ECE 6340
Intermediate EM Waves

Fall 2016

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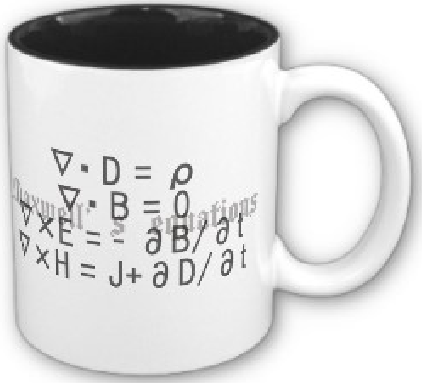
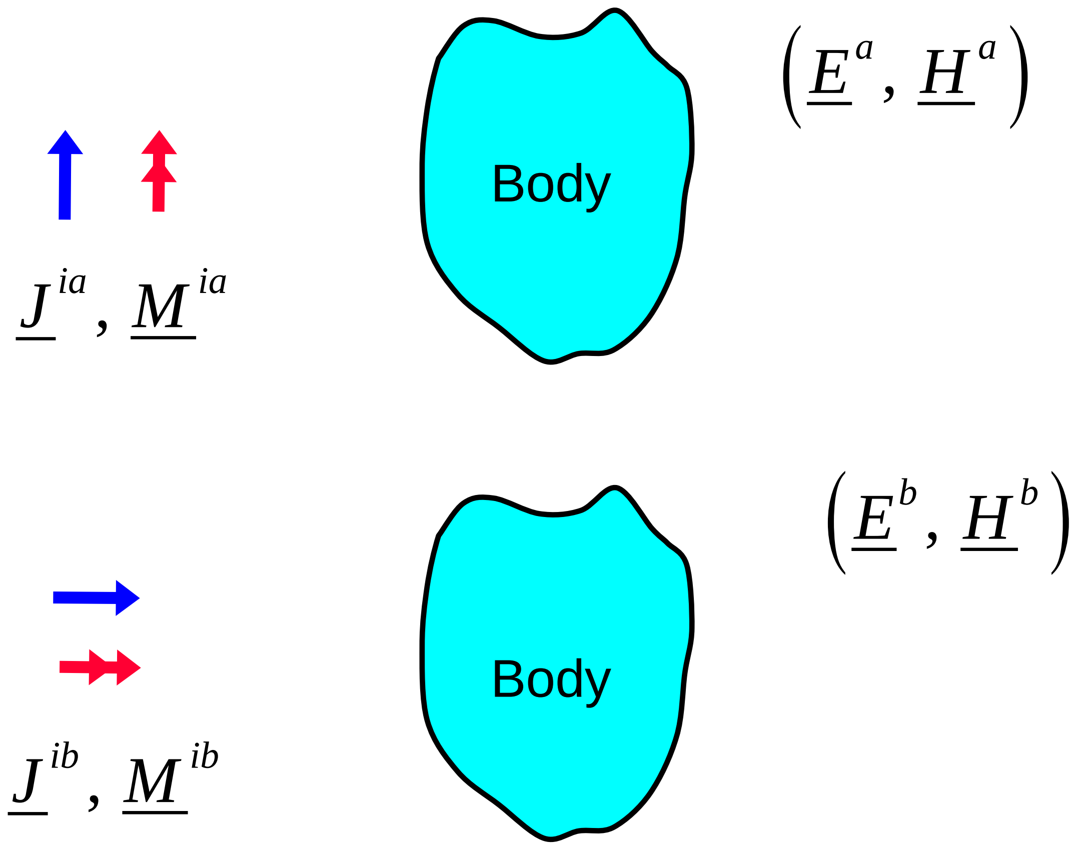
Dept. of ECE



Notes 27

Reciprocity Theorem

Consider two sets of sources, radiating in the same environment.



Note: The same “body” (dielectric or PEC) exists in both cases.

Reciprocity Theorem (cont.)

$$\nabla \times \underline{H}^{a} = \underline{J}^{ia} + j\omega\varepsilon_c \underline{E}^{a}$$

$$\underline{E}^{b} \cdot (\nabla \times \underline{H}^{a}) = \underline{E}^{b} \cdot \underline{J}^{ia} + j\omega\varepsilon_c \underline{E}^{b} \cdot \underline{E}^{a}$$

Also,

$$\nabla \times \underline{E}^{b} = -\underline{M}^{ib} - j\omega\mu \underline{H}^{b}$$

$$\underline{H}^{a} \cdot (\nabla \times \underline{E}^{b}) = -\underline{H}^{a} \cdot \underline{M}^{ib} - j\omega\mu \underline{H}^{a} \cdot \underline{H}^{b}$$

Subtract:

$$\underline{E}^{b} \cdot (\nabla \times \underline{H}^{a}) - \underline{H}^{a} \cdot (\nabla \times \underline{E}^{b}) = \underline{E}^{b} \cdot \underline{J}^{ia} + j\omega\varepsilon_c \underline{E}^{b} \cdot \underline{E}^{a} + \underline{H}^{a} \cdot \underline{M}^{ib} + j\omega\mu \underline{H}^{a} \cdot \underline{H}^{b}$$

Reciprocity Theorem (cont.)

Vector identity:

$$\nabla \cdot (\underline{H}^a \times \underline{E}^b) = \underline{E}^b \cdot (\nabla \times \underline{H}^a) - \underline{H}^a \cdot (\nabla \times \underline{E}^b)$$

Hence,

$$\nabla \cdot (\underline{H}^a \times \underline{E}^b) = \underline{E}^b \cdot \underline{J}^{ia} + j\omega\varepsilon_c \underline{E}^b \cdot \underline{E}^a + \underline{H}^a \cdot \underline{M}^{ib} + j\omega\mu \underline{H}^a \cdot \underline{H}^b$$

From duality (or repeating derivation using Faraday's Law for "a" and Ampere's Law for "b") we have:

$$-\nabla \cdot (\underline{E}^a \times \underline{H}^b) = \underline{H}^b \cdot \underline{M}^{ia} + j\omega\mu \underline{H}^b \cdot \underline{H}^a + \underline{E}^a \cdot \underline{J}^{ib} + j\omega\varepsilon_c \underline{E}^a \cdot \underline{E}^b$$

Reciprocity Theorem (cont.)

