NAME: ________________________________

ECE 6340
Fall 2008

EXAM I

INSTRUCTIONS:

This exam is open-book and open-notes. You may use any material or calculator that you wish.

Put all of your answers in terms of the parameters given in the problems, unless otherwise noted.

Include units with all answers in order to receive full credit.

Please write all of your work on the sheets attached.

Please show *all of your work* and *write neatly* in order to receive credit.

Useful identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Problem 1 (40 pts)

Ocean water has the following parameters:

$$\sigma = 4 \text{ [S/m]}$$
$$\varepsilon_r'(18 \text{ GHz}) = 43$$
$$\varepsilon_r''(18 \text{ GHz}) = 38$$
$$f_{\max} = 18 \text{ GHz.}$$

The frequency $f_{\max}$ is the frequency where the imaginary part of the permittivity is maximum.

Derive a formula for the complex permittivity ε_c of the ocean water as a function of frequency, using the Debye model to determine ε as a function of frequency. Make sure that all constants appearing in your expression have been determined.

Solution

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon - j\frac{4}{\omega}$$

From the Debye model,

$$\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0 \left[\varepsilon_r(\infty) + \frac{\varepsilon_r(0) - \varepsilon_r(\infty)}{1 + j\omega\tau} \right].$$

We now need to determine the three parameters appearing in the Debye model. To do this, we use

$$\varepsilon_r' = \varepsilon_r(\infty) + \frac{\varepsilon_r(0) - \varepsilon_r(\infty)}{1 + (\omega\tau)^2}$$
$$\varepsilon_r'' = (\omega\tau)\frac{\varepsilon_r(0) - \varepsilon_r(\infty)}{1 + (\omega\tau)^2}$$

At 18 GHz we have $\omega\tau = 1$, and hence

$$\tau = 8.8419 \times 10^{-12} \text{ [s]}$$

At 18 GHz we have

$$43 = \varepsilon_r(\infty) + \frac{\varepsilon_r(0) - \varepsilon_r(\infty)}{1 + (1)^2}$$

$$38 = (1)\frac{\varepsilon_r(0) - \varepsilon_r(\infty)}{1 + (1)^2}$$

or

$$86 = \varepsilon_r(\infty) + \varepsilon_r(0)$$

$$76 = \varepsilon_r(0) - \varepsilon_r(\infty).$$

The solution is

$$\varepsilon_r(0) = 81$$
$$\varepsilon_r(\infty) = 5.$$

Problem 2 (50 pts)

Consider the problem shown below, which has a top bar that slides upward at a constant velocity v_0. The magnetic field is given by

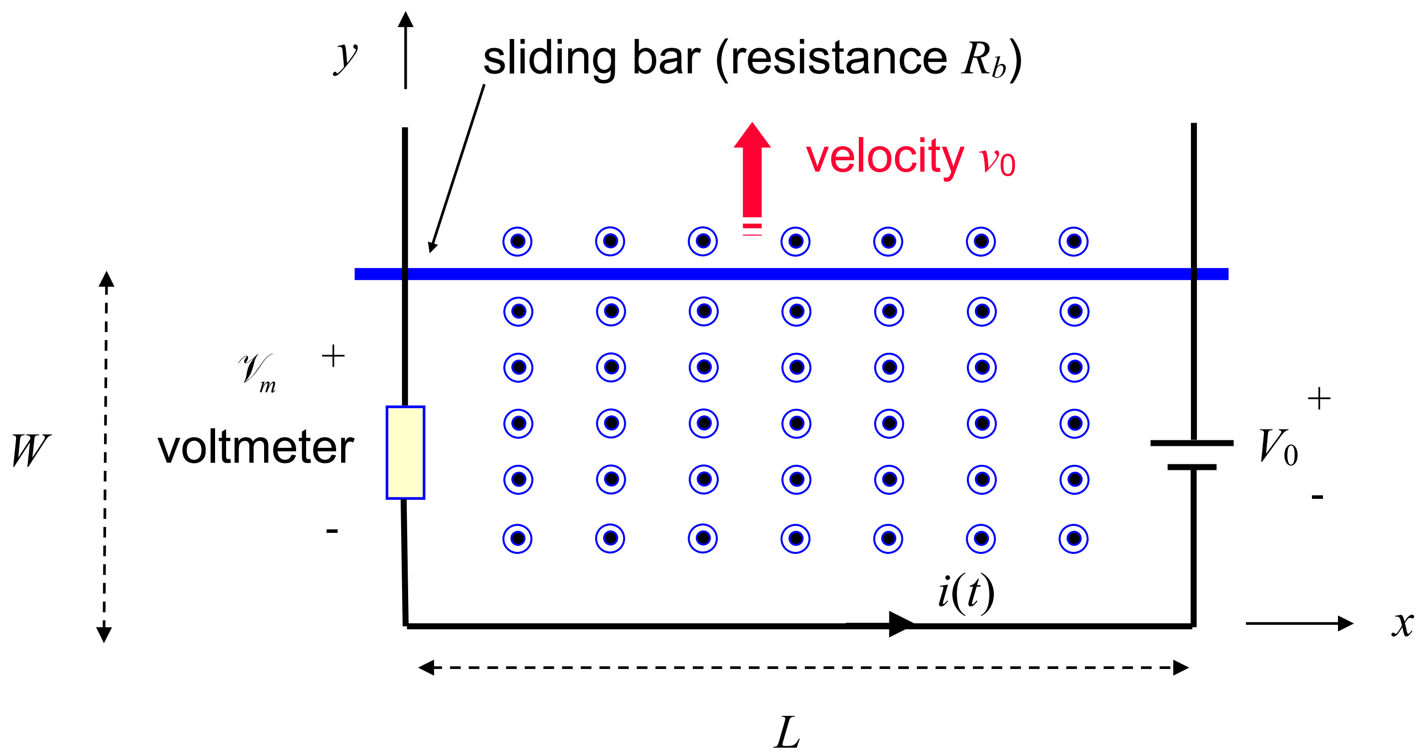
$$\underline{\mathscr{B}} = \underline{\hat{z}}\, e^{-t}\,.$$

