

ECE 6340

Fall 2016

Homework 1

- 1) Show that $\rho_v = 0$ for a time-harmonic (i.e., sinusoidal steady-state) field in a source-free homogeneous region of matter, having an arbitrary complex permittivity ϵ . Homogeneous means that the material properties do not vary with position inside the material. Source-free means that there are no current sources that have been placed inside the body. Hence, the only current that exists inside the body is conduction current, which is given by Ohm's law as

$$\underline{\mathcal{J}} = \sigma \underline{\mathcal{E}},$$

or in the phasor domain as

$$\underline{J} = \sigma \underline{E}.$$

The result ($\rho_v = 0$) is valid unless we have both zero conductivity and zero frequency.

Note: This result implies that in the time-harmonic steady state, there can never be any volume charge density inside a homogeneous source-free body!

(Hint: First use Ampere's law to show that in the time-harmonic steady-state, the divergence of the electric field must be zero inside a source-free, homogeneous material.)

- 2) Assume that we have a conductor with conductivity σ that is moving with a velocity vector \underline{v} in a magnetic field $\underline{\mathcal{B}}$. Show that the current density inside the conductor is given by

$$\underline{\mathcal{J}} = \sigma (\underline{\mathcal{E}} + \underline{v} \times \underline{\mathcal{B}}).$$

Hint: Recall that the Lorentz force law states that the force on a charged particle is

$$\underline{\mathcal{F}} = q (\underline{\mathcal{E}} + \underline{v} \times \underline{\mathcal{B}}).$$

Also, note that the current density inside of a conducting body, in general, should be proportional to the *force* on the charge carriers inside the conductor. The constant of proportionality should not depend on whether or not there is a magnetic field. If there is no magnetic field, we can assume that the usual Ohm's law $\underline{\mathcal{J}} = \sigma \underline{\mathcal{E}}$ holds.

- 3) A perfectly conducting body ($\sigma \rightarrow \infty$) is moving with a velocity \underline{v} inside of a magnetic field $\underline{\mathcal{B}}$. Show that the electric field inside the body is

$$\underline{\mathcal{E}} = -\underline{v} \times \underline{\mathcal{B}}.$$

(You may assume that the current density always remains finite.)

- 4) A resistor R is moving in a magnetic field $\underline{\mathcal{B}}$ with a velocity vector \underline{v} . (The resistor may be assumed to have a cylindrical-shaped body with cross-sectional area A for convenience, oriented along the x axis, as shown below.) The fields may be assumed to be constant inside the resistor. Show that the voltage drop across the resistor is related to the current i through the resistor by the “generalized Ohm’s law” expression

$$\Delta\text{EMF}_{AB} = Ri,$$

where ΔEMF_{AB} is the change in EMF across the ends of the resistor, defined by

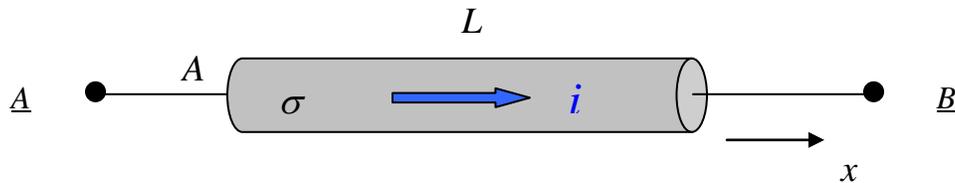
$$\Delta\text{EMF}_{AB} = \int_A^B (\underline{\mathcal{E}} + \underline{v} \times \underline{\mathcal{B}}) \cdot d\underline{r} = L(\underline{\mathcal{E}} + \underline{v} \times \underline{\mathcal{B}}) \cdot \hat{x}.$$

In this equation $\underline{\mathcal{E}}$ is the electric field inside the resistor. The term $i = i(t)$ is the current that flows through the resistor from A to B . (Note that the change in EMF is the same as the voltage drop across the resistor, if the resistor is not moving, or if there is no magnetic field.) The resistance is

$$R = \frac{L}{\sigma A},$$

where L is the length of the resistor and A is the cross-sectional area.

Hint: The derivation should parallel the derivation of the usual Ohm’s law for a non-moving resistor, but be generalized a bit. You may assume that the current i through the resistor is purely in the x direction for convenience.



- 5) A conducting rod with a resistance R slides along a set of perfectly conducting (stationary) tracks as shown below. A magnetic field $\underline{\mathcal{B}}$ exists everywhere in space. The magnetic field is given by

$$\underline{\mathcal{B}} = \hat{z} \cos(\omega t).$$

At the left end of the track is a battery with a fixed voltage V_b (see the figure below). The resistor is either at $x = L$ and is moving to the right with a fixed velocity v_0 , or it is at $x = L$ and is stationary, as indicated in each part below. Note that in either case, the area of the path is $A = hL$. If the resistor is moving, then L is a function of time, and $dL/dt = v_0$.

Note: You should not have to assume that $L = v_0 t$. That is, the velocity of the resistor does not have to be constant with respect to time (it could be, but it does not have to be). The only thing that is important for solving the problem is that when the resistor is at $x = L$ the velocity is v_0 .

- a) Use the direct form of Faraday's law to calculate the voltage drop $\mathcal{V}(t)$ around the closed path consisting of the battery, the tracks, and the resistor,

$$\mathcal{V} = \oint_C \underline{\mathcal{E}} \cdot \underline{dr},$$

where the path C is defined in the counterclockwise sense. Assume that the resistor is at $x = L$ and is not moving.

