

ECE 6340
Fall 2016

Homework 7

Assignment: Please do Probs. 1-5, 7, 9, 12, 16, 19, 24, 25 from the set below. You are welcome to do other problems for your own benefit.

- 1) A linearly-polarized plane wave is propagating in the z direction. In the $z = 0$ plane the electric field vector in the phasor domain is given by $\underline{E} = \underline{R} = \hat{x}a + \hat{y}b$ where \underline{R} is a real vector that lies in the $z = 0$ plane. Show that the magnitude of the vector \underline{E} in the phasor domain is the peak magnitude of the electric field vector in the time domain.

Next, consider a right-handed circularly-polarized plane wave propagating in the z direction. In the $z = 0$ plane the electric field vector in the phasor domain is given by $\underline{E} = \hat{x}a + \hat{y}(-ja)$. Calculate the magnitude of the electric field vector in the phasor domain and the peak magnitude of the electric field vector in the time domain. Are they equal?

- 2) Determine the polarization type of each plane wave listed below. The choices are LP, RHCP, LHCP, RHEP, and LHEP. (If the wave is circularly-polarized, you need to indicate this by saying RHCP or LHCP.)

a) $\underline{E} = [\hat{x}(2-j) + \hat{y}(3+j)] e^{-jkz}$
b) $\underline{E} = [\hat{x}(1-j) + \hat{y}(1+j)] e^{+jkz}$
c) $\underline{E} = [\hat{z}(1-j) + \hat{y}(3-j)] e^{+jkx}$
d) $\underline{E} = [\hat{y}(1-j) + \hat{z}(-4+4j)] e^{-jkx}$
e) $\underline{E} = [\hat{z}(j-1) + \hat{x}(3+3j)] e^{-jky}$.

- 3) For the plane wave in part (a) in the previous problem, determine the following:

- (a) Axial ratio AR
(b) Tilt angle τ of the polarization ellipse (in radians)
(c) Coordinates (θ, ϕ) on the Poincaré sphere (in radians).

Pick $\omega = 2\pi$ [rad/s] (a convenient choice, since the period is then one second), and make a plot of the electric field vector as a function of time. This will allow you to remove the $\pi/2$ ambiguity in the tilt angle. Include this plot as part of your solution.

- 4) A wire antenna in free space that is oriented along the x -axis is used to receive a signal from an incoming wave, which may be modeled as a plane wave. The open-circuit voltage V_T at the terminals of the antenna (the Thevenin voltage for the antenna) can be expressed as

$$V_T = h_{eff} E_x^{inc},$$

where h_{eff} is the “effective height” of the antenna and E_x^{inc} is the x component of the electric field of the incident plane wave.

Calculate the magnitude of the normalized Thevenin voltage (defined as $V_T^N = V_T / h_{eff}$) for the following incident plane waves, assuming that each plane wave has an incident power density of 1 [W/m²].

- a) A linearly polarized plane wave, polarized in the x direction and traveling in the z direction.
 - b) A RHCP wave, traveling in the z direction.
 - c) A LHCP wave, traveling in the z direction.
- 5) A RHCP antenna in free space is designed to receive a RHCP wave traveling in the z direction. It consists of two wire antennas, one oriented in the x direction and one oriented in the y direction. The signals are combined from each antenna with a 90° phase shift between the two antennas, so that the total open-circuit receive voltage is

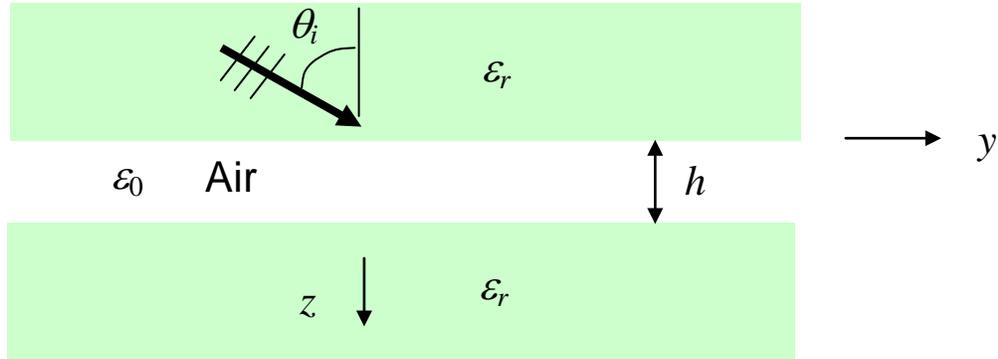
$$V_T = h_{eff} (E_x^{inc} + jE_y^{inc}).$$

Calculate the magnitude of the normalized Thevenin voltage (defined as $V_T^N = V_T / h_{eff}$) for the following incident plane waves, assuming that each plane wave has an incident power density of 1 [W/m²].