$$\nabla \cdot \left(\underline{H}^{a} \times \underline{E}^{b} \right) = \underline{E}^{b} \cdot \underline{J}^{ia} + j\omega\varepsilon_{c} \underline{E}^{b} \cdot \underline{E}^{a} + \underline{H}^{a} \cdot \underline{M}^{ib} + j\omega\mu \underline{H}^{a} \cdot \underline{H}^{b}$$

$$-\nabla \cdot \left(\underline{E}^{a} \times \underline{H}^{b} \right) = \underline{H}^{b} \cdot \underline{M}^{ia} + j\omega\mu \underline{H}^{b} \cdot \underline{H}^{a} + \underline{E}^{a} \cdot \underline{J}^{ib} + j\omega\varepsilon_{c} \underline{E}^{a} \cdot \underline{E}^{b}$$

Multiply first equation by -1 and then add:

$$-\nabla \cdot \left(\underline{H}^{a} \times \underline{E}^{b} \right) - \nabla \cdot \left(\underline{E}^{a} \times \underline{H}^{b} \right) = -\underline{E}^{b} \cdot \underline{J}^{ia} - \underline{H}^{a} \cdot \underline{M}^{ib} + \underline{H}^{b} \cdot \underline{M}^{ia} + \underline{E}^{a} \cdot \underline{J}^{ib}$$

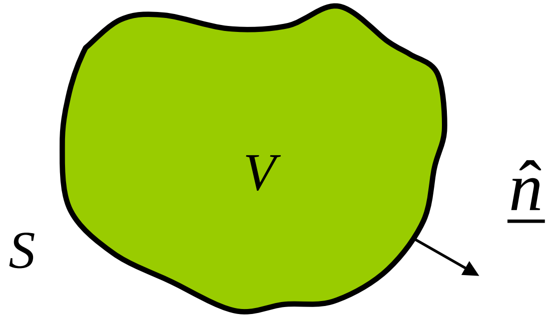
Reciprocity Theorem (cont.)

$$-\nabla \cdot (\underline{H}^a \times \underline{E}^b) - \nabla \cdot (\underline{E}^a \times \underline{H}^b) = -\underline{E}^b \cdot \underline{J}^{ia} - \underline{H}^a \cdot \underline{M}^{ib} + \underline{H}^b \cdot \underline{M}^{ia} + \underline{E}^a \cdot \underline{J}^{ib}$$

Reversing the order of the cross products in the first term on the LHS,

$$\nabla \cdot (\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b) = \underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib} - \underline{E}^b \cdot \underline{J}^{ia} + \underline{H}^b \cdot \underline{M}^{ia}$$

Next, integrate both sides over an arbitrary volume V and then apply the divergence theorem:



Reciprocity Theorem (cont.)

$$\oint_S (\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b) \cdot \underline{\hat{n}}\, dS = \int_V (\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib} - \underline{E}^b \cdot \underline{J}^{ia} + \underline{H}^b \cdot \underline{M}^{ia})\, dV$$

Now let $S \to S_\infty$

In the far-field, $\underline{H} \sim \dfrac{1}{\eta}(\underline{\hat{r}} \times \underline{E})$

Hence

$$(\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b) \sim \frac{1}{\eta}\left[\underline{E}^b \times (\underline{\hat{r}} \times \underline{E}^a) - \underline{E}^a \times (\underline{\hat{r}} \times \underline{E}^b)\right]$$

Reciprocity Theorem (cont.)

$$(\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b) \sim \frac{1}{\eta}\left[\underline{E}^b \times (\underline{\hat{r}} \times \underline{E}^a) - \underline{E}^a \times (\underline{\hat{r}} \times \underline{E}^b)\right]$$

Now use a vector identity:

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\,\underline{B} - (\underline{A} \cdot \underline{B})\,\underline{C}$$

So,

$$(\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b) \sim$$

$$\frac{1}{\eta}\left[\underline{\hat{r}}\,(\underline{E}^b \cdot \underline{E}^a) - (\underline{\hat{r}} \cdot \underline{E}^b)\underline{E}^a - \underline{\hat{r}}\,(\underline{E}^a \cdot \underline{E}^b) + (\underline{\hat{r}} \cdot \underline{E}^a)\underline{E}^b\right]$$

cancels

$$= \frac{1}{\eta}\left[-(\underline{\hat{r}} \cdot \underline{E}^b)\underline{E}^a + (\underline{\hat{r}} \cdot \underline{E}^a)\underline{E}^b\right]$$

$$= O\left(\frac{1}{r^3}\right)$$

Reciprocity Theorem (cont.)

Hence $$\oint_S \left(\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b\right) \cdot \underline{\hat{n}}\, dS \to 0$$

Therefore,

$$\oint_S \left(\underline{E}^b \times \underline{H}^a - \underline{E}^a \times \underline{H}^b\right) \cdot \underline{\hat{n}}\, dS$$
$$= \int_V \left(\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib} - \underline{E}^b \cdot \underline{J}^{ia} + \underline{H}^b \cdot \underline{M}^{ia}\right) dV$$

$$\int_V \left(\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib} - \underline{E}^b \cdot \underline{J}^{ia} + \underline{H}^b \cdot \underline{M}^{ia}\right) dV = 0$$

Reciprocity Theorem (cont.)

Final form of reciprocity theorem:

$$\int_V \left(\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib} \right) dV = \int_V \left(\underline{E}^b \cdot \underline{J}^{ia} - \underline{H}^b \cdot \underline{M}^{ia} \right) dV$$

LHS: Fields of "a" dotted with the sources of "b"

RHS: Fields of "b" dotted with the sources of "a"

Reciprocity Theorem (cont.)

$$\int_V (\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib})\, dV = \int_V (\underline{E}^b \cdot \underline{J}^{ia} - \underline{H}^b \cdot \underline{M}^{ia})\, dV$$

Define "reactions":

$$< a, b > = \int_V (\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \underline{M}^{ib})\, dV$$

$$< b, a > = \int_V (\underline{E}^b \cdot \underline{J}^{ia} - \underline{H}^b \cdot \underline{M}^{ia})\, dV$$

Then

$$< a, b > = < b, a >$$

Extension: Anisotropic Case

$$\underline{D} = \underline{\underline{\varepsilon}} \cdot \underline{E}$$

$$\underline{B} = \underline{\underline{\mu}} \cdot \underline{H}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \underline{\underline{\mu}} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}$$

If $\varepsilon_{ij} = \varepsilon_{ji}$ and $\mu_{ij} = \mu_{ji}$ (symmetric matrices) then reciprocity holds.
These are called “reciprocal” materials.

