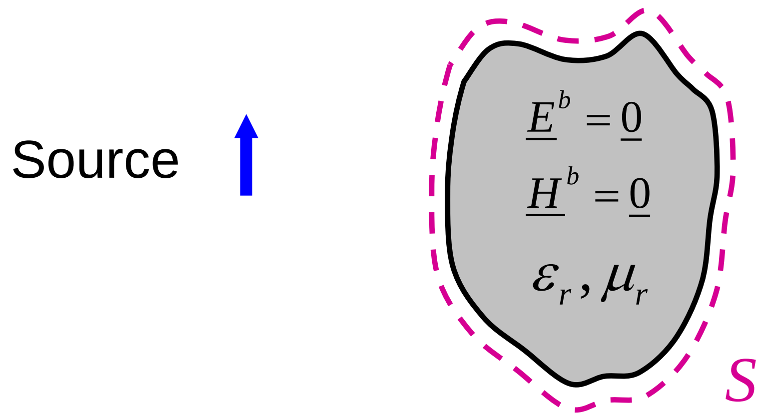
$$\underline{E} = \underline{E}^i + \underline{E}^s$$

$\underline{E}^i$ = incident field

$\underline{E}^s$ = scattered field

Note:
The body is assumed to be homogeneous.

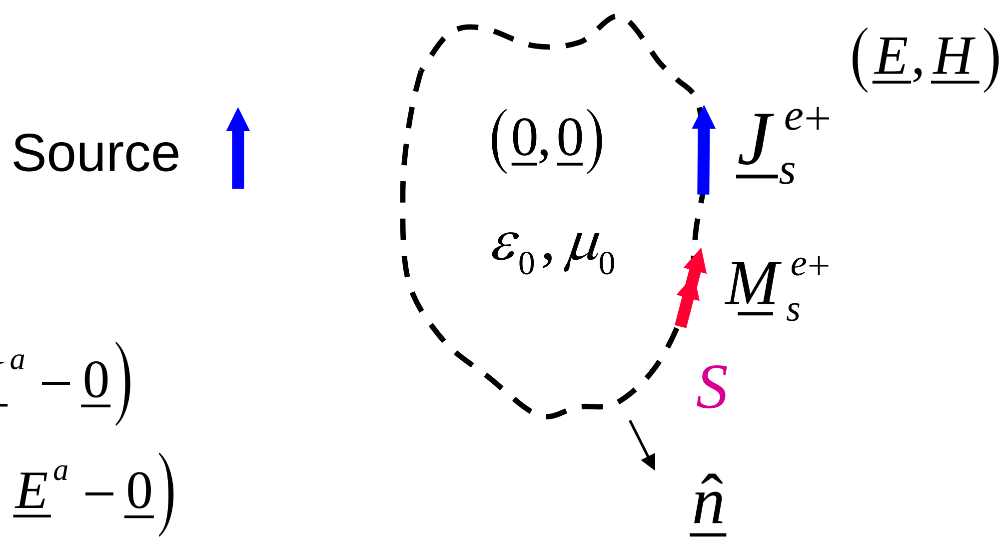
Exterior Equivalence



$$\underline{E}^a = \underline{E}$$
$$\underline{H}^a = \underline{H}$$

Replace body by free space

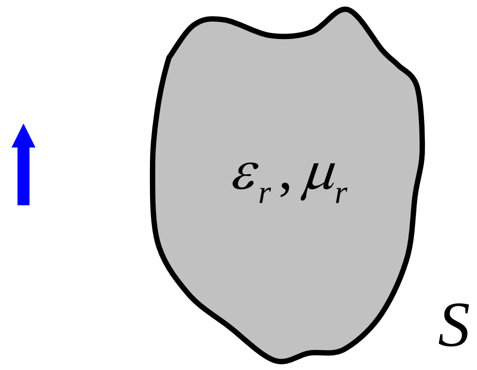
(The material doesn't matter in the zero-field region.)



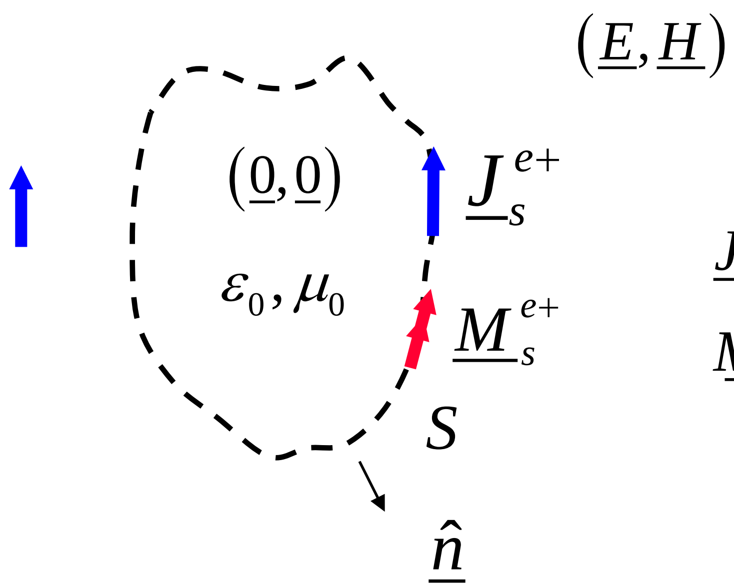
$$\underline{J}_s^{e+} = \underline{\hat{n}} \times \left(\underline{H}^a - \underline{0} \right)$$
$$\underline{M}_s^{e+} = -\underline{\hat{n}} \times \left(\underline{E}^a - \underline{0} \right)$$

Summary for Exterior

Original problem:



Free-space problem:



$$\underline{J}_s^{e+} = \underline{\hat{n}} \times \underline{H}^{+}$$

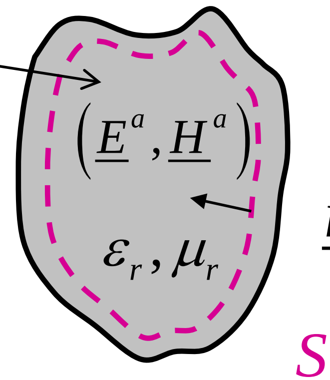
$$\underline{M}_s^{e+} = -\underline{\hat{n}} \times \underline{E}^{+}$$

Interior Equivalence

$$\underline{E}^a = \underline{E}$$

$$\underline{H}^a = \underline{H}$$

No sources or fields outside S



$$\underline{E}^b = \underline{0}$$

$$\underline{H}^b = \underline{0}$$

$$\underline{\hat{n}}' = -\underline{\hat{n}}$$

Place dielectric material in the "dead region" (region b).