- a) A linearly polarized plane wave, polarized in the x direction and traveling in the z direction.
- b) A RHCP wave, traveling in the z direction.
- c) A LHCP wave, traveling in the z direction.

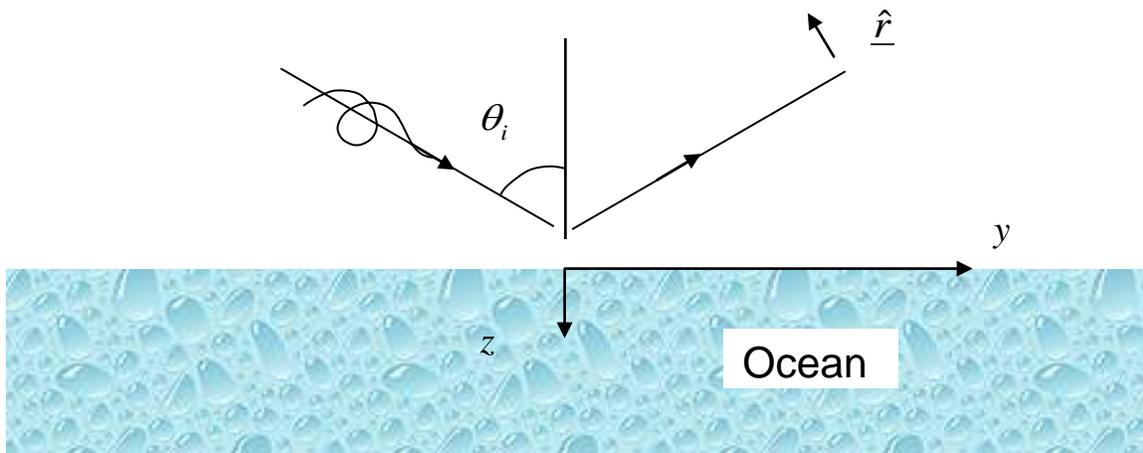
- 6) This problem explores the pros and cons of using circular polarization vs. linear polarization in a transmitted wave, when the receive antenna is linearly polarized. This problem uses the concept of effective height of a wire antenna, as explained in Prob. 4.
- a) A vertically-polarized plane wave is incident in free space on a wire antenna that is vertical. The wire antenna delivers a certain amount of power to a load that it is connected to. How many dB will be lost in the power delivered to the load if the wire antenna is then oriented at a 30° angle with respect to vertical? What about a 60° angle? What about a 90° angle? (i.e., the wire antenna is now horizontal).
 - b) Next, assume that a circularly-polarized wave is incident in free space on the same wire antenna. Again, the wire antenna delivers a certain amount of power to a load that it is connected to. How many dB will be lost in the power delivered to the load if the wire antenna is then oriented at a 30° angle with respect to vertical? What about a 60° angle? What about a 90° angle? (i.e., the wire antenna is now horizontal).
 - c) Next, assume that both the vertically-polarized plane wave and the circularly-polarized plane wave each carry $1 \text{ [W/m}^2\text{]}$ of power density. Calculate the ratio of power received by the antenna (i.e., delivered to the load) when it is illuminated by the vertically-polarized plane wave to the power received by the antenna when it is illuminated by the circularly polarized plane wave, assuming that the antenna is vertical. Then calculate this power ratio when the wire antenna is oriented at a 30° angle with respect to vertical. Repeat for a 60° angle. Finally, repeat for a 90° angle (i.e., the wire antenna is now horizontal).
- 7) A RHCP antenna delivers a certain amount of power to a load when the incident signal is RHCP. Now suppose the incident signal is elliptically polarized with an axial ratio of 2.0. The incident wave has the same power density as it did before. How much of a dB loss will there be in the received signal (i.e., in the power delivered to the load)? You may assume that the RHCP antenna is an ideal RHCP antenna that delivers no power to the load when the incident signal is LHCP. Also, assume that the elliptically-polarized wave is predominantly right-handed. That is, the magnitude of the RHCP component is larger than the magnitude of the LHCP component.
- 8) Prove that for a plane wave in a lossless region $\underline{\beta} \cdot \underline{\alpha} = 0$. Hint: Start with the separation equation, and consider the real and imaginary parts of this equation.

