NAME: ______________SOLUTION______________________________________

STUDENT NUMBER: ____

ELEE 6340
Fall 2004

EXAM I

INSTRUCTIONS:

This exam is open-book and open-notes. You may use any material or calculator that you wish.

Please write all of your work on the sheets attached.

Please show *all of your work* and *write neatly* in order to receive credit.

Put all of your answers in terms of the parameters given in the problems, unless otherwise noted.

Problem 1 (30 pts)

A resistive bar with a resistance R rotates counterclockwise around a circular track at a constant angular velocity ω as shown below. The inner edge of the resistor pivots on an electrode and the outer edge of the resistor contacts the track. The inner electrode and the track are connected to a battery with a voltage V_0. A magnetic field exists everywhere in space, which has the value

$$\underline{B} = \hat{\underline{z}}\, B_0$$

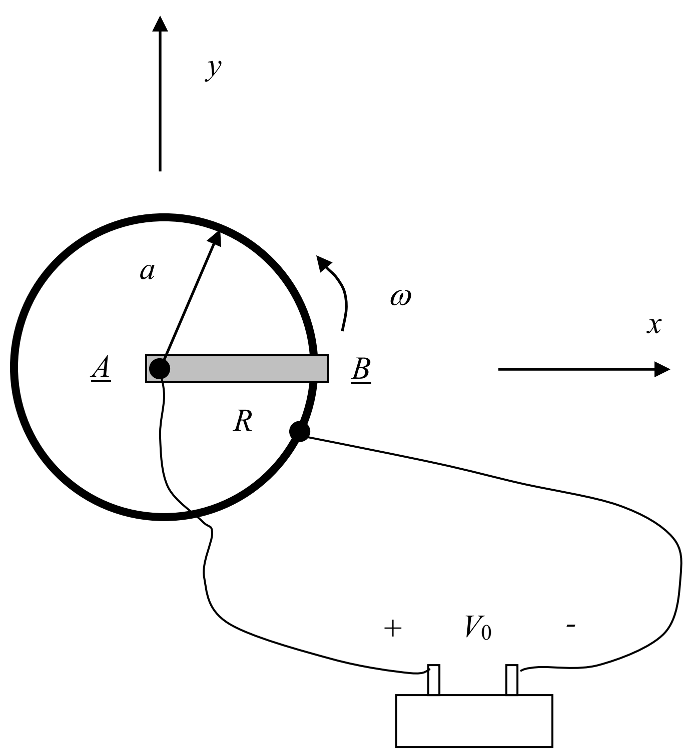
where B_0 is a constant.

a) Determine the voltage drop V_{AB} across the resistive bar, where $\underline{A}$ is the center electrode and $\underline{B}$ is the track.

b) Determine the change in EMF across the resistor, ΔEMF_{AB}. Do this by first calculating directly the term

$$I_m = \int_{\underline{A}}^{\underline{B}} \left(\underline{v} \times \underline{B}\right) \cdot d\underline{r}$$

c) Determine the current I through the resistor, flowing from the center to the outside part of the track.



ROOM FOR WORK

a) The time-derivative of the magnetic field is zero. Therefore, Faraday's law says that the line integral of the electric field around the closed path much be zero. Hence, the voltage drop across the bar is the same as the battery voltage.

Hence,

$V_{AB} = V_0$.

b) $\Delta EMF_{AB} = V_0 + I_m$

$$I_m = \int_A^B \left(\underline{v} \times \underline{B}\right) \cdot d\underline{r} = \int_A^B \left(\omega\rho\underline{\hat{\phi}} \times \underline{z}B_0\right) \cdot \underline{\hat{\rho}}\, d\rho = \omega B_0 \int_A^B \left(\rho\underline{\hat{\rho}}\right) \cdot \underline{\hat{\rho}}\, d\rho = \omega B_0 \int_0^a \rho d\rho = \omega B_0 \frac{a^2}{2}$$

Hence,

$$\Delta EMF_{AB} = V_0 + \omega B_0 \frac{a^2}{2}.$$

c) $I = \dfrac{\Delta EMF_{AB}}{R}$

so that

$$I = \frac{V_0}{R} + \omega B_0 \frac{a^2}{2R}.$$

Problem 2 (20 pts)