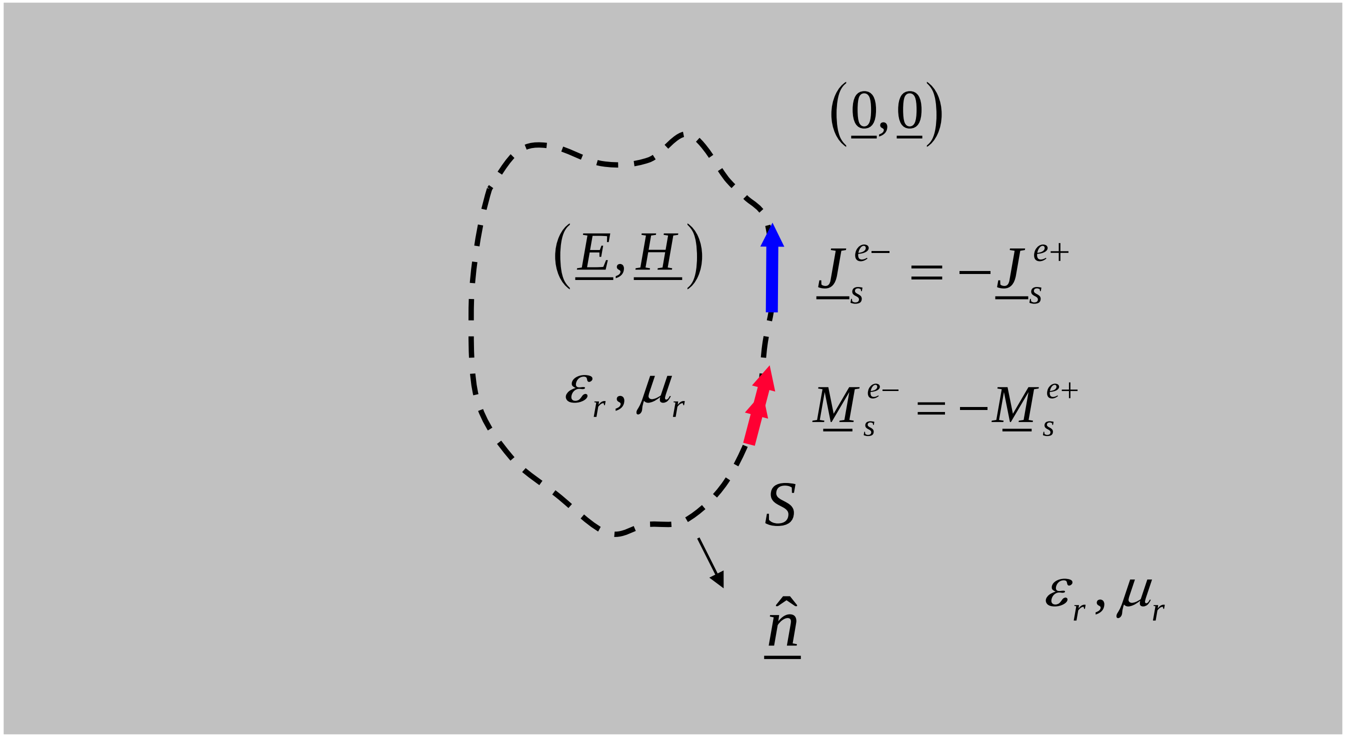
$$\underline{J}_s^{e-} = \left(-\underline{\hat{n}}\right) \times \left(\underline{H}^a - \underline{0}\right) = -\underline{\hat{n}} \times \underline{H}^- = -\underline{\hat{n}} \times \underline{H}^+ = -\underline{J}_s^{e+}$$

$$\underline{M}_s^{e-} = -\left(-\underline{\hat{n}}\right) \times \left(\underline{E}^a - \underline{0}\right) = \underline{\hat{n}} \times \underline{E}^- = \underline{\hat{n}} \times \underline{E}^+ = -\underline{M}_s^{e+}$$

$$\underline{J}_s^{e-} = -\underline{J}_s^{e+}$$

$$\underline{M}_s^{e-} = -\underline{M}_s^{e+}$$

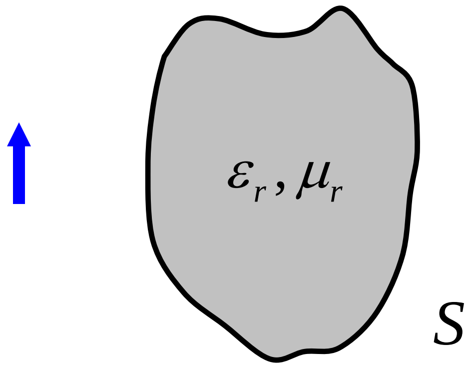
Interior Equivalence (cont.)



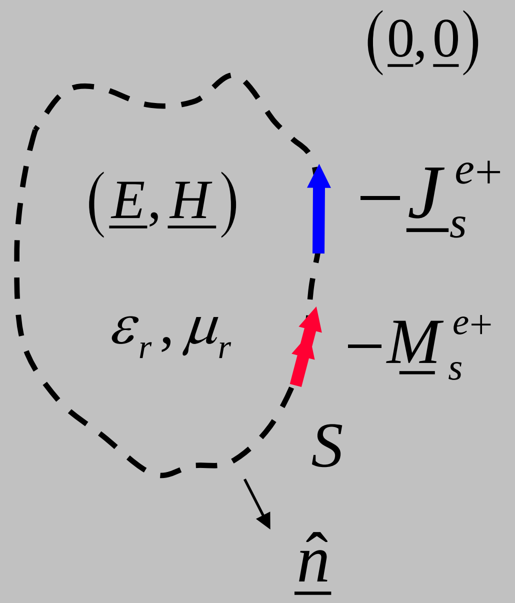
When we calculate the fields from these currents, we let them radiate in an *infinite dielectric medium*.

Summary for Interior

Original problem:

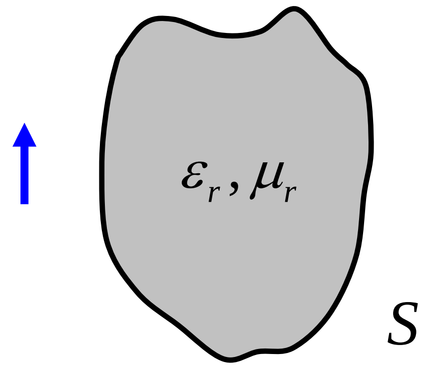


Homogeneous-medium problem:



Integral Equation

The "+" means calculate the fields just outside the surface, radiated by the sources in free space.



The "–" means calculate the fields just inside the surface, assuming an infinite dielectric region.

Boundary conditions:

$$\underline{E}_t^{+}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] + \underline{E}_t^{i} = \underline{E}_t^{-}\left[\underline{J}_s^{e-}, \underline{M}_s^{e-}\right]$$

$$\underline{H}_t^{+}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] + \underline{H}_t^{i} = \underline{H}_t^{-}\left[\underline{J}_s^{e-}, \underline{M}_s^{e-}\right]$$

Recall:

$$\underline{J}_s^{e-} = -\underline{J}_s^{e+}$$

$$\underline{M}_s^{e-} = -\underline{M}_s^{e+}$$

Hence:

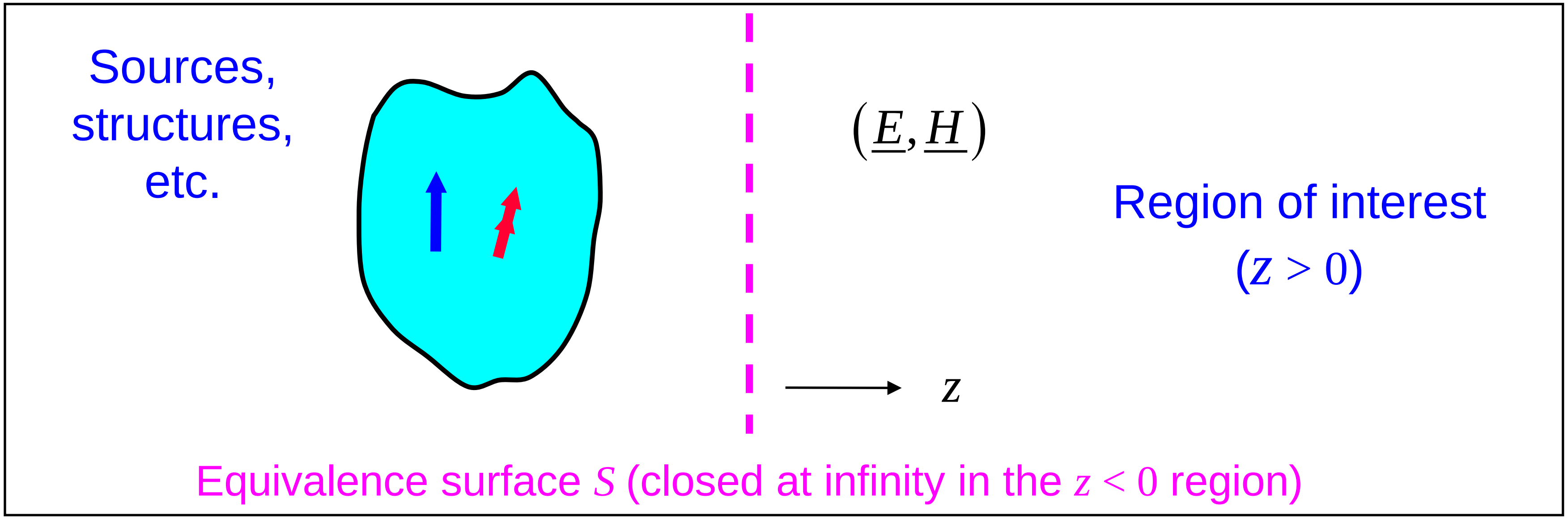
$$\underline{E}_t^{+}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] + \underline{E}_t^{-}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] = -\underline{E}_t^{i}$$

$$\underline{H}_t^{+}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] + \underline{H}_t^{-}\left[\underline{J}_s^{e+}, \underline{M}_s^{e+}\right] = -\underline{H}_t^{i}$$