The sliding bar has a resistance R_b. The other rails are perfectly conducting. A voltmeter is in the circuit, as shown below. Assume that this voltmeter has an internal resistance R_m. The circuit has an unknown current $i(t)$ flowing counterclockwise through it.

a) Determine the current $i(t)$.

b) Determine the voltage reading $\mathscr{V}_m$ on the voltmeter.

c) Determine the voltage drop $\mathscr{V}_{AB}$ across the sliding bar, where $\underline{A}$ is the left end of the bar and $\underline{B}$ is the right end.



Solution

Using the generalized Faraday's law and the generalized Ohm's law, we have

$$i(t)\left[R_b + R_m\right] - V_0 = EMF = -\frac{d\Psi}{dt}$$

We have

$$\Psi = e^{-t}(LW)$$

$$\frac{d\Psi}{dt} = -e^{-t}(LW) + e^{-t}(Lv_0).$$

Hence

$$i(t)\left[R_b + R_m\right] - V_0 = e^{-t}(LW) - e^{-t}(Lv_0).$$

We then have

$$i(t) = \frac{V_0 + e^{-t}(LW) - e^{-t}(Lv_0)}{R_b + R_m}.$$

The voltage drop across the voltmeter is then

$$v_m(t) = R_m i(t)$$

so that

$$v_m(t) = R_m\left[\frac{V_0 + e^{-t}(LW) - e^{-t}(Lv_0)}{R_b + R_m}\right].$$

To calculate the voltage drop across the bar, we use the regular Faraday's law:

$$\mathscr{V} = \oint_C \underline{\mathscr{E}} \cdot d\underline{r} = -\int_S \frac{\partial \underline{\mathscr{B}}}{\partial t} \cdot \underline{\hat{z}}\ dS\,.$$

Hence we have

$$-\mathscr{V}_0 - \mathscr{V}_b + \mathscr{V}_m = -\int_S \frac{\partial \mathscr{B}_z}{\partial t} \cdot dS = -LW\left(-e^{-t}\right).$$

We then have

$$\mathscr{V}_b = \mathscr{V}_m - \mathscr{V}_0 - LW\left(e^{-t}\right).$$

Problem 3 (40 pts)

Consider a hollow rectangular waveguide resonator as shown below. It consists of a hollow rectangular waveguide of length L with metal plates at the two ends ($z = 0$ and $z = L$). The electric field inside of the waveguide resonator at a resonant frequency ω is given by

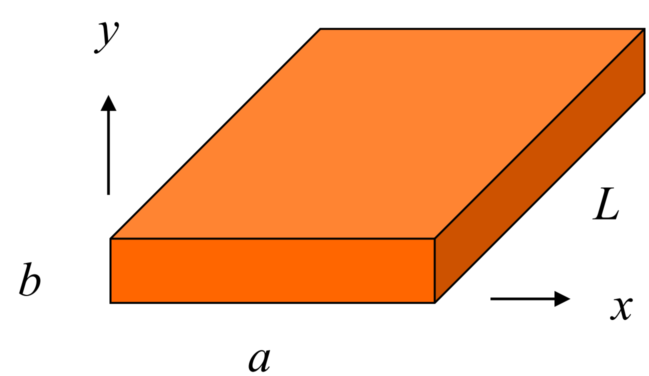
$$\underline{E} = \underline{\hat{y}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{L}\right).$$

a) Determine the magnetic field inside of the resonator.

b) Determine the time-average electric energy stored inside the resonator.

c) Determine the total complex power flowing down the waveguide in the z direction through the cross-sectional plane $z = L / 4$.

c) Determine the total time-average force on the plate at the left end of the resonator ($z = 0$).



