NAME: ____________SOLUTION________________________

ECE 6340
Fall 2012

EXAM I

INSTRUCTIONS:

This exam is open-book and open-notes. You may use any material or calculator that you wish. Laptops or other devices that may be used to communicate are not allowed.

- Put all of your answers in terms of the parameters given in the problems, unless otherwise noted.
- Include units with all numerical answers.
- Please circle your final answers.
- Please write all of your work on the sheets attached (if you need more room, you may write on the backs of the pages).

Please show *all of your work* and *write neatly* in order to receive credit.

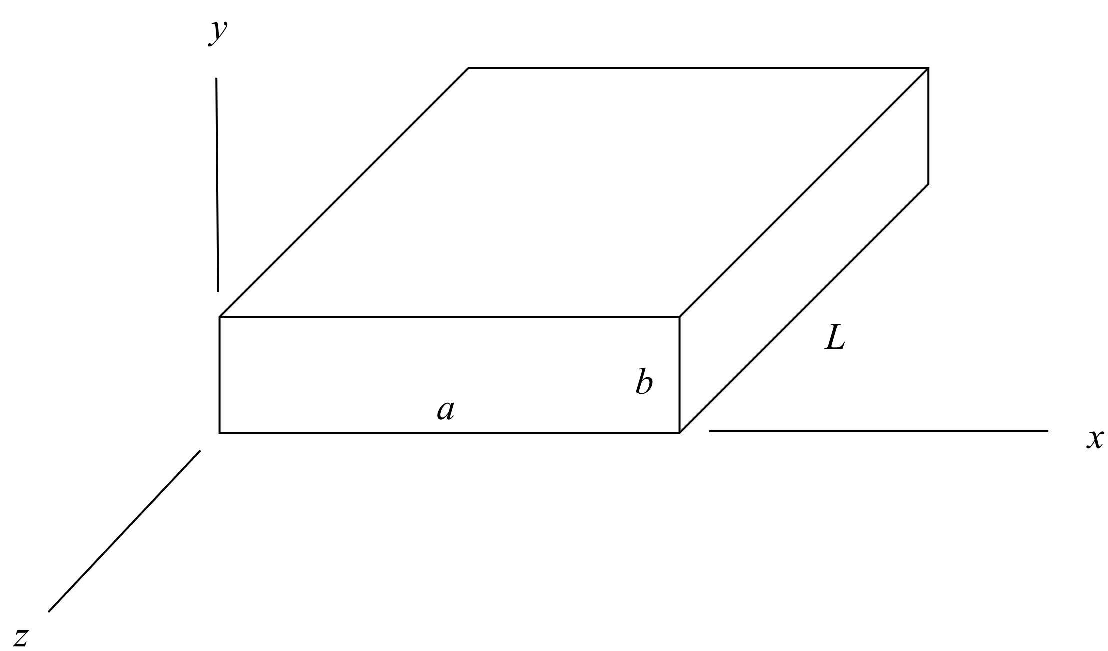
Problem 1 (25 pts)

The TE_{101} mode in a hollow perfectly conducting rectangular waveguide resonator (having dimensions $a \times b \times L$) has an electric field that is given by

$$\underline{E} = \underline{\hat{y}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{L}\right).$$

(There is also a magnetic field inside the resonator, but the formula for it is not given. Do not assume that the magnetic field is zero!)

Determine the total time-average force in the x direction $\langle \mathscr{F}_x \rangle$ on the right wall of the resonator at $x = a$.



ROOM FOR WORK

From Faraday's law, the magnetic field inside the resonator is

$$\underline{H} = \frac{1}{-j\omega\mu_0}\left[\hat{\underline{z}}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{L}\right) - \hat{\underline{x}}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{L}\right)\right].$$

The time-average force of the right wall is given in terms of the time-average pressure as

$$\langle \mathscr{F}_x \rangle = \int_0^L \int_0^b \langle \mathscr{P}_x \rangle \, dy dz.$$

The pressure is given in terms of the Maxwell stress tensor component as

$$\mathscr{P}_x = -\mathscr{T}_{xx}.$$

The stress tensor component is given by

$$\mathscr{T}_{xx} = \varepsilon_0 \mathscr{E}_x \mathscr{E}_x + \mu_0 \mathscr{H}_x \mathscr{H}_x - \left(\frac{1}{2}\varepsilon_0 |\underline{\mathscr{E}}|^2 + \frac{1}{2}\mu_0 |\underline{\mathscr{H}}|^2\right).$$