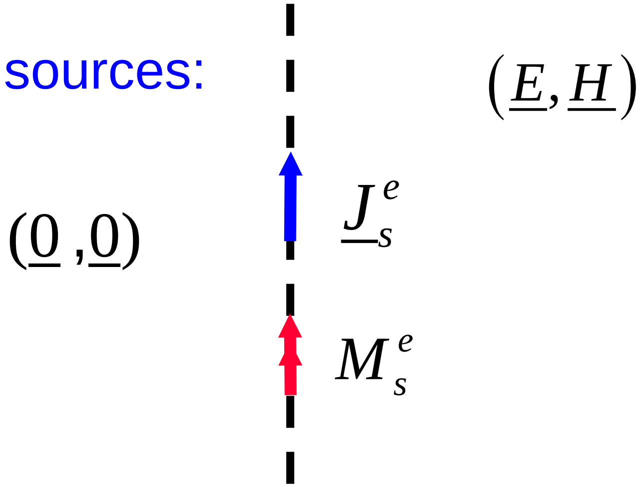
"PMCHWT" Integral Equation*

* Poggio-Miller-Chang-Harrington-Wu-Tsai

Fields in a Half Space



Equivalent sources:

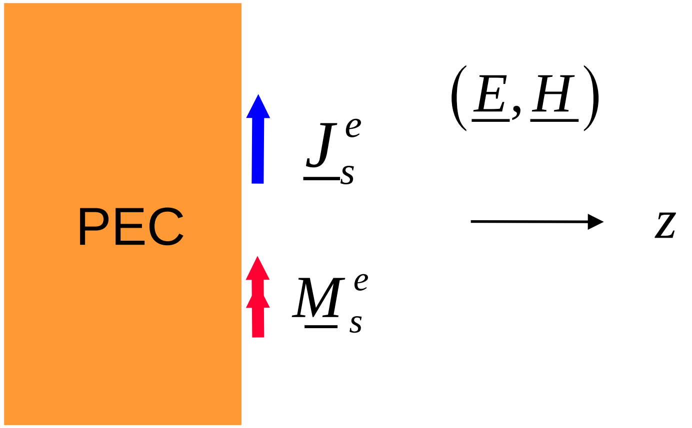


$$\underline{J}_s^e = \underline{\hat{z}} \times \underline{H}$$

$$\underline{M}_s^e = -\underline{\hat{z}} \times \underline{E}$$

Fields in a Half Space (cont.)

Put PEC in zero-field region:



$$(\underline{E}, \underline{H})$$

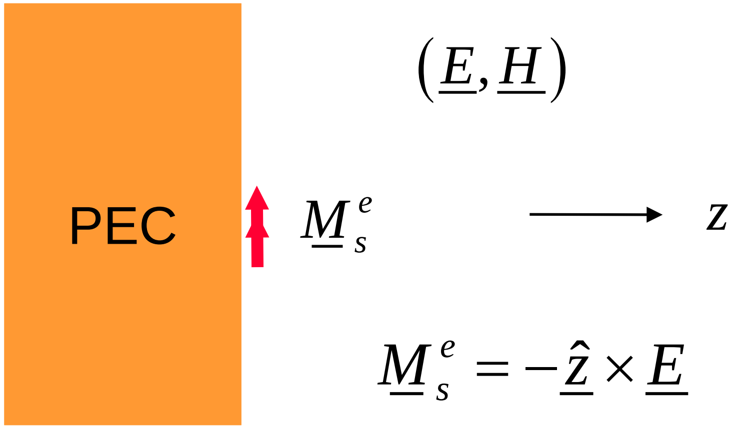
$\underline{J}_s^e$

$\underline{M}_s^e$

The electric surface current on the PEC does not radiate.

Note: The fields are only correct for $z > 0$.

Hence, we have:



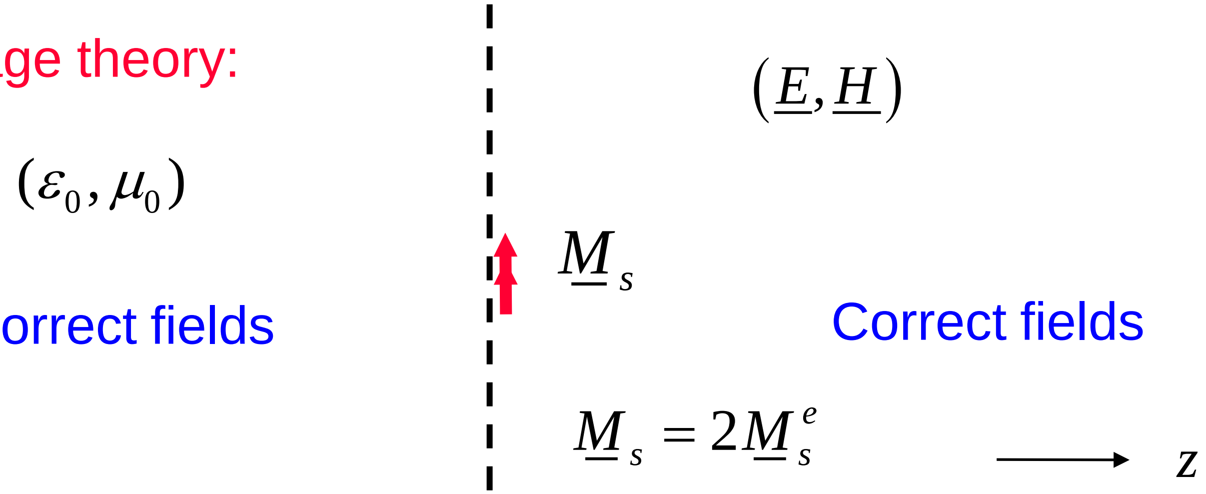
$$(\underline{E}, \underline{H})$$

$\underline{M}_s^e$

$$\underline{M}_s^e = -\underline{\hat{z}} \times \underline{E}$$

Fields in a Half Space (cont.)

Now use image theory:

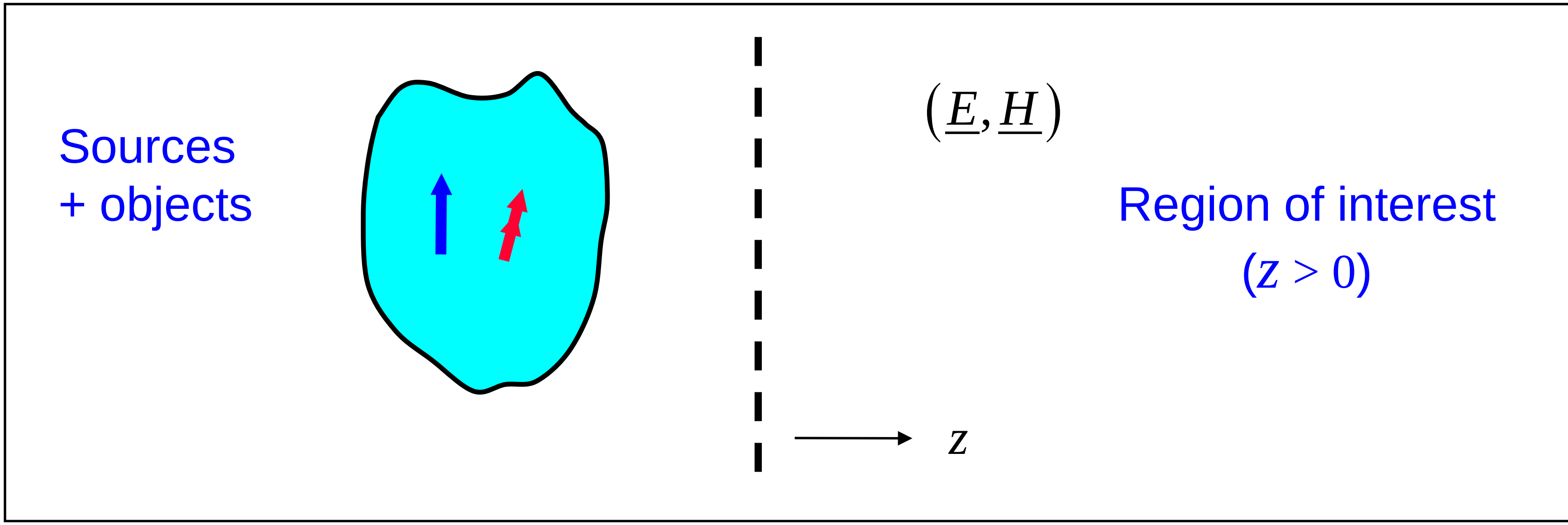


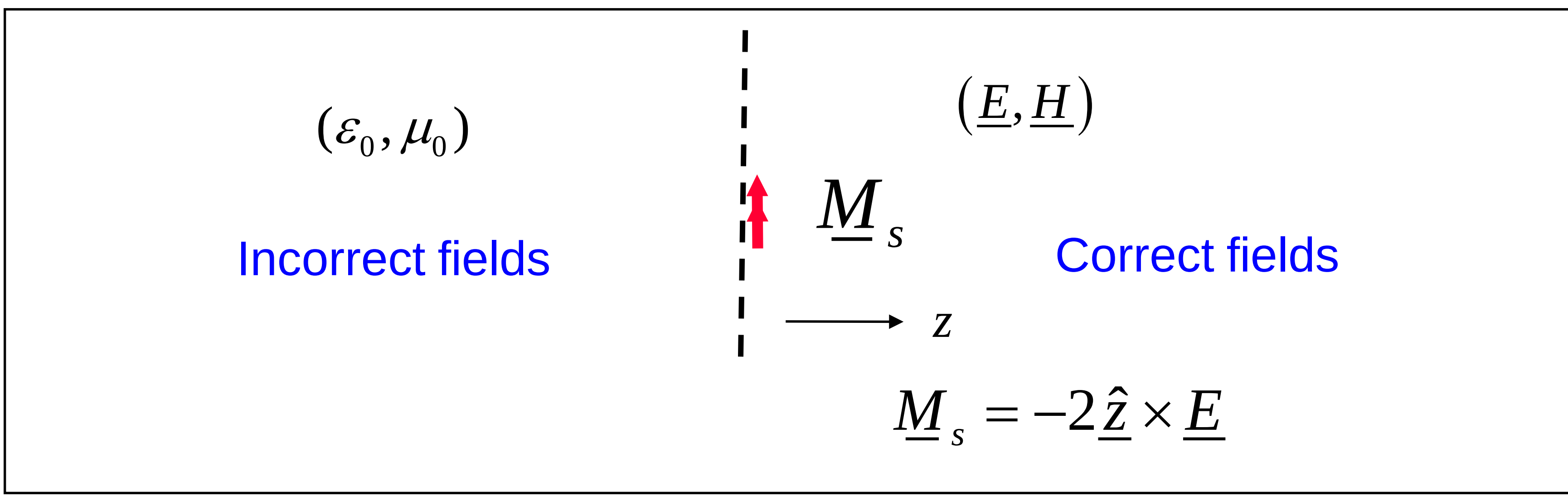
$$\underline{M}_s = -2\underline{\hat{z}} \times \underline{E}$$

Note: The fields are correct for $z > 0$.

This is useful whenever the electric field on the $z = 0$ plane is known.

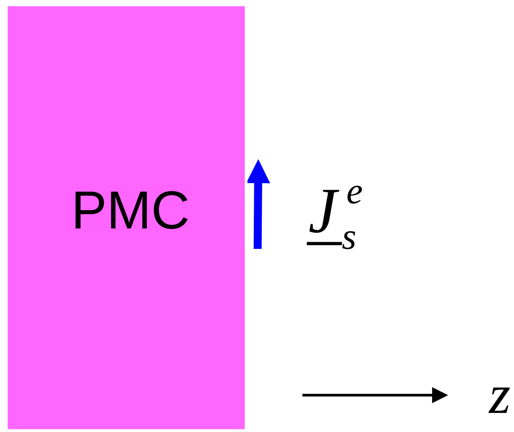
Fields in a Half Space: Summary





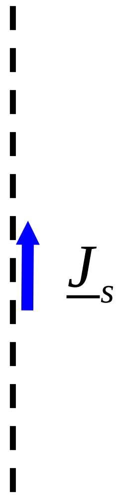
Fields in a Half Space (cont.)

Alternative (better when $\underline{H}$ is known on the interface):



The magnetic current does not radiate on PMC, and is therefore not included.

Image theory:



(ε_0, μ_0)

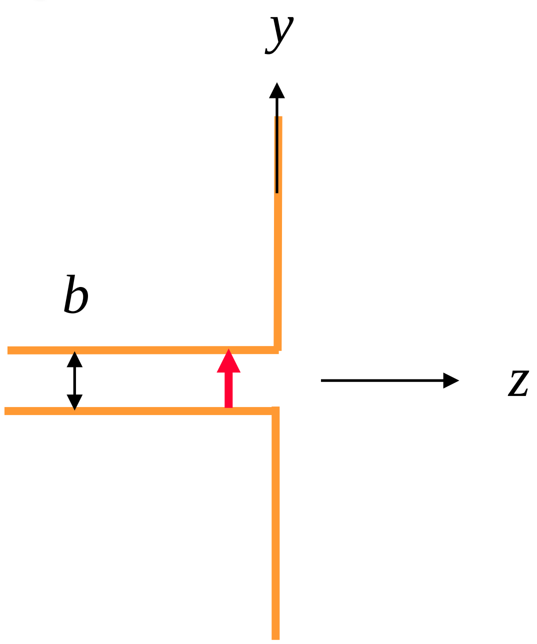
Incorrect fields

$(\underline{E}, \underline{H})$

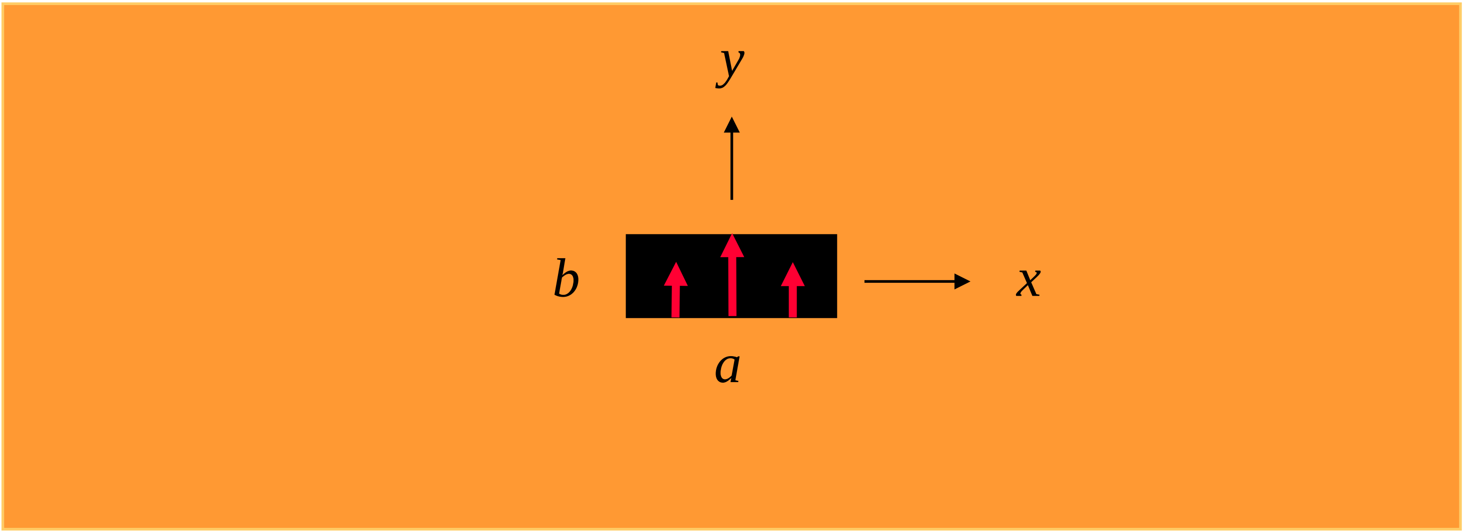
Correct fields

$$\underline{J}_s = 2\underline{J}_s^e = 2\underline{\hat{z}} \times \underline{H}$$

Example: Radiation from Waveguide



$$\underline{E}(x,y,0) = \underline{\hat{y}}\, E_0 \cos\left(\frac{\pi x}{a}\right)$$



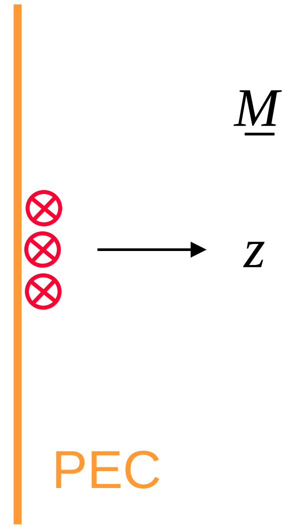
Example (cont.)