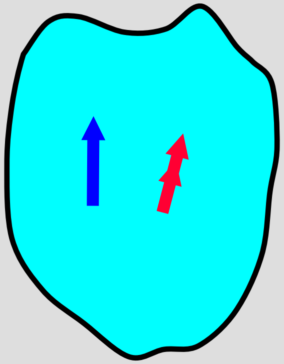
“Testing” Current

Some Basic Observations

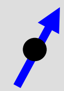
- To make the reciprocity theorem useful to us, we usually choose the “b” current to be a “testing” current or “measuring” current.
- The “b” current thus allows us to sample a quantity of interest.
- This allows us to determine some property about the quantity of interest, or in some cases, to calculate it (or at least calculate it in a simpler way).

Dipole "Testing" Current

$\underline{J}^{b}$ is a point dipole



a sources



$$\underline{J}^{ib} = \underline{\hat{p}}\,\delta\left(\underline{r} - \underline{r}_0\right)$$

b source

Recall: $\delta\left(\underline{r}\right)$ is a 3-D delta function.

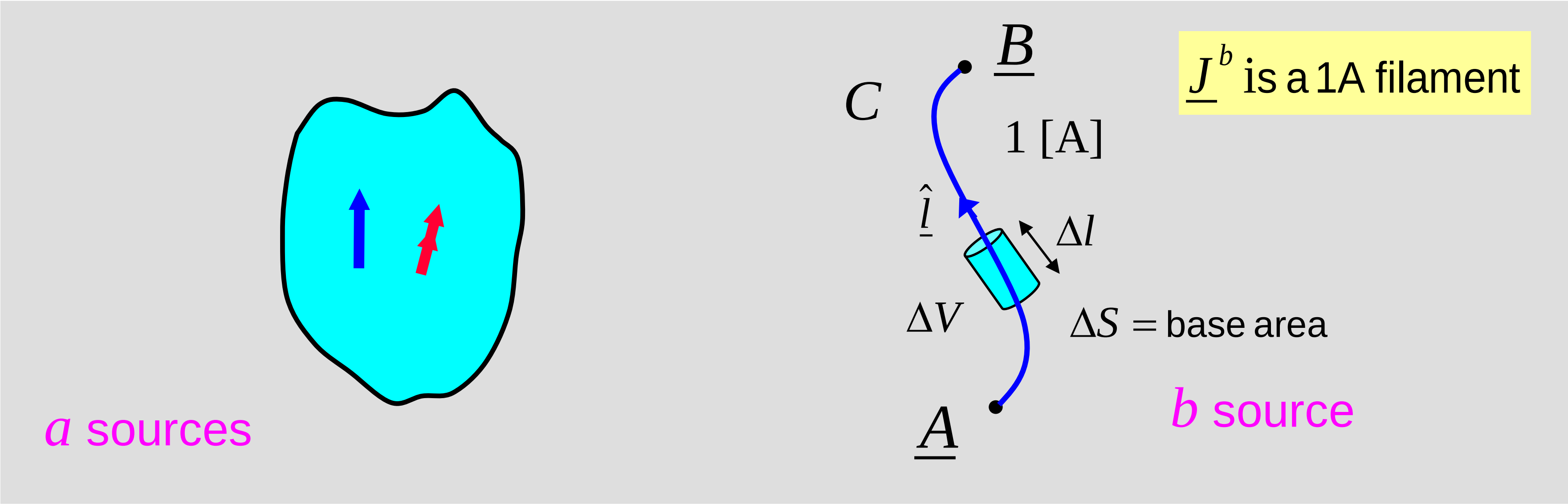
$$< a,b > = \int_V \left(\underline{E}^a \cdot \underline{J}^{ib} - \underline{H}^a \cdot \cancel{\underline{M}^{ib}} \right) dV$$

$$= \int_V \left(\underline{E}^a \cdot \underline{\hat{p}} \right) \delta\left(\underline{r} - \underline{r}_0\right) dV$$

$$= \underline{\hat{p}} \cdot \underline{E}^a\left(\underline{r}_0\right)$$

We sample a <u>field component at a point</u>.

Filament "Testing" Current



$$< a,b >= \int_V \underline{E}^a \cdot \underline{J}^b dV$$

$$= \int_C \underline{E}^a \cdot \hat{\underline{l}}\, dl$$

$$= \int_C \underline{E}^a \cdot d\underline{r}$$

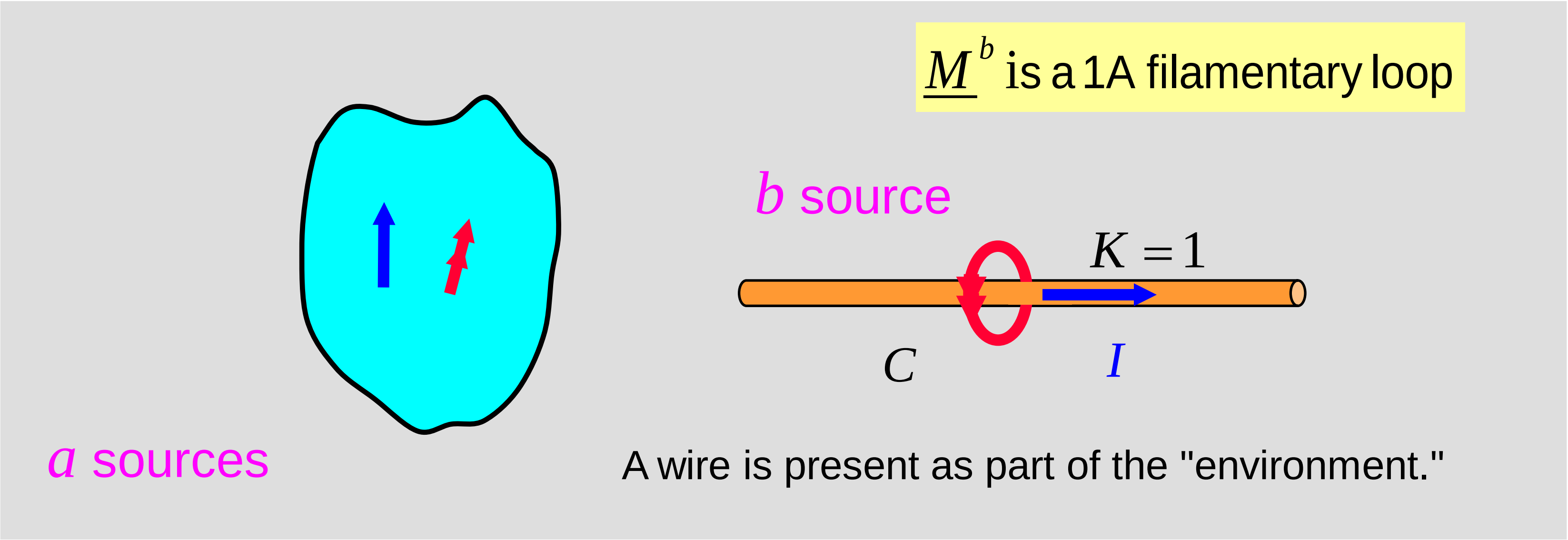
$$= V_{AB}$$

$$\underline{J}^b = \hat{\underline{l}}\, J_l^b$$

$$\int_{\Delta V} J_l^b\, dV = \int_{\Delta S} J_l^b\, dS\, \Delta l = I\, \Delta l = \Delta l$$

We sample a voltage drop between two points.

