NAME: ____________SOLUTION_________________________

ECE 6340
Fall 2005

EXAM I

INSTRUCTIONS:

This exam is open-book and open-notes. You may use any material or calculator that you wish. Please show *all of your work* and *write neatly* in order to receive credit. Put all of your answers in terms of the parameters given in the problems, unless otherwise noted. Include units with all answers in order to receive full credit.

Please write all of your work on the sheets attached.

Problem 1 (30 pts)

There is a boundary $z = 0$ between two different materials, each having a different relative permittivity and conductivity. Material 1 has properties $(\varepsilon_1, \sigma_1)$. Material 2 has properties $(\varepsilon_2, \sigma_2)$. There is no electric or magnetic surface current at the boundary.

1. Using Maxwell's equations and boundary conditions, prove that the following boundary condition must be true in the time-harmonic steady state:

   $$\underline{\hat{z}} \cdot \left(j\omega\varepsilon_{c1} \underline{E}_1 \right) = \underline{\hat{z}} \cdot \left(j\omega\varepsilon_{c2} \underline{E}_2 \right)$$

   where $\varepsilon_c = \varepsilon - j\sigma / \omega$ is the complex effective permittivity. (This boundary condition physically means that the total equivalent current coming from all mechanisms (displacement, polarization, and conduction) must be continuous across the boundary.)

2. Assume that in region 1 the electric field has a normal component $\underline{\hat{z}}\, E_{z1}$. Derive an expression for the surface charge density ρ_s at the boundary, in terms of E_{z1} and the material parameters.

ROOM FOR EXTRA WORK

Part 1

$$\nabla \times \underline{H} = j\omega\varepsilon_c \underline{E}$$

$$\hat{\underline{z}} \cdot \left(\nabla \times \underline{H}\right) = j\omega\,\varepsilon_c \left(\hat{\underline{z}} \cdot \underline{E}\right)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\varepsilon_c \left(\hat{\underline{z}} \cdot \underline{E}\right)$$

Since the tangential magnetic field is continues across an interface, the LHS of the above equation must be continuous. Hence it follows that, as long as $\omega \neq 0$,

$$\hat{\underline{z}} \cdot \left(\varepsilon_{c1} \underline{E}_1\right) = \hat{\underline{z}} \cdot \left(\varepsilon_{c2} \underline{E}_2\right).$$

Part 2

$$\begin{aligned}\rho_s &= \hat{\underline{z}} \cdot \left(\underline{D}_1 - \underline{D}_2\right) \\ &= \hat{\underline{z}} \cdot \left(\varepsilon_1 \underline{E}_1 - \varepsilon_2 \underline{E}_2\right) \\ &= \varepsilon_1 E_{z1} - \varepsilon_2 E_{z2} \\ &= \varepsilon_1 E_{z1} - \varepsilon_2 E_{z1} \left(\frac{\varepsilon_{c1}}{\varepsilon_{c2}}\right).\end{aligned}$$

Hence,

$$\rho_s = \varepsilon_1 E_{z1} \left(1 - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)\left(\frac{\varepsilon_{c1}}{\varepsilon_{c2}}\right)\right).$$

Problem 2 (30 pts)

A voltmeter is connected to a pair of perfectly conducting rails as shown below. At the end of the rails is a battery with voltage V_0. The bottom rail is fixed while the top rail is moving upward at a constant velocity v. The rails exist in a region where there is a magnetic field given by