Step #1

Apply equivalence principle

The feeding waveguide was removed from the dead region that was created, and then an infinite PEC plane was introduced.

$$\underline{M}_s^e = -\hat{\underline{z}} \times \underline{E} = \hat{\underline{x}}\, E_0 \cos\left(\frac{\pi x}{a}\right)$$

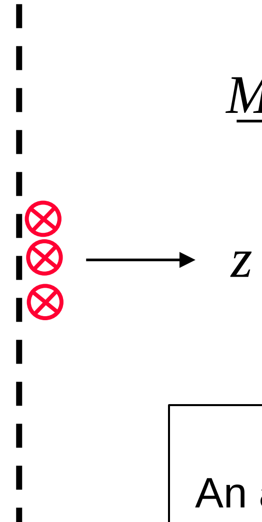


Step #2

Apply image theory

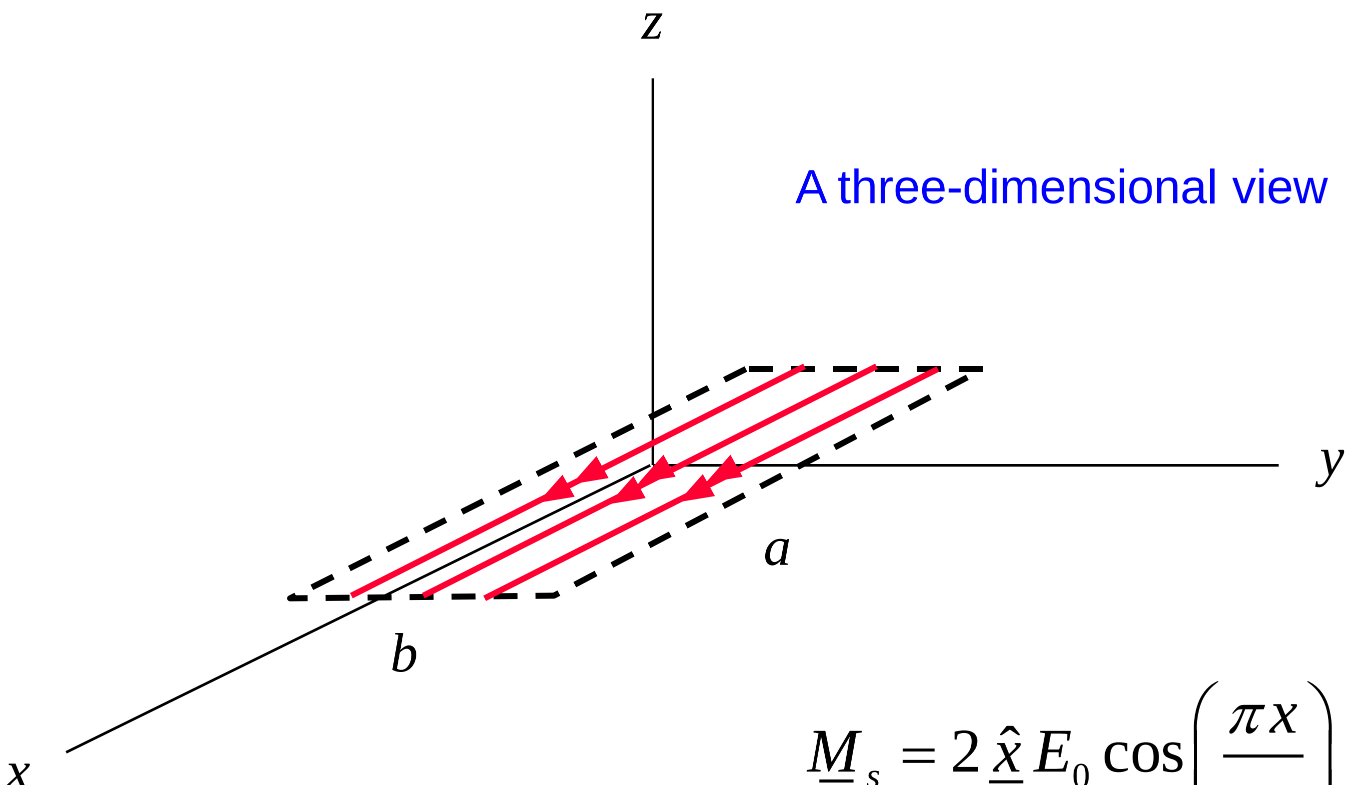
Image theory is applied to remove the ground plane and double the magnetic surface current.

$$\underline{M}_s = 2\, \hat{\underline{x}}\, E_0 \cos\left(\frac{\pi x}{a}\right)$$



Note:
An alternative way of getting to step 1 is to apply B.C.s for $\underline{M}_s$ on a PEC.

Example (cont.)



$$\underline{M}_s = 2\,\underline{\hat{x}}\,E_0 \cos\left(\frac{\pi x}{a}\right)$$

Example (cont.)

Step #3

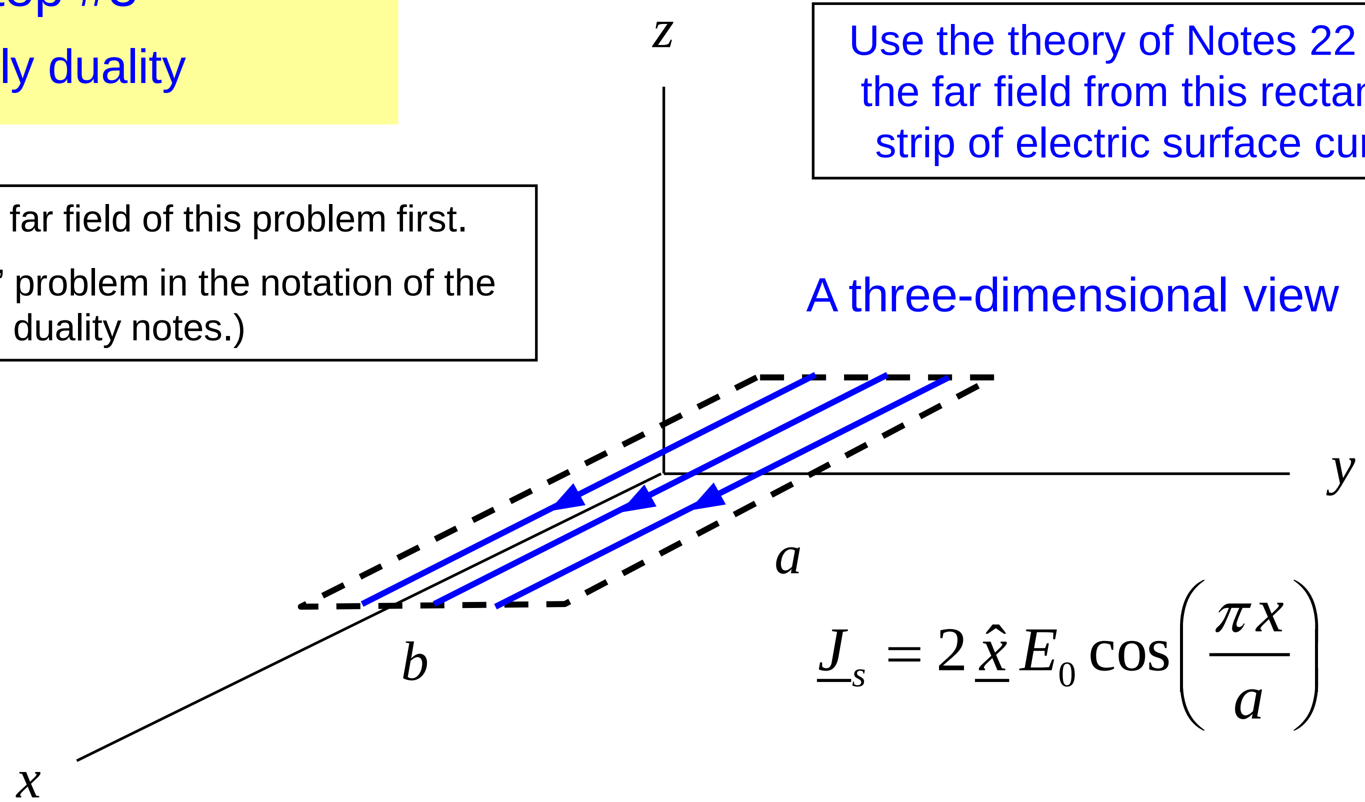
Apply duality

Solve for the far field of this problem first.