- 9) A TE_z plane wave traveling in a region of glass ($\epsilon_r = 4.0$) is incident at an angle of $\theta_i = 60^\circ$ on an air gap as shown below (note that this is beyond the critical angle for the air region). Make a plot of the % power reflected as a function of h / λ_0 .



- 10) A RHCP wave from a GPS satellite is incident on the surface of the ocean as shown below, with an incident angle of $\theta_i = 45^\circ$. The frequency is 1.575 GHz. The ocean water has a complex relative permittivity due to polarization losses that is $\epsilon_r = 80 (1 - j(0.1))$, and also a conductivity that is $\sigma = 4.0$ [S/m]. Determine the percentage of power that is reflected, the axial ratio, and the handedness of the reflected wave.

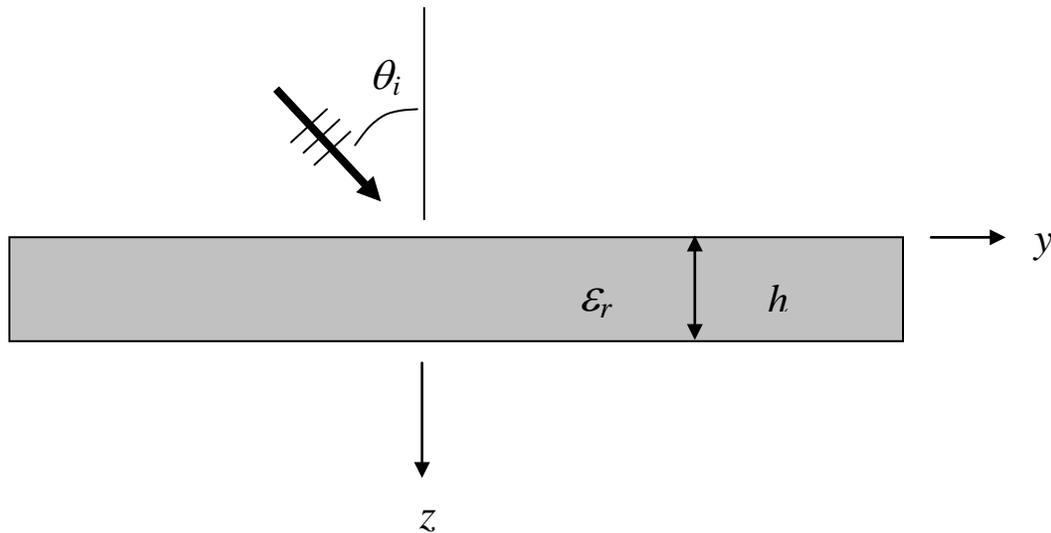
Note that in the calculation of axial ratio and tilt angle, the components E_x and E_r are now playing the roles of E_x and E_y that are used when the wave is traveling in the z direction. (The formulas for axial ratio given in the class notes assume that the direction of propagation is z).



- 11) Calculate and plot the percentage of transmitted power versus the incident angle θ_i (in degrees) for both a TE_z wave and a TM_z wave that is incident on a dielectric slab as shown below. (You will have two separate plots on the same graph.) The lossless non-magnetic slab has a relative permittivity of $\epsilon_r = 2.2$ and a thickness of $h/\lambda_0 = 0.5/\sqrt{\epsilon_r}$.

Note that the slab is exactly one-half of a wavelength thick in the dielectric. For normal incidence, can you predict what the reflection coefficients should be in view of this? Do your results confirm this prediction?

What is the Brewster angle for a wave incident on the slab material? Comment on what happens in your results at this angle.



- 12) In dynamics, we have the equation

$$\underline{E} = -j\omega\underline{A} - \nabla\Phi.$$

- (a) Show that in statics, the scalar potential function Φ can be interpreted as a voltage function. That is, show that in statics

$$V_{AB} \equiv \int_A^B \underline{E} \cdot d\underline{r} = \Phi(\underline{A}) - \Phi(\underline{B}).$$

- (b) Next, explain why this equation is not true (in general) in dynamics. Derive a formula for V_{AB} in terms of the scalar potential function and the magnetic vector potential \underline{A} .
- (c) Explain why the voltage drop (defined as the line integral of the electric field, as defined above) depends on the path from \underline{A} to \underline{B} in dynamics, using Faraday's law in integral form. Start by considering two different paths from \underline{A} to \underline{B} and form a closed path from these two paths.
- (d) Does the right-hand side of the above equation (the difference in the potential function) depend on the path, in dynamics?