Magnetic Frill "Testing" Current



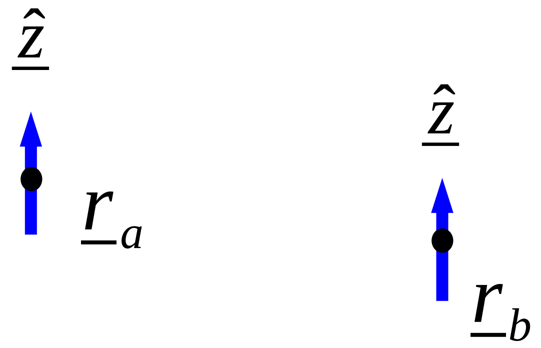
$$
\begin{aligned}
\langle a,b\rangle &= \int_V -\underline{H}^{a}\cdot\underline{M}^{b}\,dV \\
&= -K\int_C \underline{H}^{a}\cdot\underline{\hat{l}}\,dl \\
&= -\int_C \underline{H}^{a}\cdot d\underline{r} \\
&= -I
\end{aligned}
$$

We sample a current on a wire.

Note:
There is no displacement current through the loop if it hugs the PEC wire.

Example

Two infinitesimal unit-amplitude electric dipoles



$$\underline{J}^{ia} = \underline{\hat{z}}\,\delta\left(\underline{r} - \underline{r}_a\right)$$

$$\underline{J}^{ib} = \underline{\hat{z}}\,\delta\left(\underline{r} - \underline{r}_b\right)$$

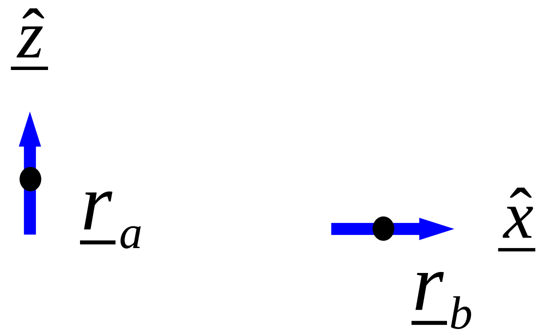
$$< a, b > = < b, a >$$

$$\int_V \underline{E}^a \cdot \underline{J}^b - \underline{H}^a \cdot \cancel{\underline{M}^b}\, dV = \int_V \underline{E}^b \cdot \underline{J}^a - \underline{H}^b \cdot \cancel{\underline{M}^a}\, dV$$

$$E_z^a(\underline{r}_b) = E_z^b(\underline{r}_a)$$

Example

Two infinitesimal unit-amplitude electric dipoles



$$\underline{J}^{ia} = \underline{\hat{z}}\,\delta\left(\underline{r} - \underline{r}_a\right)$$

$$\underline{J}^{ib} = \underline{\hat{x}}\,\delta\left(\underline{r} - \underline{r}_b\right)$$