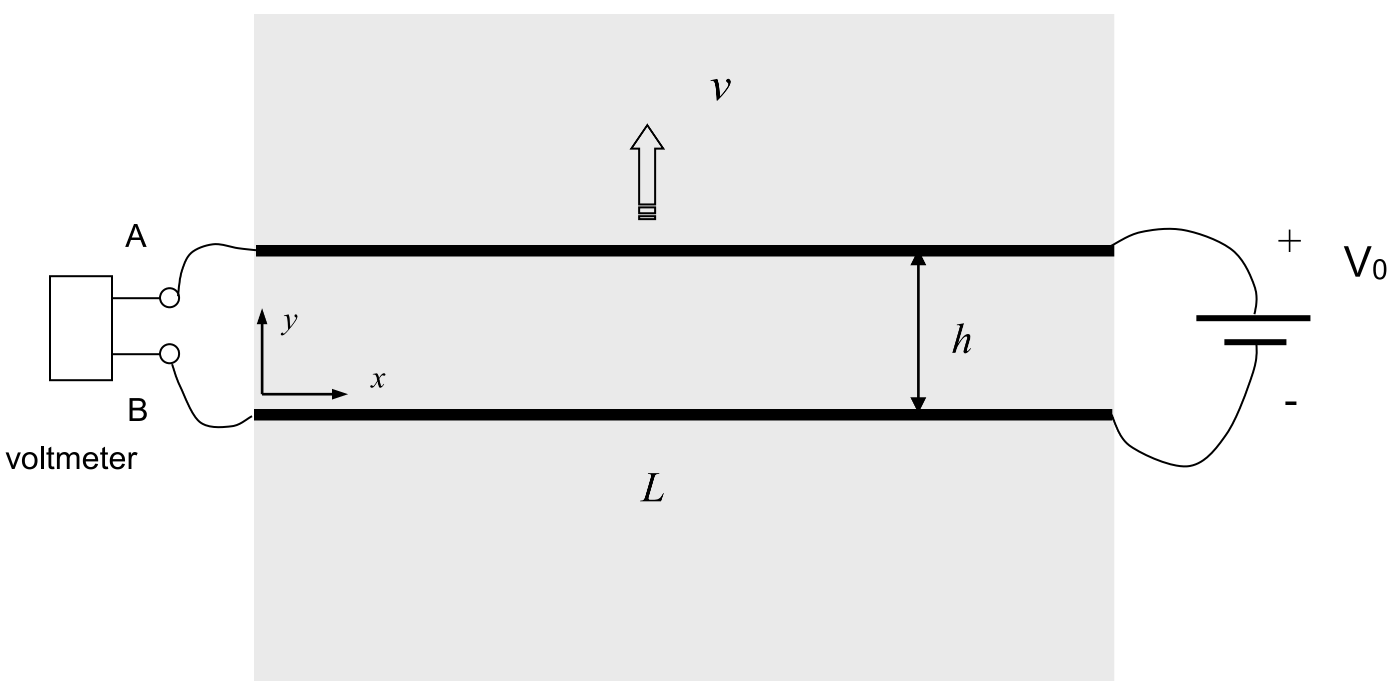
$$\underline{B} = \hat{\underline{z}}\, B_0 \cos(\omega t)$$

(this is the shaded region).

1. Determine what voltage the voltmeter will read when the separation between the two rails is h.

2. Repeat the calculation assuming that the top rail is at a height h above the bottom rail, but it is not moving.

You may assume that the voltmeter will always read correctly what the voltage drop is between the two terminals of the meter. That is, the meter will always read the correct value V_{AB}.



ROOM FOR EXTRA WORK

Part 1

$$EMF = -\frac{d\psi}{dt}$$

Hence

$$V_{AB} - V_0 = -\frac{d\psi}{dt}$$

or

$$V_{AB} = V_0 - \frac{d\psi}{dt}.$$

The magnetic flux through the closed path (in the positive z direction) is given by

$$\psi = \left(B_0 \cos(\omega t)\right)\left(Ly(t)\right)$$

where $y(t)$ is the height of the top rail. Hence

$$\frac{d\psi}{dt} = \left(B_0 \cos(\omega t)\right)\left(Ly'(t)\right) + (-\omega)\left(B_0 \sin(\omega t)\right)\left(Ly(t)\right).$$

Hence we have

$$\frac{d\psi}{dt} = \left(B_0 \cos(\omega t)\right)(Lv) + (-\omega)\left(B_0 \sin(\omega t)\right)(Lh).$$

Therefore

$$V_{AB} = V_0 - \left(B_0 \cos(\omega t)\right)(Lv) + \left(\omega B_0 \sin(\omega t)\right)(Lh)$$