A rectangular waveguide section is filled with air, and short-circuited at $z = 0$. The length of the waveguide section is h. The metal walls of the waveguide are perfectly conducting. Inside the waveguide only the TE_{10} mode exists (above the cutoff frequency) at an angular frequency ω. The electric field inside the waveguide is given by

$$\underline{E} = \underline{\hat{y}} \sin\left(\frac{\pi x}{a}\right) \sin\left(k_z z\right).$$

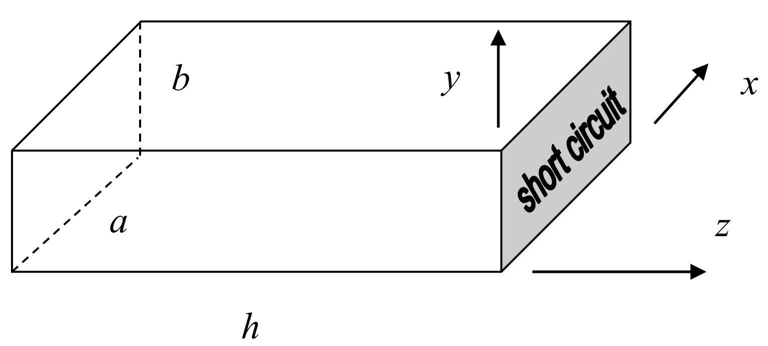
a) Calculate the magnetic field component H_x inside the waveguide.

b) Calculate the complex power P_c flowing into the waveguide section through the cross-sectional plane $z = -h$.

c) Calculate the time-average energy stored in the electric field (denoted as $\langle U_E \rangle$) inside the waveguide section.

d) Use the complex Poynting theorem to determine the time-average energy stored in the magnetic field inside the waveguide section (denoted as $\langle U_H \rangle$). You may leave the answer in terms of P_c, $\langle U_E \rangle$, and ω.

Useful identity: $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$



ROOM FOR WORK

a) $$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left(-\hat{\underline{x}}\frac{\partial E_y}{\partial z} + \hat{\underline{z}}\frac{\partial E_y}{\partial x}\right)$$

$$E_y = \sin\left(\frac{\pi x}{a}\right)\sin\left(k_z z\right)$$

so

$$\underline{H} = \hat{\underline{x}}\frac{k_z}{j\omega\mu_0}\sin\left(\frac{\pi x}{a}\right)\cos\left(k_z z\right) - \hat{\underline{z}}\frac{1}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(k_z z\right)$$

Hence

$$H_x = \frac{k_z}{j\omega\mu_0}\sin\left(\frac{\pi x}{a}\right)\cos\left(k_z z\right)$$

b) $$P_c = \frac{1}{2}\int_S\left(\underline{E}\times\underline{H}^*\right)\cdot\hat{\underline{z}}\,dS = -\frac{1}{2}\int_0^a\int_0^b E_y H_x^*\,dy\,dx = -\frac{b}{2}\int_0^a E_y H_x^*\,dx$$

or

$$P_c = j\frac{b}{2}\left(\frac{k_z}{\omega\mu_0}\right)\sin\left(k_z h\right)\cos\left(k_z h\right)\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx$$

so that

$$P_c = j\frac{b}{2}\left(\frac{k_z}{\omega\mu_0}\right)\sin\left(k_z h\right)\cos\left(k_z h\right)\left(\frac{a}{2}\right)$$

Hence

$$P_c = j\frac{ab}{4}\left(\frac{k_z}{\omega\mu_0}\right)\sin\left(k_z h\right)\cos\left(k_z h\right).$$