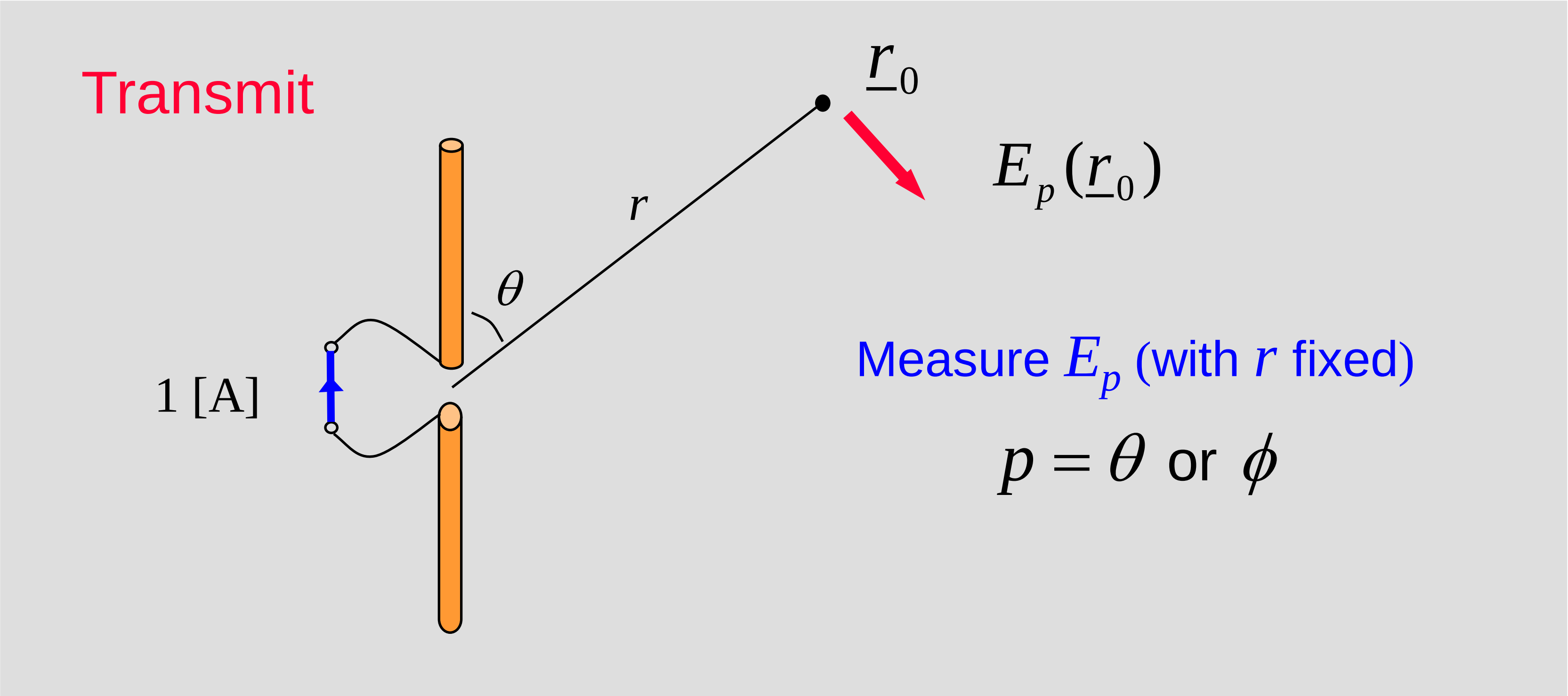
$$< a,b > = < b,a >$$

$$\int_V \underline{E}^a \cdot \underline{J}^b - \underline{H}^a \cdot \cancel{\underline{M}^b}\, dV = \int_V \underline{E}^b \cdot \underline{J}^a - \underline{H}^b \cdot \cancel{\underline{M}^a}\, dV$$

$$E_x^a(\underline{r}_b) = E_z^b(\underline{r}_a)$$

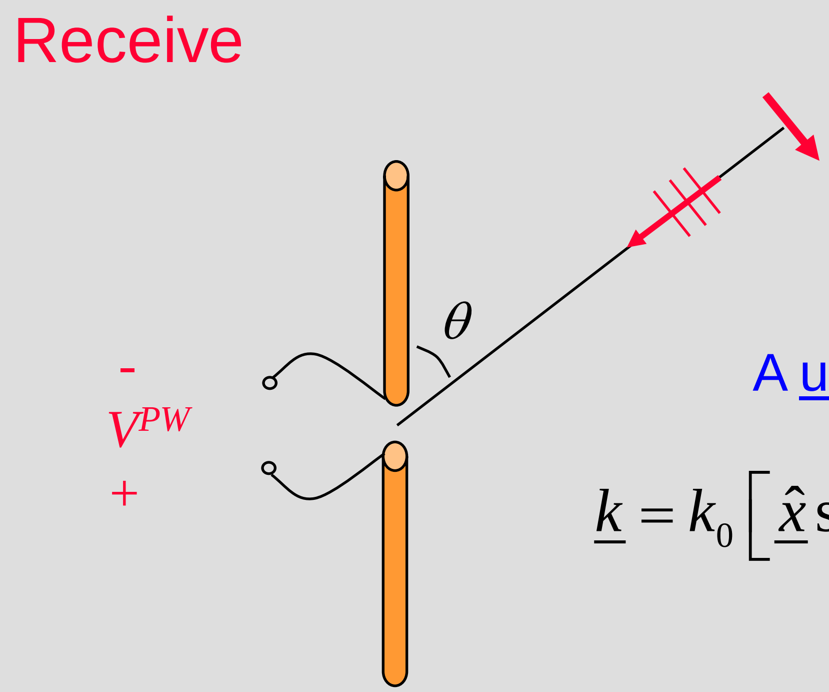
Example

The far-field transmit and receive patterns of any antenna are the same.



$$T(\theta,\phi) \equiv E_p(\underline{r}_0)$$

Example (cont.)



$$\underline{E}^i = \hat{\underline{p}}\, e^{+j\underline{k}\cdot\underline{r}}$$

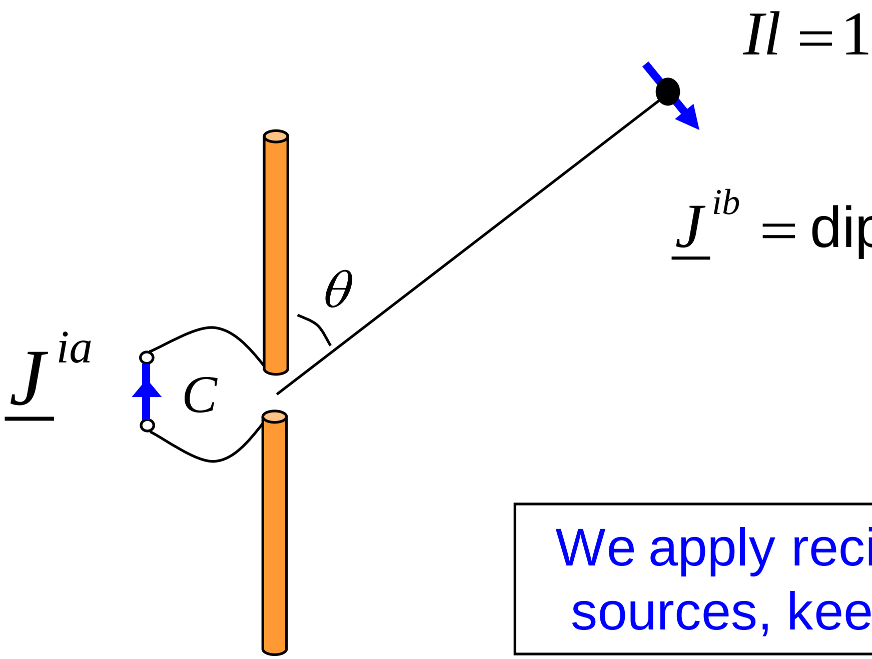
A unit-strength plane wave is incident.

$$\underline{k} = k_0\left[\hat{\underline{x}}\sin\theta\cos\phi + \hat{\underline{y}}\sin\theta\sin\phi + \hat{\underline{z}}\cos\theta\right]$$

$$R(\theta,\phi) \equiv V^{PW}$$

Example (cont.)

Next, define two sources:



$\underline{J}^{ib}$ = dipole in $\underline{\hat{p}}$ direction

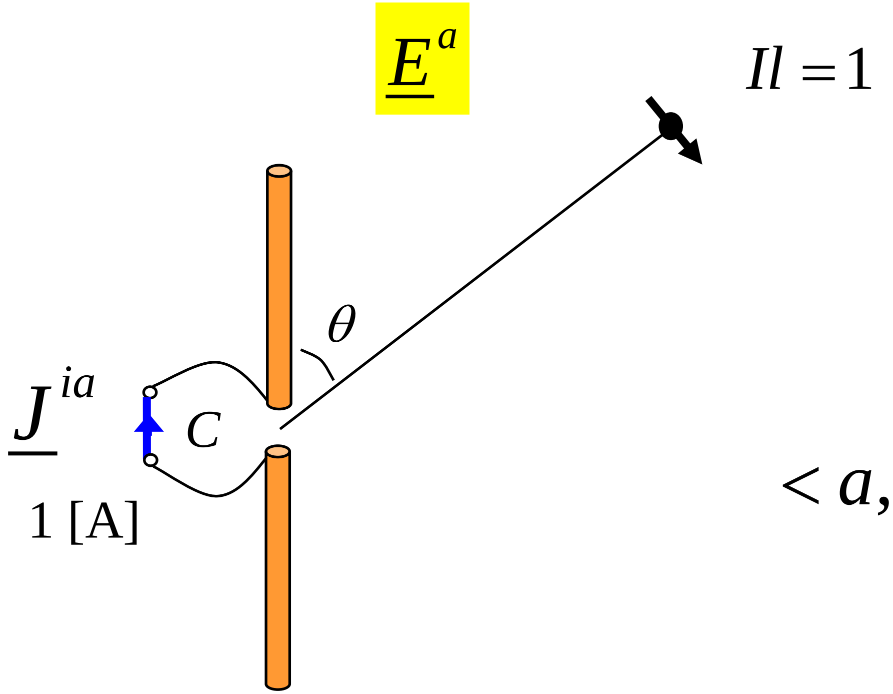
$\underline{\hat{p}} = \underline{\hat{\theta}}$ or $\underline{\hat{\phi}}$

We apply reciprocity between these two sources, keeping the antenna present.

$\underline{J}^{ia} = 1[\mathrm{A}]$ filament

The antenna (and feed wires) is the “body.”

Example (cont.)



The field $\underline{E}^a$ is the field produced by the 1[A] feed current exciting the antenna.

$$< a,b > = \int_V \underline{E}^a \cdot \underline{J}^{ib} dV$$

$$= \int_V \underline{E}^a \cdot \left[\underline{\hat{p}} \delta(\underline{r} - \underline{r}_0) \right] dV$$

$$= E_p^a(\underline{r}_0)$$

The antenna is the "body."

Note: The black color is used to show where dipole "b" is, even though it is not radiating here.

Example (cont.)

Hence, we have

$$< a,b > = T\left(\theta,\phi\right)$$

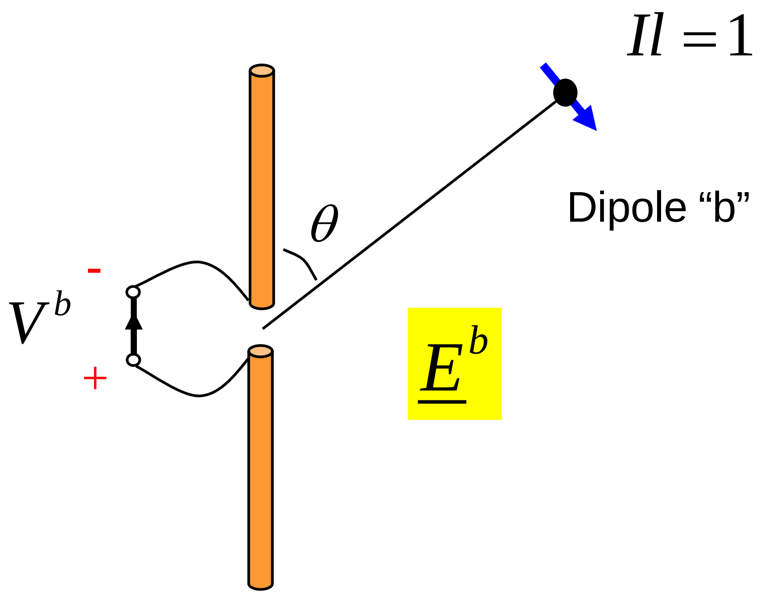
Example (cont.)

The field $\underline{E}^b$ is the field produced by dipole "b" in the far field.

$$< b, a > = \int_V \underline{E}^b \cdot \underline{J}^{ia} \, dV$$

$$= \int_C \underline{E}^b \cdot \underline{\hat{l}} \, dl = \int_C \underline{E}^b \cdot \underline{dr}$$

$$= V^b$$

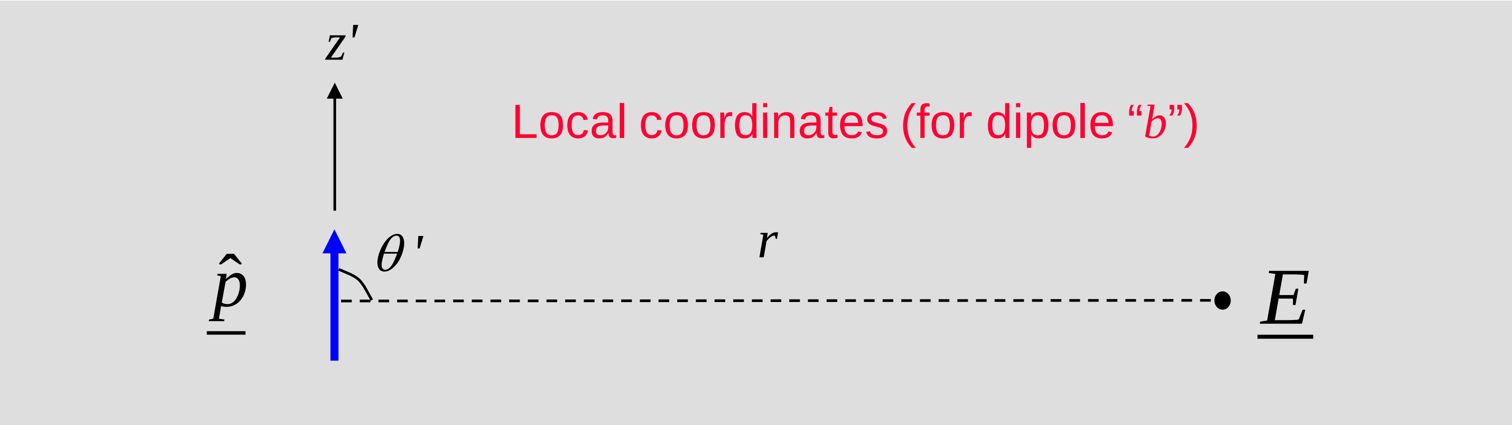


The voltage V^b is the open-circuit voltage due to a unit-amplitude dipole in the far field.

Note: The black color is used to show where filament "a" is, even though it is not radiating here.

Example (cont.)

As $r \to \infty$, the incident field from dipole "b" becomes a plane-wave field.