At the right wall ($x = a$) there is no electric field, and only a z component of the magnetic field. hence we have

$$\mathscr{T}_{xx} = -\left(\frac{1}{2}\mu_0 |\mathscr{H}_z|^2\right).$$

Taking the time average, we have, from the time-average theorem,

$$\langle \mathscr{T}_{xx} \rangle = -\left(\frac{1}{4}\mu_0 |H_z|^2\right) = -\frac{1}{4}\mu_0 \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{L}\right) = -\frac{1}{4}\mu_0 \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \sin^2\left(\frac{\pi z}{L}\right).$$

Hence we have

$$\langle \mathscr{F}_x \rangle = \frac{1}{4}\left(\frac{\pi}{a}\right)^2 \left(\frac{1}{\omega^2\mu_0}\right) \int_0^L \int_0^b \cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{L}\right) dy dz.$$

Hence we have

$$\left\langle \mathscr{F}_x \right\rangle = \frac{bL}{8}\left(\frac{\pi}{a}\right)^2 \frac{1}{\omega^2 \mu_0} \text{ [N]}.$$

Problem 2 (30 pts)

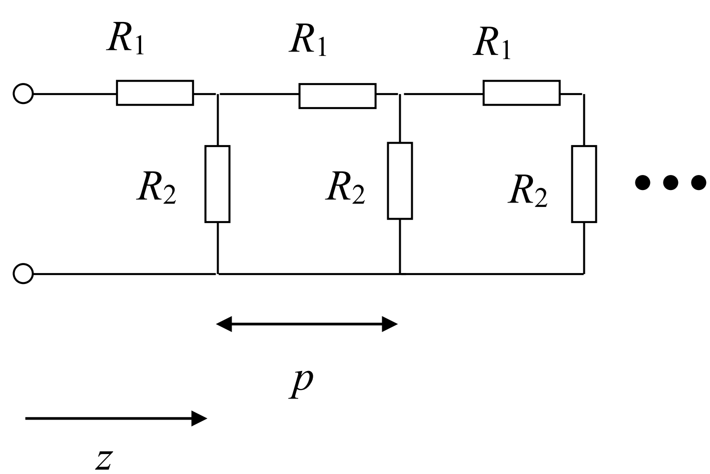
A semi-infinite cascade of resistors is shown below. Assume that the change in voltage across a unit cell of length p is small.

(a) Determine an approximate formula for the DC input impedance seen at $z = 0$ looking into the cascade of resistors, using transmission line theory (assuming that the length p is small).

(b) Assuming that a DC voltage of 1.0 Volt is applied at $z = 0$, determine an approximate formula for the voltage at $z = Np$ where N is an integer, using transmission line theory (assuming that the length p is small).

(c) Give a mathematical constraint that will ensure that the voltage variation across a unit cell is small, and hence ensure that your results are accurate. A good starting point would be to assume that the voltage magnitude changes by less than 10% as you go across a unit cell.

(d) Derive an exact expression for the DC input impedance seen looking into the cascade. Do this by taking advantage of the fact that the input impedance at $z = 0$ should be the same as the impedance seen looking to the right at $z = p$ (just to the right of the first resistor R_2), since the cascade is infinite.



ROOM FOR WORK

Part (a)

Using artificial transmission-line theory, we have that

$$Z = R_1\left(\frac{1}{p}\right).$$

$$Y = R_2\left(\frac{1}{p}\right).$$

The input impedance is the same as the characteristic impedance Z_0. Hence,

$$Z_{in} = Z_0 = \sqrt{ZY}.$$

We thus have

$$Z_{in} = \sqrt{R_1 R_2}.$$

Part (b)

The propagation wavenumber is given by

$$\gamma = \sqrt{Z/Y} = \frac{1}{p}\sqrt{\frac{R_1}{R_2}}.$$

Hence,

$$V\left(Np\right) = e^{-N\sqrt{R_1/R_2}} \quad \text{[V]}.$$

Part (c)

We should require that

$$e^{-\sqrt{R_1/R_2}} > 0.9 .$$

Part (d)

The exact input impedance is

$$\begin{aligned} Z_{in} &= R_1 + R_2 \parallel Z_{in} \\ &= R_1 + \frac{R_2 Z_{in}}{R_2 + Z_{in}}. \end{aligned}$$

This give us

$$Z_{in}\left(R_2 + Z_{in}\right) = R_1\left(R_2 + Z_{in}\right) + R_2 Z_{in}$$

or

$$Z_{in}^2 + \left(-R_1\right) Z_{in} - R_1 R_2 = 0.$$

The solution of this quadratic equation is

$$Z_{in} = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2}.$$

Note that the plus sign is chosen in the quadratic equation solution to give a positive result.