ROOM FOR WORK

c) $E_y = \sin\left(\frac{\pi x}{a}\right)\sin(k_z z)$

$$\langle U_E\rangle = \frac{1}{4}\int_V \varepsilon_0 \left|\underline{E}\right|^2 dV = \frac{1}{4}\varepsilon_0 b\int_0^a\int_{-h}^0 \sin^2\left(\frac{\pi x}{a}\right)\sin^2(k_z z)\,dz\,dx$$

so

$$\langle U_E\rangle = \frac{1}{4}\varepsilon_0 b\left(\frac{a}{2}\right)\int_{-h}^0 \sin^2(k_z z)\,dz = \frac{1}{4}\varepsilon_0 b\left(\frac{a}{2}\right)\int_{-h}^0\left[\frac{1+\cos(2k_z z)}{2}\right]dz$$

or

$$\langle U_E\rangle = \frac{1}{4}\varepsilon_0 b\left(\frac{a}{2}\right)\left[\frac{z}{2}+\frac{\sin(2k_z z)}{4k_z}\right]_{-h}^0$$

or

$$\langle U_E\rangle = \frac{1}{4}\varepsilon_0 b\left(\frac{a}{2}\right)\left[\frac{h}{2}+\frac{\sin(2k_z h)}{4k_z}\right]$$

or

$$\langle U_E\rangle = \varepsilon_0\left(\frac{ab}{16}\right)\left[h+\frac{\sin(2k_z h)}{2k_z}\right]$$

d) From the complex Poynting theorem,

$$\operatorname{Im} P_c = 2\omega\left[\langle U_H\rangle - \langle U_E\rangle\right]$$

so

$$\langle U_H\rangle = \langle U_E\rangle + \frac{1}{2\omega}\operatorname{Im} P_c$$

Problem 3 (20 pts)

A certain waveguide is air filled. The wall of the waveguide is perfectly conducting. The cross-sectional shape of the waveguide is arbitrary. At a frequency of f_0, a particular waveguide mode propagating in the z direction has a phase change of ϕ_0 radians per meter in the z direction.

Please answer the questions below, leaving all answers in terms of the parameters given above and in the questions below, and the speed of light in air, denoted as c.

a) What is the cutoff frequency of this waveguide mode?

b) What will the attenuation of this same waveguide mode be, in dB/m, at a frequency f that is one half of the cutoff frequency?

c) What is the phase change of the waveguide mode for a one meter length of waveguide at frequency f_0, if the waveguide is now filled with a lossless dielectric material having a relative permittivity of ε_r ?

d) What is the complex wavenumber k_z at frequency f_0, if the waveguide is now filled with a lossy material having an effective relative permittivity $\varepsilon_{re} = \varepsilon_r' - j\varepsilon_r''$?

ROOM FOR WORK

a) The phase constant is $\beta = \phi_0$. Hence

$$\phi_0 = \sqrt{k_0^2 - k_c^2}$$

so

$$k_c = \sqrt{k_0^2 - \phi_0^2}$$

so

$$k_c = \sqrt{\left(\frac{2\pi f_0}{c}\right)^2 - \phi_0^2}\ .$$

Hence

$$\frac{2\pi f_c}{c} = \sqrt{\left(\frac{2\pi f_0}{c}\right)^2 - \phi_0^2}$$

or

$$f_c = \sqrt{f_0^2 - \phi_0^2\left(\frac{c}{2\pi}\right)^2}$$

or

$$f_c = f_0\sqrt{1 - \phi_0^2\left(\frac{c}{2\pi f_0}\right)^2}\ .$$

b) At a frequency lower than the cutoff frequency, we have

$$\alpha = \sqrt{k_c^2 - k_0^2} = k_c\sqrt{1 - \left(\frac{f}{f_c}\right)^2}\ .$$

Hence

$$\alpha = \sqrt{\left(\frac{2\pi f_0}{c}\right)^2 - \phi_0^2}\ \sqrt{1-\left(\frac{f}{f_c}\right)^2}\ .$$

If the operating frequency is one-half the cutoff frequency we have

$$\alpha = \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2\pi f_0}{c}\right)^2 - \phi_0^2}\ .$$