- b) How would the answer to part (a) change if the resistor were at $x = L$ but was moving to the right with a velocity v_0 ? Leave your answer in terms of L .
- c) Use the EMF form of Faraday's law to calculate the EMF around the same closed path. Assume that the resistor is at $x = L$ and is moving to the right with a velocity v_0 .
- d) Assume that the resistor is at $x = L$ and is moving to the right with a velocity v_0 . Calculate the EMF around the same closed path by calculating the term

$$\oint_C (\underline{v}_s \times \underline{\mathcal{B}}) \cdot \underline{dr},$$

and adding it to the expression obtained in part (b). (The velocity \underline{v}_s is the velocity at any point on the closed path.) Your answer for the EMF should be the same as that found in part (c).

- e) Using your answer to part (c), find the drop in EMF, ΔEMF_{AB} , across the resistor when it is at $x = L$ and is moving to the right with a velocity v_0 . Point A is on top of the resistor and point B is on the bottom.
- f) Use your answer to part (a) to find the voltage drop $\mathcal{V}_R(t)$ across the resistor (note the reference direction for voltage defined in the figure), assuming that the resistor is located at $x = L$ and is not moving.
- g) Use your answer to part (b) to find the voltage drop $\mathcal{V}_R(t)$ across the resistor, assuming it is located at $x = L$ and it is moving to the right with a constant velocity v_0 .
- h) Find the current $i(t)$ flowing in the downward direction through the resistor when it is located at $x = L$ and is not moving. Do this by using Ohm's law together with your answer to part (f).
- i) Find the current $i(t)$ flowing in the downward direction through the resistor when it is located at $x = L$ and it is moving to the right with a constant velocity v_0 . Do this by using the generalized Ohm's law together with your answer to part (e).

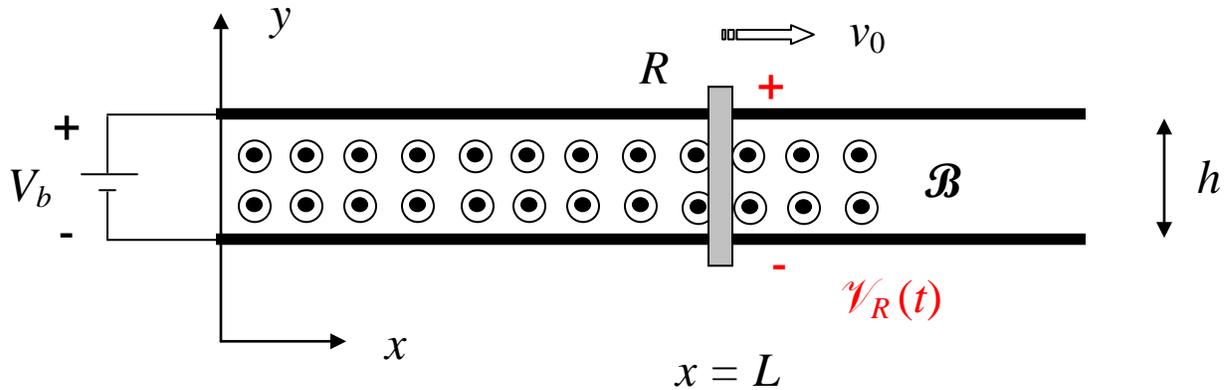
Note: Both Faraday's law and the "generalized Faraday's law" are both valid laws that apply in all situations, regardless of whether or not the path is moving. If the path is not moving, the two laws immediately reduce to the same equation. If the path is moving (i.e., path C is a function of time), then the two laws appear to be different, but (as shown in class) they are equivalent.

The direct form of Faraday's law:

$$\mathcal{V} = \oint_C \underline{\mathcal{E}} \cdot \underline{dr} = \int_S -\frac{\partial \underline{\mathcal{B}}}{\partial t} \cdot \underline{\hat{n}} dS$$

The EMF form of Faraday's law:

$$\text{EMF} = \oint_C \left(\underline{\mathcal{E}} + \underline{v}_s \times \underline{\mathcal{B}} \right) \cdot \underline{dr} = -\frac{d}{dt} \int_S \underline{\mathcal{B}} \cdot \underline{\hat{n}} dS = -\frac{d\psi}{dt}$$



- 6) A rectangular loop of perfectly conducting wire is inside a uniform magnetic field that is given by

$$\underline{\mathcal{B}} = \hat{x} B_0.$$

The loop is rotating about the z axis with an angular velocity ω [rad/s]. (The direction of rotation matches the fingers of the right hand, if the thumb points along the z axis.) At $t = 0$ the loop lies in the xz plane as shown below. Calculate the voltage drop across the two output terminals \mathcal{V}_{AB} of the loop. (Note that this is the basic principle on which an AC generator works.) Please pay attention to get the sign right! You may assume that the output terminals AB are outside the region of magnetic field. Only the rectangular loop is inside the region of magnetic field.

Use the EMF form of Faraday's law.

- 7) Repeat the previous problem using the direct form of Faraday's law. (Hint: This will involve calculating the voltage drops along the different segments of the loop. This calculation will involve calculating the electric field inside the wire segments. See Prob. 3 to help with this.)

