ECE 6341

Spring 2016

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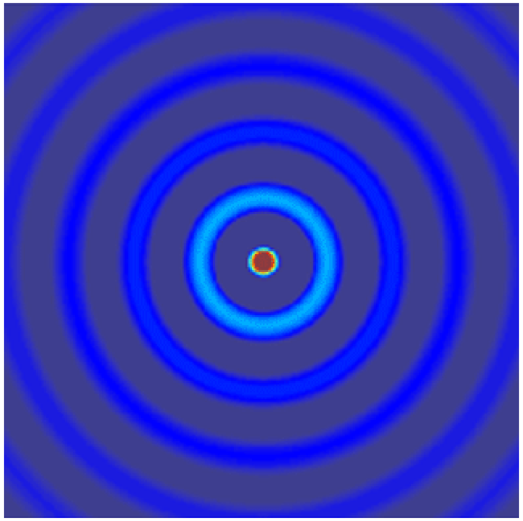
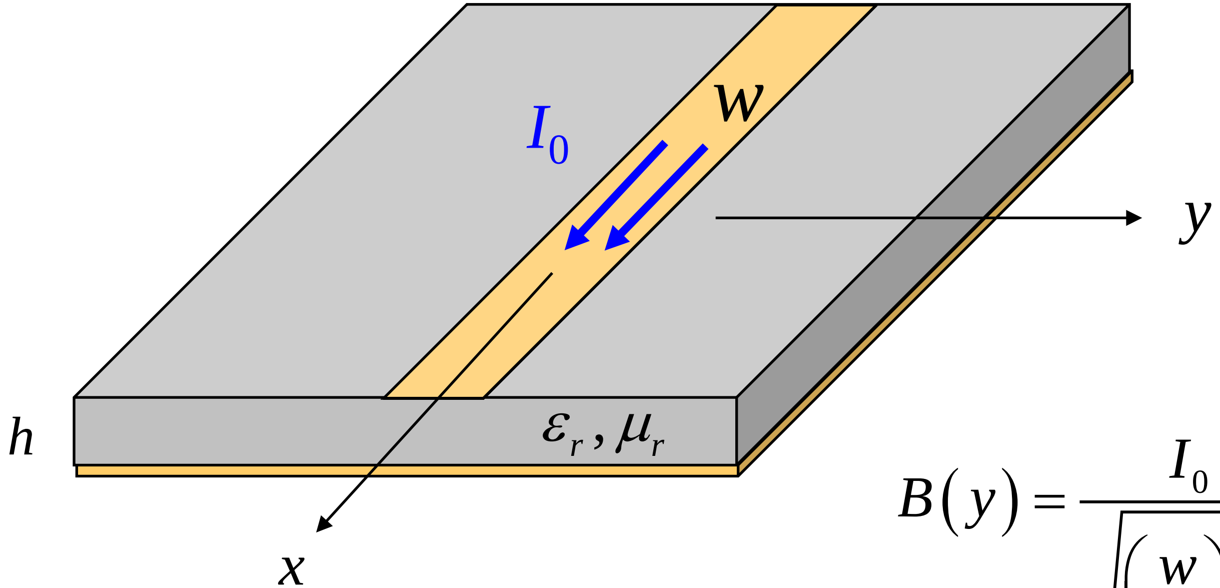
ECE Dept.



Notes 40

Microstrip Line

Microstrip Line



$$B(y) = \frac{I_0 / \pi}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}}$$

Dominant quasi-TEM mode:

$$J_{sx}(x, y) = B(y)\, e^{-jk_{x0}x}$$

We assume a purely x-directed current and a real wavenumber k_{x0}.

Note: The wavenumber is unknown, but we will solve for it later.

Microstrip Line (cont.)