In dB/m we have

$$\alpha_{dB} = 8.686\ \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2\pi f_0}{c}\right)^2 - \phi_0^2}\ \text{[dB/m]}$$

c) If the waveguide is now filled with a material having a relative permittivity ε_r, the phase constant at a frequency f_0 will be

$$\beta = \sqrt{k_0^2\varepsilon_r - k_c^2} = \sqrt{k_0^2\varepsilon_r - \left(\frac{2\pi f_0}{c}\right)^2 + \phi_0^2}$$

or

$$\beta = \frac{2\pi f_0}{c}\sqrt{\varepsilon_r - \left(1-\phi_0^2\left(\frac{c}{2\pi f_0}\right)^2\right)}.$$

d) If the material inside the waveguide is now a lossy material, we have

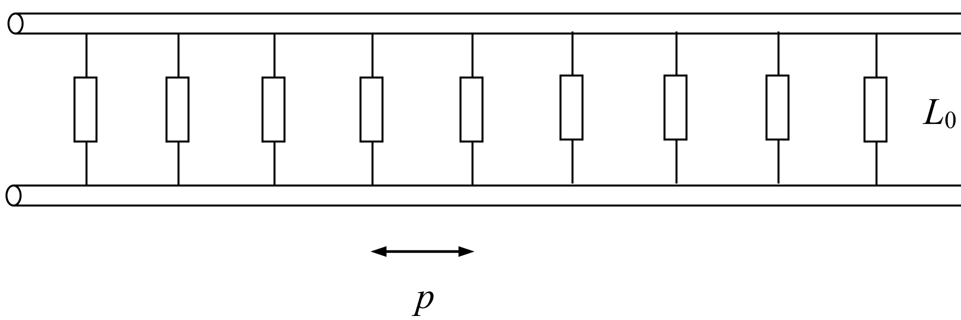
$$k_z = \sqrt{k_0^2\varepsilon_{re} - k_c^2} = \sqrt{k_0^2\varepsilon_{re} - \left(\frac{2\pi f_0}{c}\right)^2 + \phi_0^2}$$

$$k_z = \frac{2\pi f_0}{c}\sqrt{\varepsilon_{re} - \left(1-\phi_0^2\left(\frac{c}{2\pi f_0}\right)^2\right)}.$$

Problem 4 (30 pts)

An artificial transmission lines is made from an air-filled transmission line that has a characteristic impedance Z_0. The wires of the transmission line are perfectly conducting. A periodic series of inductances are added in parallel across the line as shown below. Each inductance has L_0 Henrys and the spacing between inductors is p. Assume that p is small compared with a wavelength.

1. Calculate the characteristic impedance Z_0^p and the propagation wavenumber k_z^p of the line with the periodic inductive loading. Your answer should be in terms of Z_0, ω, L_0, p, and the speed of light c.
2. At what frequency will $\beta = 0$?



ROOM FOR WORK

a) The characteristic impedance is given by

$$Z_0 = \sqrt{\frac{L}{C}}$$

and the wavenumber is given by

$$\mu_0 \varepsilon_0 = LC = \frac{1}{c^2}.$$

Hence,

$$L = \frac{Z_0}{c}.$$

and

$$C = \frac{1}{cZ_0}.$$

We then have that

$$Z_0^p = \sqrt{\frac{j\omega L}{j\omega C + \frac{1}{j\omega L_0 p}}}.$$

Therefore,

$$Z_0^p = \sqrt{\frac{j\omega \frac{Z_0}{c}}{\frac{j\omega}{cZ_0} + \frac{1}{j\omega L_0 p}}}.$$

Simplifying, we have

$$Z_0^p = Z_0 \sqrt{\frac{1}{1 - \frac{cZ_0}{\omega^2 L_0 p}}}.$$

The propagation wavenumber is

$$k_z^p = -j\sqrt{(j\omega L)\left(j\omega C + \frac{1}{j\omega L_0 p}\right)}.$$

Hence we have

$$k_z^p = -j\sqrt{\left(j\omega \frac{Z_0}{c}\right)\left(\frac{j\omega}{cZ_0} + \frac{1}{j\omega L_0 p}\right)}.$$

Simplifying, we have

$$k_z^p = k_0\sqrt{\left(1 - \frac{cZ_0}{\omega^2 L_0 p}\right)}.$$

b) In order to make the propagation wavenumber zero,

$$\omega = \sqrt{\frac{cZ_0}{L_0 p}}.$$