Solution

Use Faraday's law to find the magnetic field:

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \,.$$

This gives us

$$H = -\frac{1}{j\omega\mu}\left[\underline{\hat{x}}\left(-\frac{\partial E_y}{\partial z}\right) + \underline{\hat{z}}\left(\frac{\partial E_y}{\partial x}\right)\right]$$

So that

$$H = -\frac{1}{j\omega\mu}\left[\underline{\hat{x}}\left(-\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{L}\right)\right) + \underline{\hat{z}}\left(\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{L}\right)\right)\right].$$

The time-average electric field stored inside the resonator is

$$\left\langle \mathcal{U}_E \right\rangle = \int_V \frac{1}{4}\varepsilon_0 \left|E_y\right|^2 dV = \frac{1}{4}\varepsilon_0\, b \int_0^a \int_0^L \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{L}\right) dz\, dx \,.$$

Performing the integrations, we have

$$\left\langle \mathcal{U}_E \right\rangle = \frac{1}{4}\varepsilon_0\, b\left(\frac{a}{2}\right)\left(\frac{L}{2}\right).$$

Hence

$$\left\langle \mathcal{U}_E \right\rangle = \varepsilon_0\left(\frac{abL}{16}\right).$$

Note: because the waveguide resonator is lossless, the time-average energy stored in the magnetic field is the same as that stored in the electric field.

The complex power flowing down the waveguide is

$$P_z = \int_S \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right)\cdot \underline{\hat{z}}\, dS = -\int_S \frac{1}{2} E_y H_x^*\, dS \,.$$

Hence we have

$$P_z = -\int_S \frac{1}{2}\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{L}\right)\right]\left[\frac{1}{j\omega\mu_0}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{L}\right)\right]^* dS$$

or

$$P_z = -j\frac{\pi / L}{2\omega\mu_0}\left[\sin\left(\frac{\pi z}{L}\right)\right]\left[\cos\left(\frac{\pi z}{L}\right)\right]\int_S \sin^2\left(\frac{\pi x}{a}\right) dS \,.$$

Performing the integration, we have

$$P_z = -j\frac{\pi / L}{2\omega\mu_0}\left[\sin\left(\frac{\pi z}{L}\right)\right]\left[\cos\left(\frac{\pi z}{L}\right)\right]\left(\frac{ab}{2}\right).$$

Substituting in for the value of z, we have

$$P_z = -j\frac{(\pi / L)}{2\omega\mu_0}\left[\frac{1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{2}}\right]\left(\frac{ab}{2}\right).$$

The final result is then

$$P_z = -j\frac{\pi}{8\omega\mu_0}\left(\frac{ab}{L}\right).$$

The complex power is purely imaginary, as expected, since the waveguide is lossless and terminated in a short circuit at the end.

To calculate the force we use the Maxwell stress tensor in free space,

$$T_{ij} = \varepsilon_0 \mathscr{E}_i \mathscr{E}_j + \mu_0 \mathscr{H}_i \mathscr{H}_j - \delta_{ij}\left[\frac{1}{2}\varepsilon_0 \left|\underline{\mathscr{E}}\right|^2 + \frac{1}{2}\mu_0 \left|\underline{\mathscr{H}}\right|^2\right].$$

The only nonzero component for the plate at $z = 0$ is

$$T_{zz} = -\frac{1}{2}\mu_0 \left|\mathscr{H}_x\right|^2.$$

Hence we have

$$T_{zz} = -\frac{1}{2}\mu_0 \left|\frac{1}{\omega\mu_0}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{a}\right)\cos\left(\omega t - \pi / 2\right)\right|^2.$$

The time average force per unit area is then

$$\langle T_{zz}\rangle = -\frac{1}{4}\mu_0 \left|\frac{1}{\omega\mu_0}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{a}\right)\right|^2.$$

Integrating over the plate at z = 0, we have

$$\langle \mathscr{F}_z\rangle = -\frac{1}{4}\mu_0 \left|\frac{1}{\omega\mu_0}\left(\frac{\pi}{L}\right)\right|^2\left(\frac{ab}{2}\right).$$

Hence we have

$$\langle \mathscr{F}_z\rangle = -\frac{1}{8}\left(\frac{1}{\omega^2\mu_0}\right)(ab)\left(\frac{\pi}{L}\right)^2.$$