We need to determine the incident field $\underline{E}$ from the dipole:

Let $\quad \underline{\hat{p}} \to \underline{\hat{z}}'$

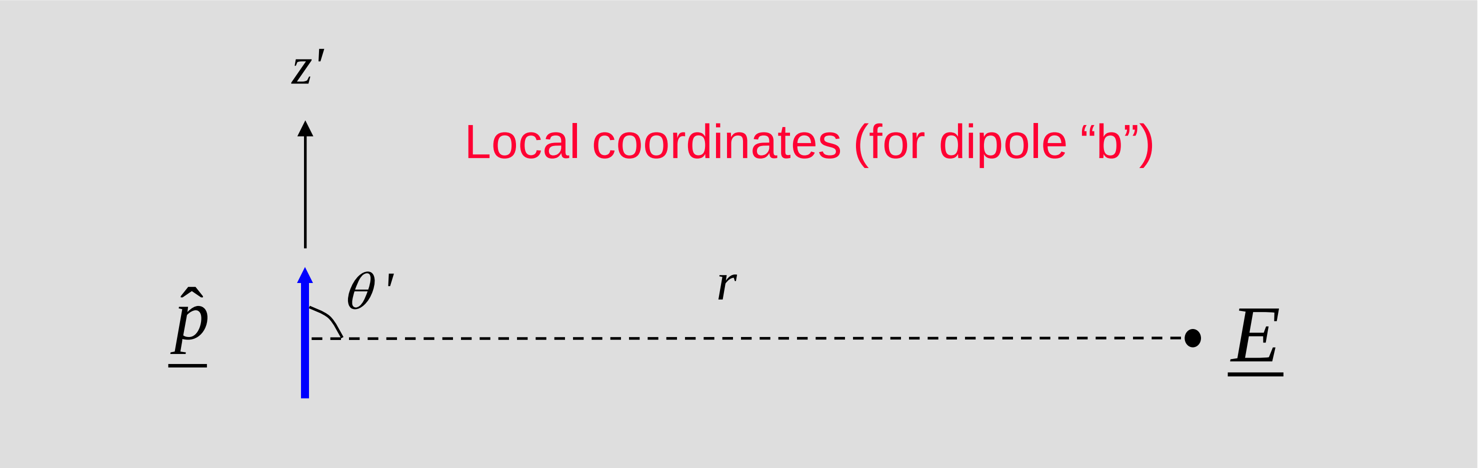
$$\underline{E} \sim \underline{\hat{\theta}}' \left(\frac{j\omega\mu_0}{4\pi r} \right) e^{-jkr} \sin\theta'$$

$\theta' = \dfrac{\pi}{2}$ so

$$\underline{E} \sim \underline{\hat{\theta}}' \left(\frac{j\omega\mu_0}{4\pi r} \right) e^{-jkr} = -\underline{\hat{p}} \left(\frac{j\omega\mu_0}{4\pi r} \right) e^{-jkr}$$

Example (cont.)

Define $$E_0 \equiv \left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jkr}$$



$$r \to \infty \qquad \underline{E} \sim \underline{\hat{p}}\, E_0$$

Example (cont.)

Hence

$$\underline{E}^{b,inc}(0,0,0) \approx E_0\, \underline{\hat{p}}$$

More generally,

$$\underline{E}^{b,inc}(x,y,z) \approx E_0\, \underline{\hat{p}}\, e^{+j\underline{k}\cdot\underline{r}}$$

The incident field from the "testing" dipole thus acts as a plane wave polarized in the $\underline{\hat{p}}$ direction, with amplitude E_0 at the origin.

Hence

$$\langle b,a \rangle = V^b = E_0 V^{PW} = E_0\, R(\theta,\phi)$$

Example (cont.)

Summarizing, we have:

$$< a,b > = T\left(\theta,\phi\right)$$

$$< b,a > = E_0\, R(\theta,\phi)$$

From reciprocity:

$$< a,b > = < b,a >$$

so

$$T\left(\theta,\phi\right) = E_0 R(\theta,\phi)$$

Example (cont.)

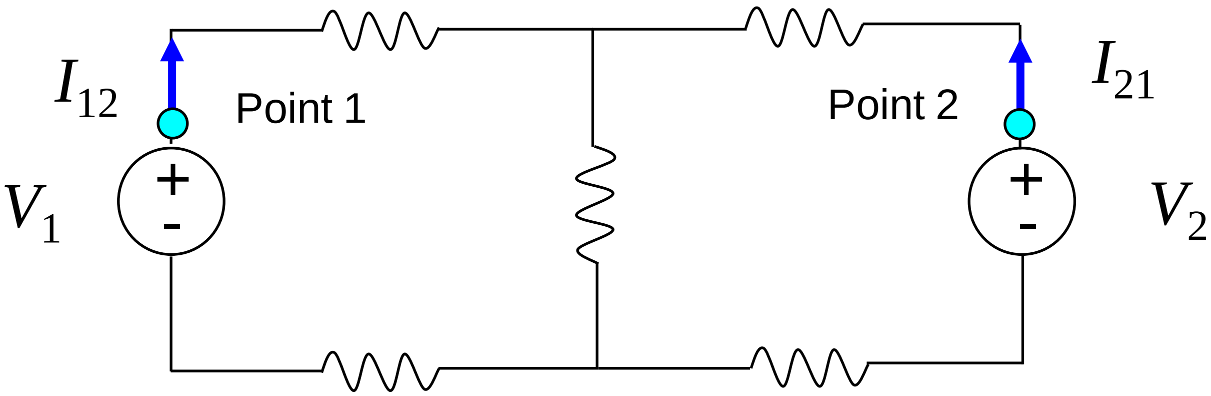
Hence, in summary we have:

$$T(\theta,\phi) = R(\theta,\phi)\left[\left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jkr}\right]$$

The shape of the far-field transmit and receive patterns are the same.