Fourier transform of current:

$$\tilde{J}_{sx}\left(k_x,k_y\right)=I_0\int_{-w/2}^{w/2}\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2-y^2}}e^{jk_y y}\,dy\int_{-\infty}^{\infty}e^{-jk_{x0}x}\,e^{jk_x x}\,dx$$

$$\tilde{J}_{sx}\left(k_x,k_y\right)=I_0\,J_0\left(\frac{k_y w}{2}\right)\int_{-\infty}^{\infty}e^{-jk_{x0}x}\,e^{jk_x x}\,dx$$

Please see the first appendix for the transform of the Maxwell function.

Using the integral representation of the delta-function, we have:

$$\tilde{J}_{sx}\left(k_x,k_y\right)=I_0\,J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]$$

Note : $\delta(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\alpha x}\,dx$

Microstrip Line (cont.)

$$\tilde{J}_{sx}\left(k_x,k_y\right)=I_0\,J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]$$

We then have

$$E_x(x,y;0,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,\tilde{J}_{sx}\left[k_x^2 V_i^{TM}\left(0,0\right)+k_y^2 V_i^{TE}\left(0,0\right)\right]\cdot e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

$z = 0,\ z' = 0$

Hence we have:

$$E_x(x,y;0,0)=\frac{I_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]\left[k_x^2 V_i^{TM}\left(0,0\right)+k_y^2 V_i^{TE}\left(0,0\right)\right]\cdot e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Microstrip Line (cont.)

$$E_x(x,y;0,0) = \frac{I_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{k_t^2}\, J_0\left(\frac{k_y w}{2}\right)\left[\delta\left(k_x - k_{x0}\right)\right]\left[k_x^2\, V_i^{TM}\left(0,0\right) + k_y^2\, V_i^{TE}\left(0,0\right)\right] \cdot e^{-j\left(k_x x + k_y y\right)} dk_x dk_y$$

Integrating over the δ-function, we have:

$$E_x(x,y;0,0) = \frac{I_0}{2\pi} e^{-jk_{x0}x}\int_{-\infty}^{\infty} -\frac{1}{k_t^2}\, J_0\left(\frac{k_y w}{2}\right)\left[k_{x0}^2\, V_i^{TM}\left(0,0\right) + k_y^2\, V_i^{TE}\left(0,0\right)\right] e^{-jk_y y} dk_y$$

where we now have $k_t^2 = k_{x0}^2 + k_y^2$

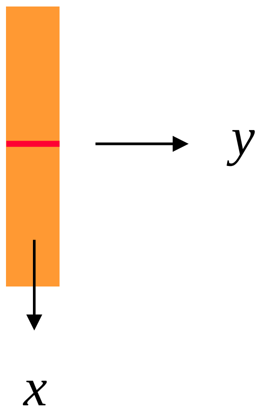
$$k_{z0} = \sqrt{k_0^2 - k_t^2} = \sqrt{\left(k_0^2 - k_{x0}^2\right) - k_y^2} \qquad k_{z1} = \sqrt{k_1^2 - k_t^2} = \sqrt{\left(k_1^2 - k_{x0}^2\right) - k_y^2}$$

Microstrip Line (cont.)

Enforce EFIE using Galerkin's method:

$$\int_{-w/2}^{w/2} E_x(0,y;0,0)\,T(y)\,dy = 0$$

The EFIE is enforced on the red line.



where $T(y) = B(y)$ (testing function = basis function)

Recall that

$$E_x(x,y;0,0) = \frac{I_0}{(2\pi)} e^{-jk_{x0}x} \int_{-\infty}^{\infty} -\frac{1}{k_t^2}\, J_0\!\left(\frac{k_y w}{2}\right) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] e^{-jk_y y}\, dk_y$$

Substituting into the EFIE integral, we have

$$\frac{I_0}{(2\pi)} \int_{-\infty}^{\infty} -\frac{1}{k_t^2}\, J_0\!\left(\frac{k_y w}{2}\right) \tilde{T}(-k_y) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Microstrip Line (cont.)

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) \tilde{T}\left(-k_y\right) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Since the testing function is the same as the basis function,

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) J_0\left(\frac{-k_y w}{2}\right) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Since the Bessel function is an even function,

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0^2\left(\frac{k_y w}{2}\right) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Microstrip Line (cont.)