Part 2

The only difference here is that $v = 0$. Hence we have

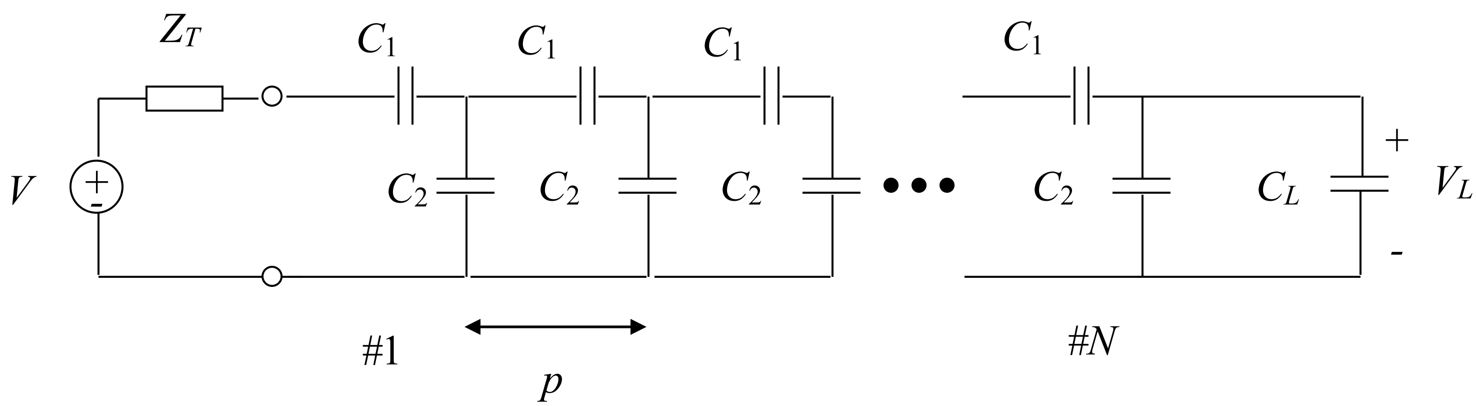
$$V_{AB} = V_0 + \left(\omega B_0 \sin(\omega t)\right)(Lh).$$

Problem 3 (40 pts)

A filter is made by cascading identical sections of capacitors together as shown below. At the left end a voltage source is connected, consisting of a time-harmonic voltage V at a frequency of ω with a Thevenin impedance Z_T. At the right end a load capacitor C_L is connected. Assume that the dimension p is very small compared with a wavelength, and the length of the filter is $N p$ (the filter consists of N sections), where N is a large number. Also, assume that the value of C_L is chosen so that no reflections occur at the load.

1. Find an approximate expression for the voltage V_L across the load capacitor.

2. Find the value of C_L.



ROOM FOR EXTRA WORK

Part 1

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{\frac{1}{j\omega C_1}\frac{1}{p}}{j\omega C_2 / p}} = \frac{1}{j\omega\sqrt{C_1 C_2}}$$

$$\gamma = \sqrt{ZY} = \sqrt{\left(\frac{1}{j\omega C_1}\frac{1}{p}\right)\left(j\omega C_2 \frac{1}{p}\right)} = \frac{1}{p}\sqrt{\frac{C_2}{C_1}}\,.$$

Note that γ is real, so that $\alpha = \gamma$ and $\beta = 0$.

$$V_L = V\left(\frac{Z_0}{Z_T + Z_0}\right)e^{-\alpha(Np)}\,.$$

Hence,

$$V_L = V\left(\frac{1}{1 + j\omega\sqrt{C_1 C_2}Z_T}\right)e^{-N\sqrt{\frac{C_2}{C_1}}}\,.$$

Part 2

For a perfect match, we require that

$$Z_L = Z_0\,.$$

Hence,

$$\frac{1}{j\omega C_L} = \frac{1}{j\omega\sqrt{C_1 C_2}}\,.$$

Therefore, we have

$$C_L = \sqrt{C_1 C_2}\,.$$