Example

Reciprocity in circuit theory



$$\frac{V_1}{I_{21}} = \frac{V_2}{I_{12}}$$

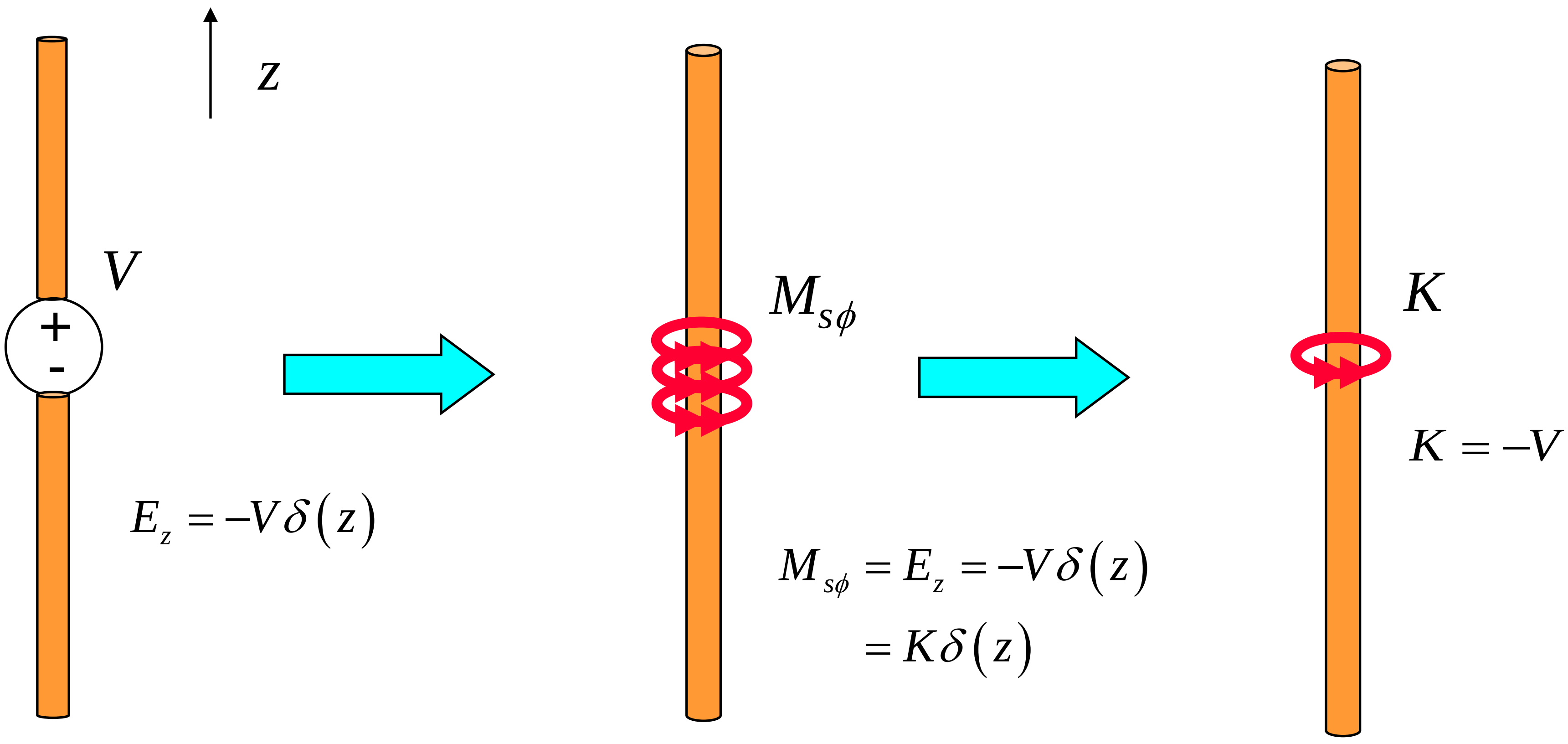
I_{21} = current at location 2 produced by V_1

I_{12} = current at location 1 produced by V_2

Note: If $V_1 = V_2$, then the result becomes $I_{12} = I_{21}$.

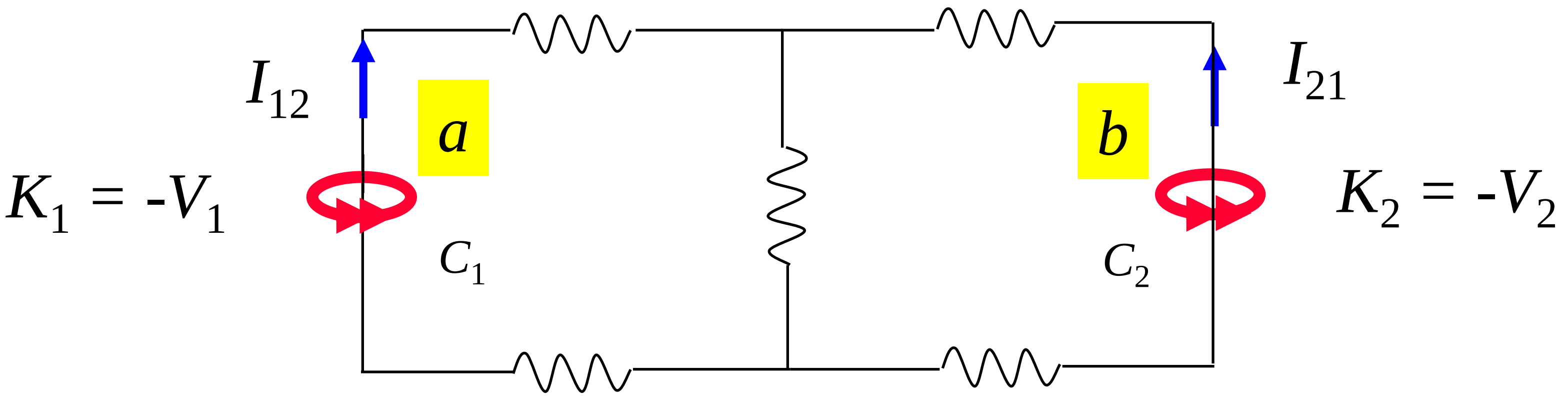
Example (cont.)

Magnetic frill modeling of voltage source



K is the amplitude of the magnetic frill current that flows in the positive ϕ direction.

Example (cont.)



The sources (*a* and *b*) are the magnetic frills; the "environment" is the circuit.

$$< a,b > = \int_V -\underline{H}^a \cdot \underline{M}^b dV = -K_2 \int_{C_2} \underline{H}^a \cdot \hat{\underline{l}}\, dl = V_2 \int_{C_2} \underline{H}^a \cdot \hat{\underline{l}}\, dl = V_2 I_{21}$$

$$< b,a > = \int_V -\underline{H}^b \cdot \underline{M}^a dV = -K_1 \int_{C_2} \underline{H}^b \cdot \hat{\underline{l}}\, dl = V_1 \int_{C_1} \underline{H}^b \cdot \hat{\underline{l}}\, dl = V_1 I_{12}$$

From reciprocity: $< a,b > = < b,a >$

Example (cont.)

Hence

$$V_2 I_{21} = V_1 I_{12}$$

or

$$\frac{V_1}{I_{21}} = \frac{V_2}{I_{12}}$$