Using symmetry, we have

$$\int_0^\infty J_0^2\left(\frac{k_y w}{2}\right)\frac{1}{k_t^2}\left[k_{x0}^2 V_i^{TM}(0,0)+k_y^2 V_i^{TE}(0,0)\right]dk_y=0$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

Note:
The Michalski functions are calculated in closed form later.

This is a transcendental equation of the following form:

$$F\left(k_{x0}\right)=0$$

We have to solve this equation numerically.

Note: $k_0 < k_{x0} < k_1$

Microstrip Line (cont.)

Branch points:

$$k_{z0}^2 = k_0^2 - k_t^2 = k_0^2 - \left(k_{x0}^2 + k_y^2\right)$$

Hence
$$k_{z0} = \left(\left(k_0^2 - k_{x0}^2\right) - k_y^2\right)^{1/2}$$
$$= -j\left(k_y^2 - \left(k_0^2 - k_{x0}^2\right)\right)^{1/2}$$
$$= -j\left(k_y^2 + \left(k_{x0}^2 - k_0^2\right)\right)^{1/2}$$
$$= -j\left(k_y - j\sqrt{k_{x0}^2 - k_0^2}\right)^{1/2}\left(k_y + j\sqrt{k_{x0}^2 - k_0^2}\right)^{1/2}$$

Note: The wavenumber k_{z0} causes branch points to arise.

$$\Rightarrow \quad k_{yb} = \pm j\sqrt{k_{x0}^2 - k_0^2}$$

Microstrip Line (cont.)

Poles ($k_y = k_{yp}$):