(This is the "A" problem in the notation of the duality notes.)

Use the theory of Notes 22 to find the far field from this rectangular strip of electric surface current.

A three-dimensional view



$$\underline{J}_s = 2\,\underline{\hat{x}}\,E_0 \cos\left(\frac{\pi x}{a}\right)$$

Then use:

$$\underline{J}_s \rightarrow \underline{M}_s$$
$$\underline{E} \rightarrow \underline{H}$$
$$\underline{H} \rightarrow -\underline{E}$$

$$\varepsilon_0 \rightarrow \mu_0$$
$$\mu_0 \rightarrow \varepsilon_0$$

Example (cont.)

The vector array factor for the "case A" problem is then

$$\underline{a}(\theta,\phi) = \int_{-a}^{a}\int_{-b}^{b} 2\,\underline{\hat{x}}\,E_0 \cos\left(\frac{\pi x}{a}\right) e^{j\left(k_x x' + k_y y'\right)}\, dx' dy'$$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$

We then have, for case A, that the far field is

$$\underline{E} \sim -j\omega A_t\left(r,\theta,\phi\right)$$

$$\underline{A}(r,\theta,\phi) \sim \frac{\mu}{4\pi}\psi\left(r\right)\underline{a}\left(\theta,\phi\right)$$

$$\psi(r) = \frac{e^{-jkr}}{r}$$

Example (cont.)

For the original waveguide-fed aperture problem we then have

$$\underline{f}(\theta,\phi) = \int_{-a}^{a}\int_{-b}^{b} 2\,\underline{\hat{x}}\,E_0 \cos\left(\frac{\pi x}{a}\right) e^{j\left(k_x x' + k_y y'\right)}\, dx'dy'$$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$

The far field then

$$\underline{H} \sim -j\omega \underline{F}_t(r,\theta,\phi)$$

$$\underline{F} \sim \left(\frac{\varepsilon_0}{4\pi}\right)\psi(r)\,\underline{f}(\theta,\phi)$$

$$\underline{E} \sim -\eta_0\left(\underline{\hat{r}} \times \underline{H}\right) \qquad \psi(r) = \frac{e^{-jkr}}{r}$$

Example (cont.)

Performing the integration, we have

$$\underline{f}\left(\theta,\phi\right)=2\,\underline{\hat{x}}\,E_0\left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x\frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_x a}{2}\right)^2}\right)\left(b\,\text{sinc}\left(\frac{k_y b}{2}\right)\right)$$

$$k_x=k_0\sin\theta\cos\phi$$
$$k_y=k_0\sin\theta\sin\phi$$

We also have

$$\begin{aligned}\underline{f}_t\left(\theta,\phi\right)&=\underline{\hat{\theta}}\left(\underline{\hat{\theta}}\cdot\left(\underline{\hat{x}}\,f_x\right)\right)+\underline{\hat{\phi}}\left(\underline{\hat{\phi}}\cdot\left(\underline{\hat{x}}\,f_x\right)\right)\\&=f_x\left(\underline{\hat{\theta}}\left(\cos\theta\cos\phi\right)+\underline{\hat{\phi}}\left(-\sin\phi\right)\right)\end{aligned}$$

Example (cont.)

The far field of the waveguide-fed aperture is then:

$$\underline{H}(r,\theta,\phi) \sim -j\omega \underline{F}_t(r,\theta,\phi)$$

$$\underline{E} \sim -\eta_0 (\underline{\hat{r}} \times \underline{H})$$

$$\underline{F}_t \sim \left(\frac{\varepsilon_0}{4\pi}\right) \psi(r)\, \underline{f}_t(\theta,\phi) \qquad \psi(r) = \frac{e^{-jkr}}{r}$$

$$\underline{f}_t(\theta,\phi) = \left(\underline{\hat{\theta}}\left(\cos\theta\cos\phi\right) + \underline{\hat{\phi}}\left(-\sin\phi\right)\right) f_x(\theta,\phi)$$

$$f_x(\theta,\phi) = 2E_0 \left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x \frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x a}{2}\right)^2} \right) \left(b\,\mathrm{sinc}\left(\frac{k_y b}{2}\right) \right)$$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$

Example (cont.)

The final result is then:

$$E_\theta(r,\theta,\phi) \sim -\,j\omega\eta_0\left(\frac{\varepsilon_0}{4\pi}\right)\frac{e^{-jkr}}{r}(-\sin\phi)\,2E_0\left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x\frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_x a}{2}\right)^2}\right)\left(b\,\mathrm{sinc}\left(\frac{k_y b}{2}\right)\right)$$

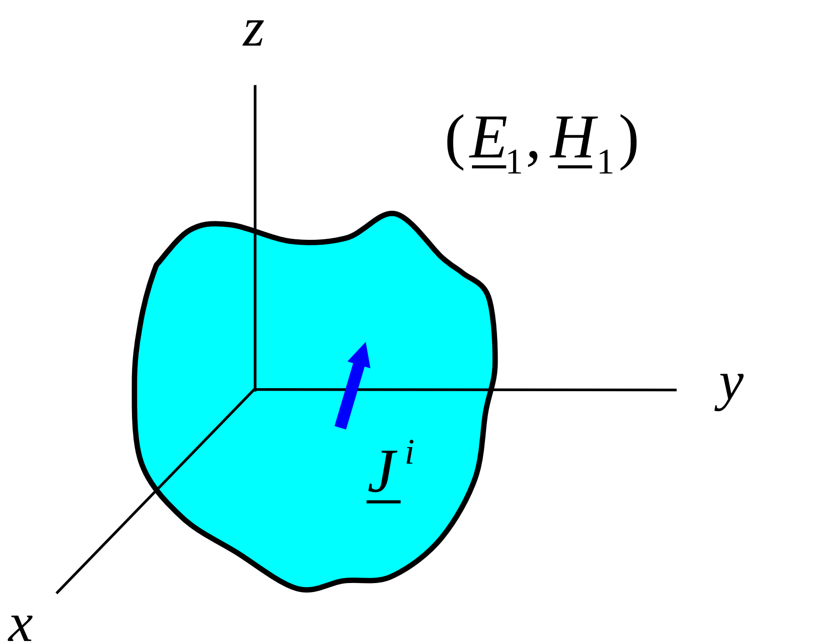
$$E_\phi(r,\theta,\phi) \sim j\omega\eta_0\left(\frac{\varepsilon_0}{4\pi}\right)\frac{e^{-jkr}}{r}(\cos\theta\cos\phi)\,2E_0\left(\frac{\left(\frac{\pi a}{2}\right)\cos\left(k_x\frac{a}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_x a}{2}\right)^2}\right)\left(b\,\mathrm{sinc}\left(\frac{k_y b}{2}\right)\right)$$