Hint: Note that, according to calculus, for any function ψ we have

$$\nabla \psi \cdot d\underline{r} = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz = d\psi .$$

- 13) Starting with Maxwell's equations, show that the electric field radiated by an impressed current density source \underline{J}^i in an infinite homogeneous region satisfies the equation

$$\nabla^2 \underline{E} + k^2 \underline{E} = \nabla(\nabla \cdot \underline{E}) + j\omega\mu \underline{J}^i .$$

Then use Ampere's law (or, if you prefer, the continuity equation and the electric Gauss law) to show that this equation may be written as

$$\nabla^2 \underline{E} + k^2 \underline{E} = -\frac{1}{\sigma + j\omega\epsilon} \nabla(\nabla \cdot \underline{J}^i) + j\omega\mu \underline{J}^i .$$

Note that the total current density is the sum of the impressed current density and the conduction current density, the latter obeying Ohm's law ($\underline{J}^c = \sigma \underline{E}$).

Explain why this equation for the electric field would be harder to solve than the equation that was derived in class for the magnetic vector potential.

- 14) Show that magnetic field radiated by an impressed current density source satisfies the equation

$$\nabla^2 \underline{H} + k^2 \underline{H} = -\nabla \times \underline{J}^i .$$

Explain why this equation for the magnetic field would be harder to solve than the equation that was derived in class for the magnetic vector potential.

15) Show that in a homogenous region of space the scalar electric potential satisfies the equation

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho_v^i}{\epsilon_c},$$

where ρ_v^i is the impressed (source) charge density, which is the charge density that goes along with the impressed current density, being related by

$$\nabla \cdot \underline{J}^i = -j\omega\rho_v^i$$

Hint: Start with $\underline{E} = -j\omega\underline{A} - \nabla\Phi$ and take the divergence of both sides. Also, take the divergence of both sides of Ampere's law and use the continuity equation for the impressed current (given above) to show that

$$\nabla \cdot \underline{E} = -\frac{1}{j\omega\epsilon_c} \nabla \cdot \underline{J}^i = \frac{\rho_v^i}{\epsilon_c}.$$

Note: It is also true from the electric Gauss law that

$$\nabla \cdot \underline{E} = \frac{\rho_v}{\epsilon},$$

but we prefer to have only an impressed (source) charge density on the right-hand side of the equation for the potential Φ . In the time-harmonic steady state, assuming a homogeneous and isotropic region, it follows that $\rho_v = \rho_v^i$. That is, there is no charge density arising from the conduction current. (If there were no impressed current sources, the total charge density would therefore be zero everywhere.)

16) Assume that we have an infinite homogenous region of space with sources inside of it. Show that the electric potential is given in terms of the impressed (source) charge density ρ_v^i by

$$\Phi(\underline{r}) = \int_V \frac{\rho_v^i(\underline{r}')}{4\pi\epsilon_c R} e^{-jkR} dV',$$

where $R \equiv |\underline{r} - \underline{r}'|$.

Do this by comparing the differential equation for Φ (from the previous problem) with that derived in class for A_z , using the known solution for A_z to the maximum extent possible to avoid any unnecessary re-derivation.

- 17) Assume a z -directed dipole centered at the origin having a constant current I (in the phasor domain) and a length d . The length d is very small compared to a wavelength. Use the conservation of charge equation to calculate the amplitude of the point charge q (in the phasor domain) that sits on the top of the dipole (at $z = d / 2$), in terms of the current I . (An equal and opposite charge $-q$ will sit on the bottom of the dipole at $z = - d / 2$).

Note that the conservation of charge equation (continuity equation) in the time domain states that the current $i(t)$ that enters a region of space must equal the rate of change of total charge Q that is inside the region. That is,

$$i = \frac{dQ}{dt}.$$

- 18) Use the result of Prob. 5 to show that the potential produced by the z -directed dipole of length d in the previous problem is given as

$$\Phi(\underline{r}) = \left(\frac{q}{4\pi\epsilon_c} \right) \left(\frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2} \right),$$

where R_1 and R_2 are the distances from the observation point $\underline{r} = (x, y, z)$ to the top and bottom points of the dipole, respectively, and r is the distance from the origin to the observation point.

Next, approximate the terms R_1 and R_2 in the above result assuming that d is small compared to r . Obtain the approximate result

$$\Phi(\underline{r}) = e^{-jkr} \left(\frac{qd}{4\pi\epsilon_c r^2} \right) (1 + jkr) \cos \theta,$$

where θ is the usual angle in spherical coordinates. (This approximate result becomes exact in the limit that d tends to zero, corresponding to an infinitesimal dipole.)