$$k_{tp}^2 = k_{x0}^2 + k_{yp}^2 = k_{\mathrm{TM}_0}^2$$

$$\Rightarrow \quad k_{yp}^2 = k_{\mathrm{TM}_0}^2 - k_{x0}^2$$

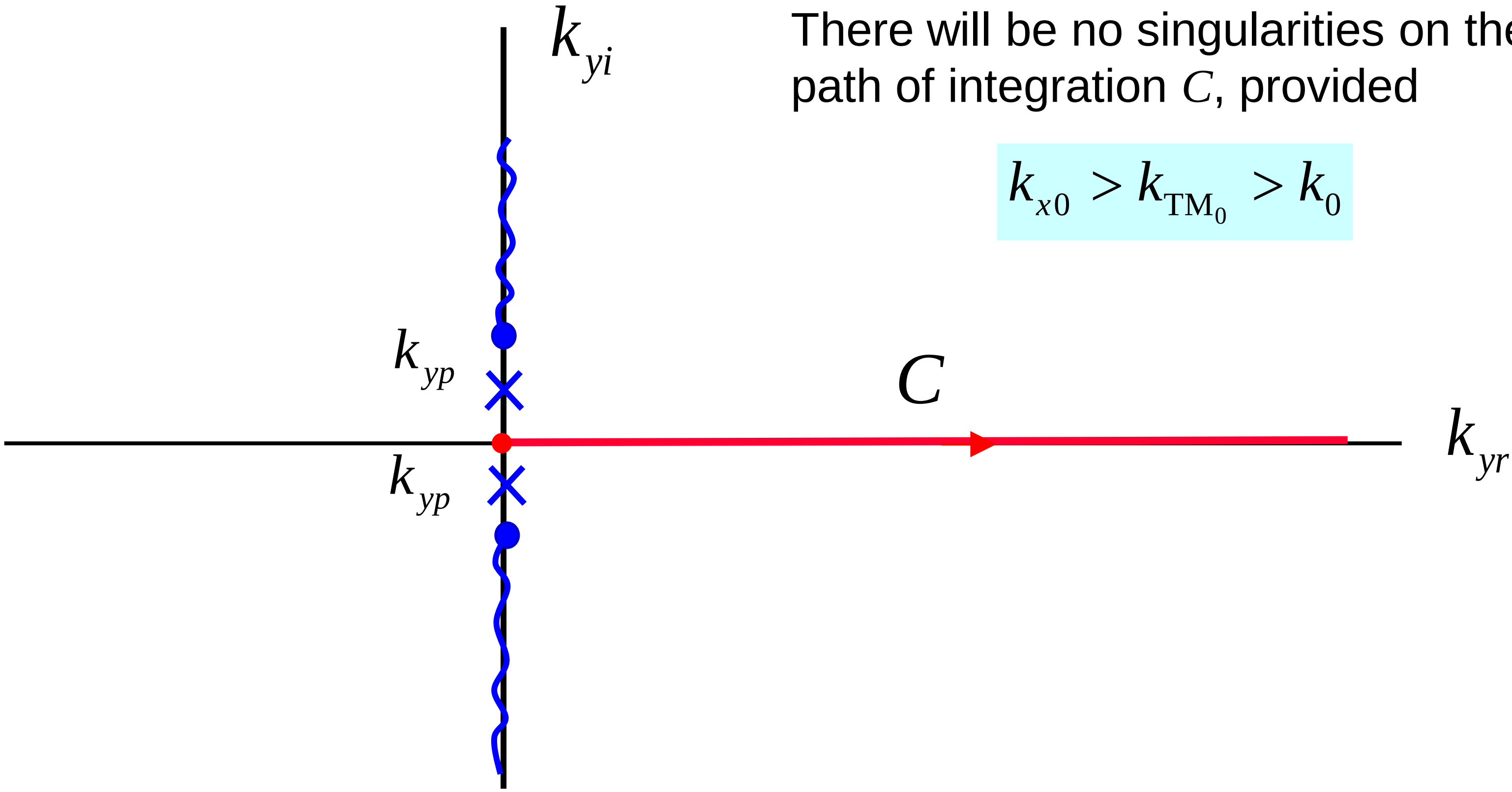
or

$$k_{yp} = \pm j\sqrt{k_{x0}^2 - k_{\mathrm{TM}_0}^2}$$

Microstrip Line (cont.)

Branch points: $k_{yb} = \pm j\sqrt{k_{x0}^2 - k_0^2}$

Poles: $k_{yp} = \pm j\sqrt{k_{x0}^2 - k_{\mathrm{TM}_0}^2}$



There will be no singularities on the path of integration C, provided

$$k_{x0} > k_{\mathrm{TM}_0} > k_0$$

Microstrip Line (cont.)

Note on wavenumber k_{x0}

$$k_{yp} = \pm j\left(k_{x0}^2 - k_{\mathrm{TM}_0}^2\right)^{1/2}$$

For a real wavenumber k_{x0}, we must have that $k_{x0} > k_{\mathrm{TM}_0}$

Otherwise, there would be poles on the real axis, and this would correspond to leakage into the TM_0 surface-wave mode of the grounded substrate.

The mode would then be a leaky mode with a complex wavenumber k_{x0}, which contradicts the assumption that the pole is on the real axis.

Hence

$$k_0 < k_{\mathrm{TM}_0} < k_{x0} < k_1$$

Microstrip Line (cont.)

If we wanted to use multiple basis functions, we could consider the following choices:

Fourier-Maxwell Basis Function Expansion:

$$J_{sx}(x,y) = e^{-jk_{x0}x} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \left[\sum_{m=0}^{M-1} a_m \cos\left(\frac{2m\pi y}{w}\right) \right]$$

$$J_{sy}(x,y) = e^{-jk_{x0}x} \sqrt{\left(\frac{w}{2}\right)^2 - y^2} \left[\sum_{n=1}^{N} b_n \sin\left(\frac{(2n-1)\pi y}{w}\right) \right]$$

Chebyshev-Maxwell Basis Function Expansion:

$$J_{sx}(x,y) = e^{-jk_{x0}x} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \left[\sum_{m=0}^{M-1} a_m T_{2m}\left(\frac{2y}{w}\right) \right] \left(\frac{2(1+\delta_{m0})}{\pi w} \right)$$

$$J_{sy}(x,y) = e^{-jk_{x0}x} \sqrt{\left(\frac{w}{2}\right)^2 - y^2} \left[\sum_{n=1}^{N} b_n U_{2n-1}\left(\frac{2y}{w}\right) \right] \left(\frac{j4}{\pi w} \right)$$