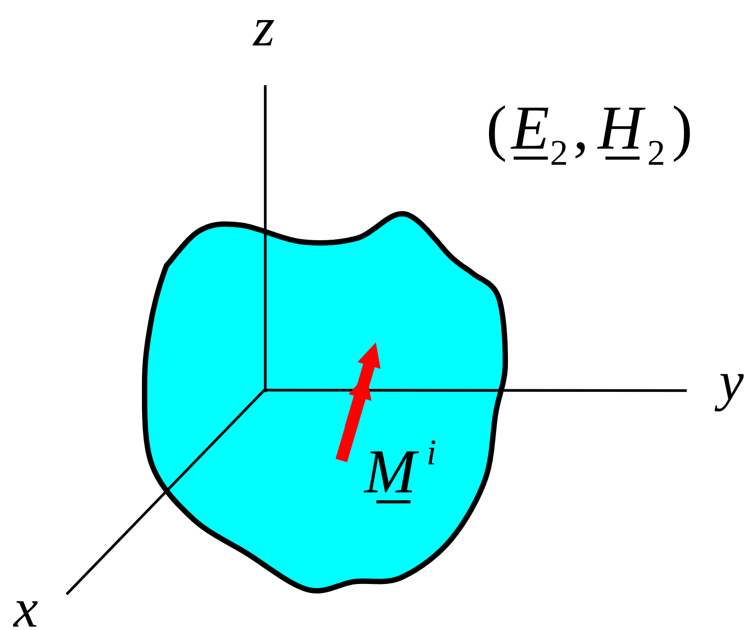
where

$$k_x = k_0\sin\theta\cos\phi$$
$$k_y = k_0\sin\theta\sin\phi$$

Volume Equivalence Principle

A radiating electric current can be replaced by a magnetic current, and vice versa.





We wish to have the same set of radiated fields.

P. E. Mayes, "The equivalence of electric and magnetic sources", *IEEE Trans. Antennas Propag*., vol. 6, pp. 295–296, 1958.

Volume Equivalence Principle (Cont.)

Set 1 (electric current source):

$$\nabla \times \underline{E}_1 = -j\omega\mu_0 \underline{H}_1$$
$$\nabla \times \underline{H}_1 = \underline{J}^i + j\omega\varepsilon_0 \underline{E}_1$$

Hence

$$\nabla \times \left(\nabla \times \underline{E}_1 \right) = -j\omega\mu_0 \left(\nabla \times \underline{H}_1 \right)$$
$$= -j\omega\mu_0 \left(\underline{J}^i + j\omega\varepsilon_0 \underline{E}_1 \right)$$

Therefore, we have

$$\nabla \times \left(\nabla \times \underline{E}_1 \right) - k_0^2 \underline{E}_1 = -j\omega\mu_0 \underline{J}^i$$

Volume Equivalence Principle (Cont.)

Set 2 (magnetic current source):

$$\nabla \times \underline{E}_2 = -j\omega\mu_0 \underline{H}_2 - \underline{M}^i$$
$$\nabla \times \underline{H}_2 = j\omega\varepsilon_0 \underline{E}_2$$

Hence

$$\begin{aligned}\nabla \times \left(\nabla \times \underline{E}_2\right) &= -j\omega\mu_0 \left(\nabla \times \underline{H}_2\right) - \nabla \times \underline{M}^i \\ &= -j\omega\mu_0 \left(j\omega\varepsilon_0 \underline{E}_2\right) - \nabla \times \underline{M}^i\end{aligned}$$

Therefore, we have

$$\nabla \times \left(\nabla \times \underline{E}_2\right) - k_0^2 \underline{E}_2 = -\nabla \times \underline{M}^i$$

Volume Equivalence Principle (Cont.)

Compare:

$$\nabla \times \left(\nabla \times \underline{E}_1 \right) - k_0^2 \underline{E}_1 = - j\omega\mu_0 \underline{J}^{\,i}$$

$$\nabla \times \left(\nabla \times \underline{E}_2 \right) - k_0^2 \underline{E}_2 = - \nabla \times \underline{M}^{\,i}$$

Set

$$- j\omega\mu_0 \underline{J}^{\,i} = - \nabla \times \underline{M}^{\,i}$$

Hence

$$\underline{J}^{\,i} = \frac{1}{j\omega\mu_0} \nabla \times \underline{M}^{\,i} \quad \Rightarrow \quad \underline{E}_1 = \underline{E}_2$$

Volume Equivalence Principle (Cont.)

Next, examine the difference in the two Faraday laws:

$$\nabla \times \underline{E}_1 = -j\omega\mu_0 \underline{H}_1$$

$$\nabla \times \underline{E}_2 = -j\omega\mu_0 \underline{H}_2 - \underline{M}^i$$

so

$$\nabla \times \left(\underline{E}_1 - \underline{E}_2 \right) = -j\omega\mu_0 \underline{H}_1 - \left(-j\omega\mu_0 \underline{H}_2 - \underline{M}^i \right)$$

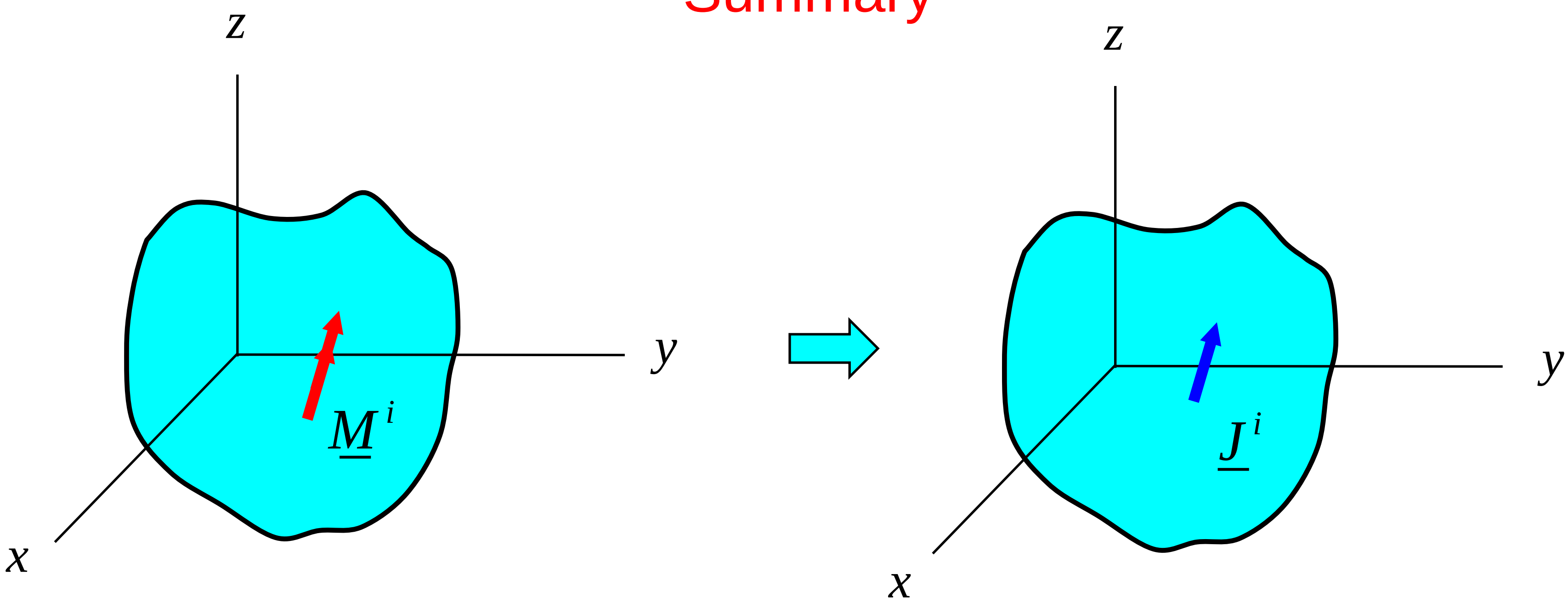
This gives us

$$\underline{H}_1 - \underline{H}_2 = \frac{1}{j\omega\mu_0} \underline{M}^i$$

Hence, the two electric fields are equal everywhere, but the magnetic fields are only the same <u>outside</u> the source region.

Volume Equivalence Principle (Cont.)