- 19) Solve for the potential function $\Phi(r, \theta)$ produced by the infinitesimal dipole. Do this by using the solution for the magnetic vector potential in spherical coordinates, and the Lorenz Gauge. Simplify the answer as much as possible.

Verify that your result is the same as what you obtained in Prob. 18, using Prob. 17 to relate q and I . (Note that in this problem the length of the dipole is called l (using the class notation), while in Prob. 18 it is called d .)

- 20) Derive the complete fields of the infinitesimal dipole, as given in the class notes “Radiation.” Do this by converting A_z to spherical coordinates and then performing the curl operations in spherical coordinates.
- 21) Starting with the frequency-domain result derived in class, show that in the time domain the far field radiated by a small z -directed dipole of length l and current $i(t)$ at the origin is given by

$$\underline{E}(r, \theta, t) = \underline{\hat{\theta}} \left(\frac{\mu l}{4\pi r} \right) \sin \theta \frac{d}{dt} i(t - r/c).$$

- 22) A rectangular microstrip patch antenna has an approximate length

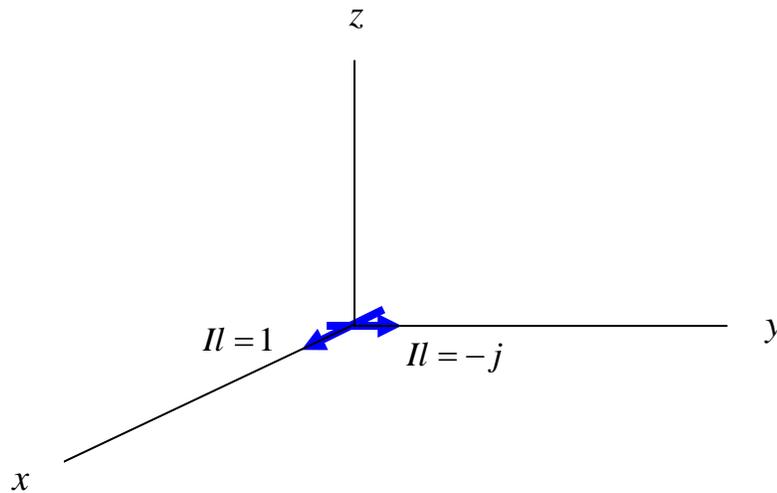
$$L = \lambda_0 / (2\sqrt{\epsilon_r}).$$

The substrate relative permittivity is $\epsilon_r = 2.94$. The width of the microstrip antenna is chosen as 1.5 times the length. The antenna operates at 2.0 GHz. The substrate and ground plane are assumed to be infinite.

- a) What is the distance from the center of the antenna that one must go in order to be in the far field, using the usual far-field criterion? Assume that the far-field formula for the radiation from a microstrip antenna already accounts for the infinite substrate and ground plane (e.g., this is the case when using the reciprocity formulation as given in Notes 28). Hence, the ground plane and substrate are not considered to be part of the antenna “body” when calculating the antenna size in the far-field formula. Only the metal patch is considered.
- b) How would the answer change if the microstrip antenna is on a finite-size ground plane that is square in shape with an edge length of 5 wavelengths? Now the finite ground plane is considered to be part of the radiating structure and we wish to be in the far field of the entire structure.
- 23) Make a plot of the radiation resistance of a wire antenna versus the normalized electrical half-length of the antenna, h/λ_0 . Plot up to $h/\lambda_0 = 1.0$, on a scale that goes from zero to a maximum of 300 [Ω].

- 24) A RHCP antenna consists of two unit-amplitude infinitesimal electric dipoles at the origin. (The antenna radiates RHCP in the positive z direction, but LHCP in the negative z direction.) One dipole is in the x direction, with a dipole moment $I\ell = 1$. The other dipole is in the y direction, with a dipole moment $I\ell = -j$. Determine the far-zone electric field for this antenna (both E_θ and E_ϕ). Do this by first deriving the magnetic vector potential in the far field and then using far-field theory to obtain the far-zone electric field from the magnetic vector potential.

(One can also obtain the solution for the far-zone electric field of the RHCP antenna by starting with the far-zone electric field of a dipole in the z direction, as given in class, followed by a rotation of coordinates, but do not proceed this way.)



- 25) An electric surface current lies in the $z = 0$ plane, inside of a rectangular region of dimensions $(a \times b)$, and is described by

$$\underline{J}_s = \hat{x} A_0 \cos\left(\frac{\pi x}{a}\right), \quad x \in (-a/2, a/2), \quad y \in (-b/2, b/2).$$

Find the far-zone electric field for this antenna (both E_θ and E_ϕ).