Continuing further (though not asked for in the problem), we can also write this as

$$\begin{aligned} Z_{in} &= \frac{1}{2} R_1 \left(1 + \sqrt{1 + 4R_2 / R_1}\right) \\ &= \frac{1}{2} R_1 \left(1 + \frac{R_2}{R_1} \sqrt{\left(\frac{R_1}{R_2}\right)^2 + 4\left(\frac{R_1}{R_2}\right)}\right) \\ &= \frac{1}{2} R_1 \left(1 + 2\frac{R_2}{R_1} \sqrt{\frac{R_1}{R_2}} \sqrt{1 + \frac{1}{4}\left(\frac{R_1}{R_2}\right)}\right). \end{aligned}$$

Under the assumption (see part c) that

$$\frac{R_1}{R_2} << 1,$$

we have

$$
\begin{aligned}
Z_{in} &\approx \frac{1}{2}R_1\left(1+2\frac{R_2}{R_1}\sqrt{\frac{R_1}{R_2}}\right) \\
&= \frac{1}{2}R_1\left(1+2\sqrt{\frac{R_2}{R_1}}\right) \\
&= \frac{R_1}{2}+\sqrt{R_1R_2} \\
&= R_2\left(\frac{1}{2}\frac{R_1}{R_2}+\sqrt{\frac{R_1}{R_2}}\right) \\
&\approx R_2\left(\sqrt{\frac{R_1}{R_2}}\right) \\
&= \sqrt{R_1R_2}.
\end{aligned}
$$

This agrees with the result from part (a).

Problem 3 (20 pts)

A shallow dish of water is placed in a microwave oven as shown below. The frequency of the oven is $f = 2.54$ [GHz]. The volume of the water is $V = 100$ [cm^3] = 10^{-4} [m^3]. Immediately above the water, in the air region, the electric field is perfectly vertical (in the z direction) and uniform, and is given by

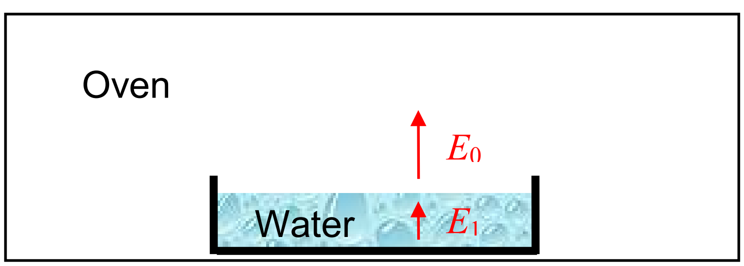
$$\underline{\mathscr{E}}_0 = \underline{\hat{z}}\, A_0 \cos(\omega t),$$

where $A_0 = 3000$ [V/m]. Assume that the electric field $\underline{\mathscr{E}}_1$ inside the water is also uniform and vertical (but is not necessarily the same as the electric field $\underline{\mathscr{E}}_0$ above the water). The water has a complex permittivity given by

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0 (80 - j10).$$

(a) Determine the electric field $\underline{\mathscr{E}}_1(t)$ inside the water as a function of time.

(b) Determine the time-average power dissipated inside the water.



ROOM FOR WORK

Part (a)

The general boundary condition is that

$$\varepsilon_0 E_{z0} = \varepsilon_{c1} E_{z1}.$$

Hence we have that

$$\underline{E}_1 = \underline{\hat{z}}\left(\frac{1}{\varepsilon_{rc}} E_{z0}\right).$$

The water is assumed to be pure water, and hence there is no conductivity, only polarization loss. Hence, we have that

$$\varepsilon_{rc} = \varepsilon_r = 80 - j10.$$

In the phasor domain, we have

$$E_{z0} = A_0.$$

Hence, in the phasor domain, the electric field inside the water is

$$\underline{E}_1 = \underline{\hat{z}}\left(\frac{A_0}{80 - j10}\right).$$

We convert this into the time domain by using

$$\underline{\mathscr{E}}_1(t) = \mathrm{Re}\left(\underline{E}_1\, e^{j\omega t}\right).$$

Hence we have

$$\begin{aligned}\underline{\mathscr{E}}_1(t) &= \underline{\hat{z}} A_0 \,\mathrm{Re}\left(\left(\frac{1}{80 - j10}\right) e^{j\omega t}\right) \\ &= \underline{\hat{z}} A_0 \,\mathrm{Re}\left(\left(0.01240 e^{j(0.1244)}\right) e^{j\omega t}\right).\end{aligned}$$

Hence,

$\underset{\sim}{\mathscr{E}}_4(t) = \hat{\underline{z}}\, A_0 (0.01240)\cos(\omega t + 0.1244) \quad$ [V/m].