Summary



$$\underline{J}^i = \frac{1}{j\omega\mu_0}\nabla \times \underline{M}^i$$

$$\underline{E}\left[\underline{J}^i\right] = \underline{E}\left[\underline{M}^i\right]$$

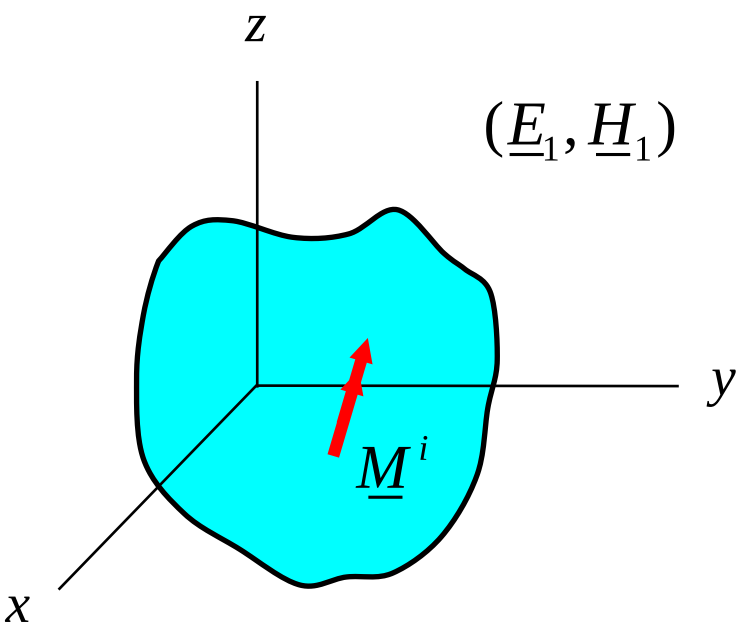
$$\underline{H}\left[\underline{J}^i\right] - \underline{H}\left[\underline{M}^i\right] = \frac{1}{j\omega\mu_0}\underline{M}^i$$

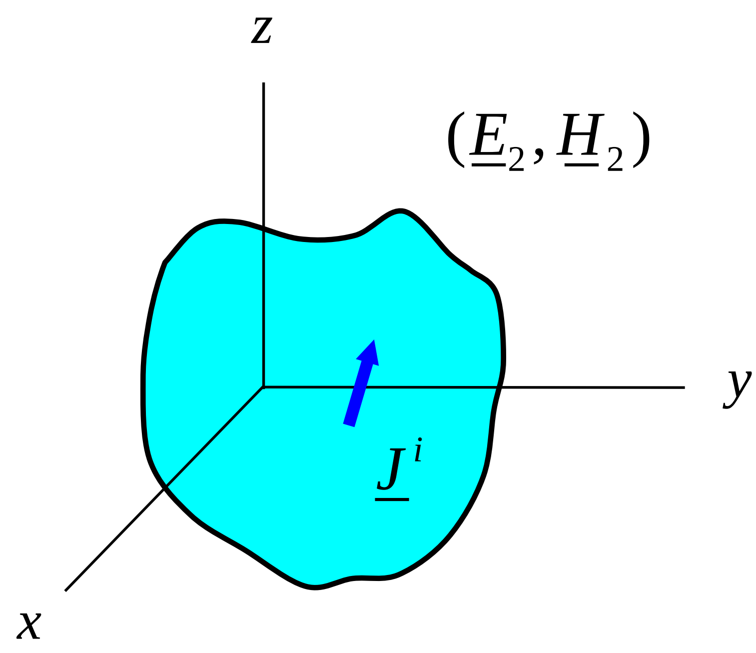
Volume Equivalence Principle (Cont.)

Apply duality on the two curl equations to get two new equations:

$$\nabla \times \left(\nabla \times \underline{H}_1 \right) - k_0^2 \underline{H}_1 = -j\omega\varepsilon_0 \underline{M}^i$$

$$\nabla \times \left(\nabla \times \underline{H}_2 \right) - k_0^2 \underline{H}_2 = \nabla \times \underline{J}^i$$





Volume Equivalence Principle (Cont.)

$$\nabla \times \left(\nabla \times \underline{H}_1 \right) - k_0^2 \underline{H}_1 = - j\omega\varepsilon_0 \underline{M}^i$$

$$\nabla \times \left(\nabla \times \underline{H}_2 \right) - k_0^2 \underline{H}_2 = \nabla \times \underline{J}^i$$

Hence

$$\underline{M}^i = -\frac{1}{j\omega\varepsilon_0} \nabla \times \underline{J}^i \quad \Rightarrow \quad \underline{H}_1^i = \underline{H}_2^i$$

Similarly, from duality we have

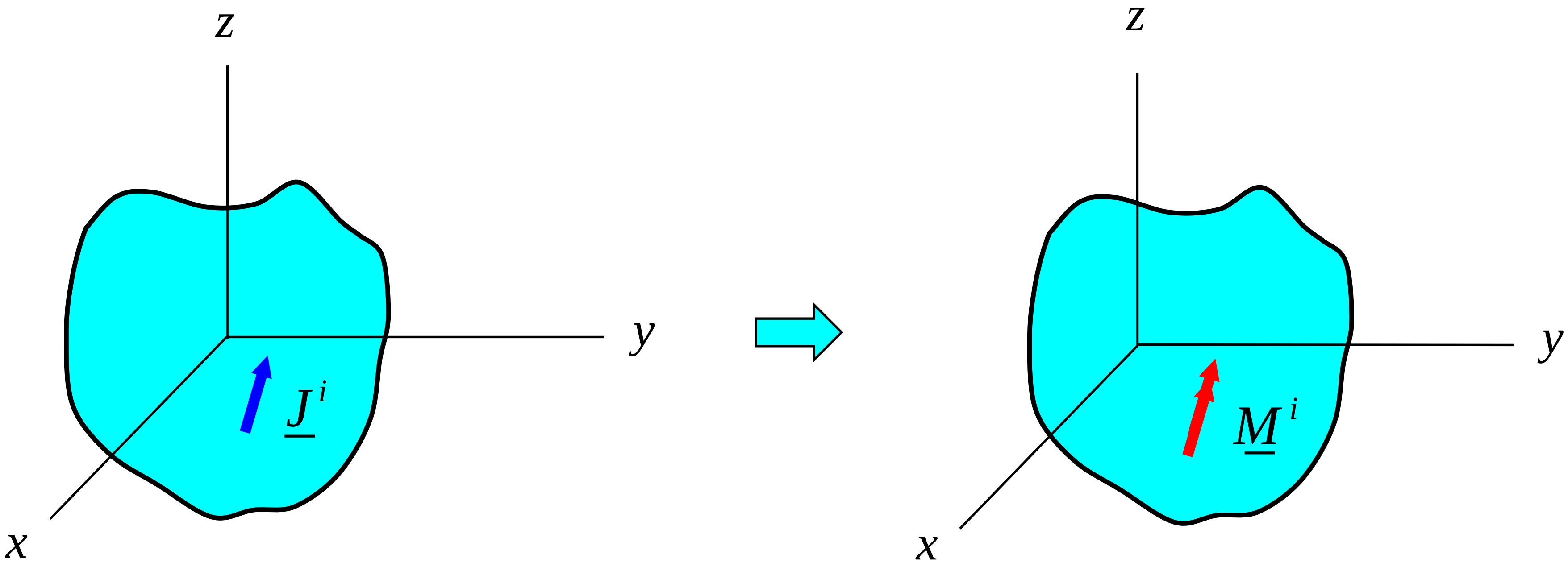
$$\nabla \times \underline{H}_1 = j\omega\varepsilon_0 \underline{E}_1$$

$$\nabla \times \underline{H}_2 = j\omega\varepsilon_0 \underline{E}_2 + \underline{J}^i$$

$$\Rightarrow \quad \nabla \times \left(\underline{H}_1 - \underline{H}_2 \right) = j\omega\varepsilon_0 \left(\underline{E}_1 - \underline{E}_2 \right) - \underline{J}^i$$

Volume Equivalence Principle (Cont.)

Summary



$$\underline{M}^{i} = -\frac{1}{j\omega\varepsilon_0}\nabla\times\underline{J}^{i}$$

$$\underline{H}\left[\underline{M}^{i}\right] = \underline{H}\left[\underline{J}^{i}\right]$$

$$\underline{E}\left[\underline{M}^{i}\right] - \underline{E}\left[\underline{J}^{i}\right] = \frac{1}{j\omega\varepsilon_0}\underline{J}^{i}$$