Inserting the numerical values,

$$\underset{\sim}{\mathscr{E}}_4(t) = \hat{\underline{z}}(37.2)\cos\left(2\pi\left(2.54\times10^9\right)t + 0.1244\right) \quad \text{[V/m]}.$$

Part (b)

The power dissipated is

$$\langle \mathscr{P}_d \rangle = \int_V \frac{1}{2}(\omega\varepsilon'')|\underline{E}|^2\, dV$$
$$= \frac{1}{2}(\omega\varepsilon'')|\underline{E}|^2\, V.$$

Hence,

$$\langle \mathscr{P}_d \rangle = \frac{1}{2}(\omega\varepsilon'')|E_{z1}|^2\, V.$$

or

$$\langle \mathscr{P}_d \rangle = \frac{1}{2}(\omega\varepsilon'')\left|\frac{E_{z0}}{\varepsilon}\right|^2 V.$$

Inserting the values, we have

$$\langle \mathscr{P}_d \rangle = \frac{1}{2}\left(2\pi\left(2.54\times10^9\right)\left(8.854\times10^{-12}\right)(10)\right)\left|\frac{3000}{80 - j10}\right|^2\left(10^{-4}\right) \text{[W]}.$$

The numerical result is

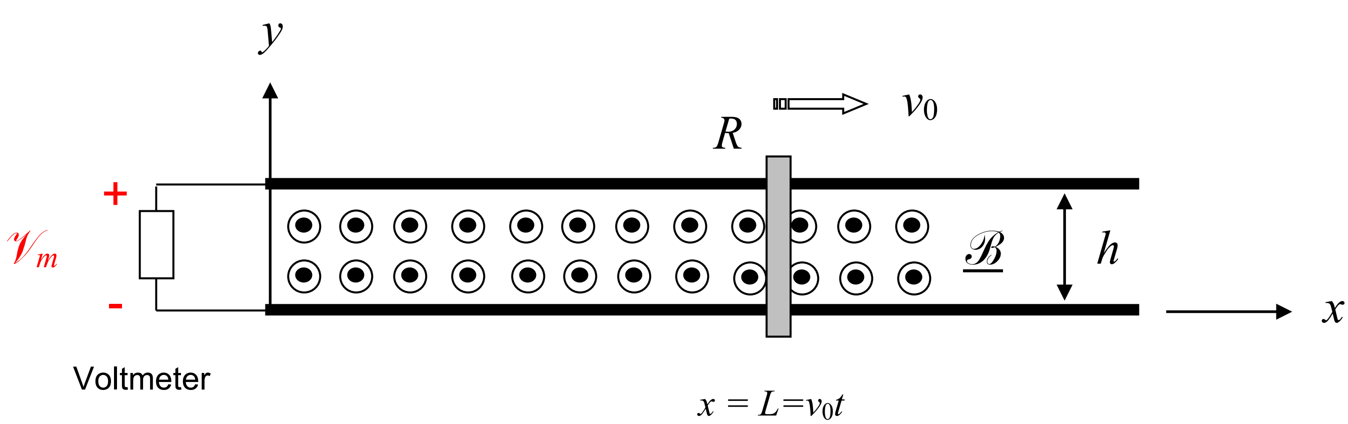
$$\langle \mathscr{P}_d \rangle = 0.0978 \text{ [W]}.$$

Problem 4 (25 pts)

A voltmeter is connected to a set of perfectly conducting rails as shown below. The voltmeter has an internal resistance of R_m, and hence can be modeled as this resistance. Between the two rails is a sliding resistor with a resistance of R. At time t the resistor is located at $x = L = v_0 t$ and is moving to the right with a constant velocity v_0. Between the rails there is a magnetic field that is given by

$$\underline{\mathscr{B}}(x,t) = \underline{\hat{z}}\, e^{-x} \cos(\omega t).$$

Determine the voltage reading on the voltmeter $\mathscr{V}_m(t)$ as a function of time.



ROOM FOR WORK

The generalized Faraday's law states that

$$EMF = -\frac{d\psi}{dt}.$$

From the generalized Ohm's law, we also know that

$$EMF = i(t)(R + R_m).$$

Hence,

$$i(t) = \frac{EMF}{R + R_m}.$$

The voltmeter reading is then given by

$$\mathcal{V}_R(t) = R_m i(t) = \left(\frac{R_m}{R + R_m}\right) EMF.$$

Therefore, we have

$$\mathcal{V}_R(t) = -\left(\frac{R_m}{R + R_m}\right)\frac{d\psi}{dt}.$$

The magnetic flux through the path is

$$\psi = h\int_0^L e^{-x}\cos(\omega t)\,dx = h\cos(\omega t)\left(1 - e^{-L}\right) = h\cos(\omega t)\left(1 - e^{-v_0 t}\right).$$

Hence, taking the derivative, we have

$$\frac{d\psi}{dt} = -h\omega\sin(\omega t)\left(1 - e^{-v_0 t}\right) + hv_0\cos(\omega t)e^{-v_0 t}.$$