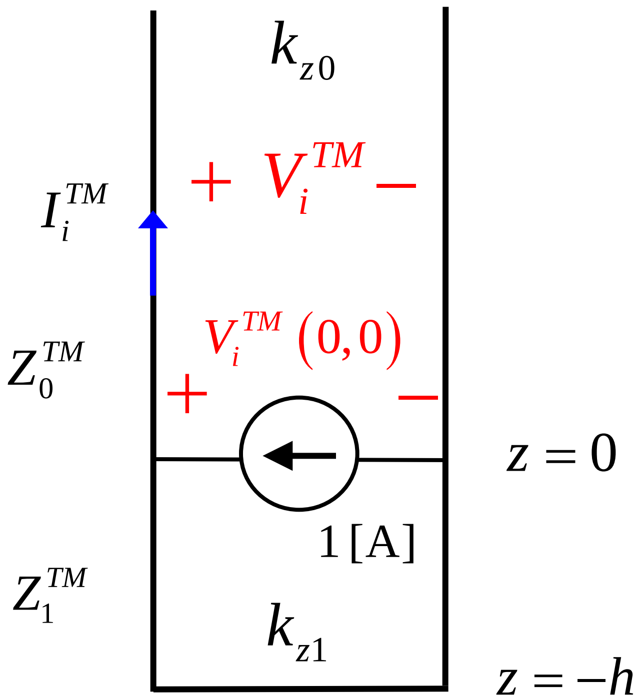
Microstrip Line (cont.)

We next proceed to calculate the Michalski voltage functions explicitly.

Microstrip Line (cont.)

TM$_z$

$V_i^{TM}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{1/2}$$

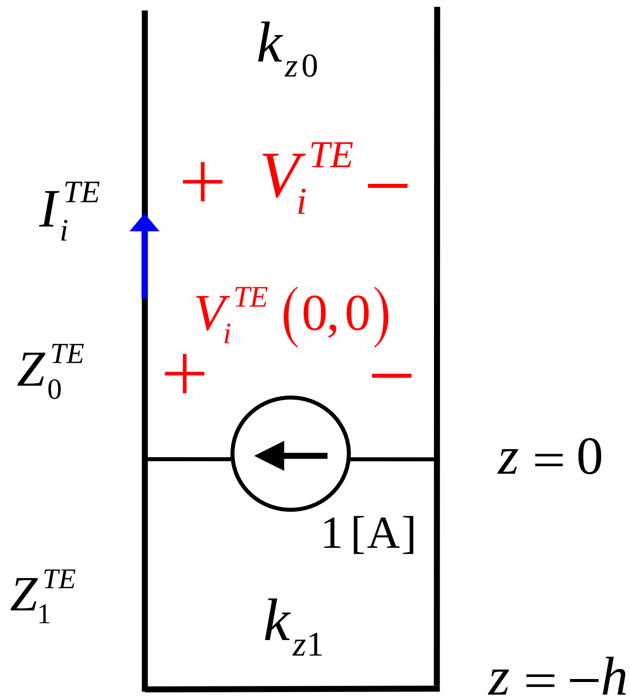
$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{1/2}$$

$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} = \eta_0\left(\frac{k_{z0}}{k_0}\right) \qquad Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} = \frac{\eta_0}{\varepsilon_r}\left(\frac{k_{z1}}{k_0}\right)$$

Microstrip Line (cont.)

TE$_z$

$V_i^{TE}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{1/2}$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{z0}} = \frac{\eta_0}{\left(k_{z0}/k_0\right)}$$

$$Z_1^{TE} = \frac{\omega\mu_1}{k_{z1}} = \frac{\eta_0\mu_r}{\left(k_{z1}/k_0\right)}$$

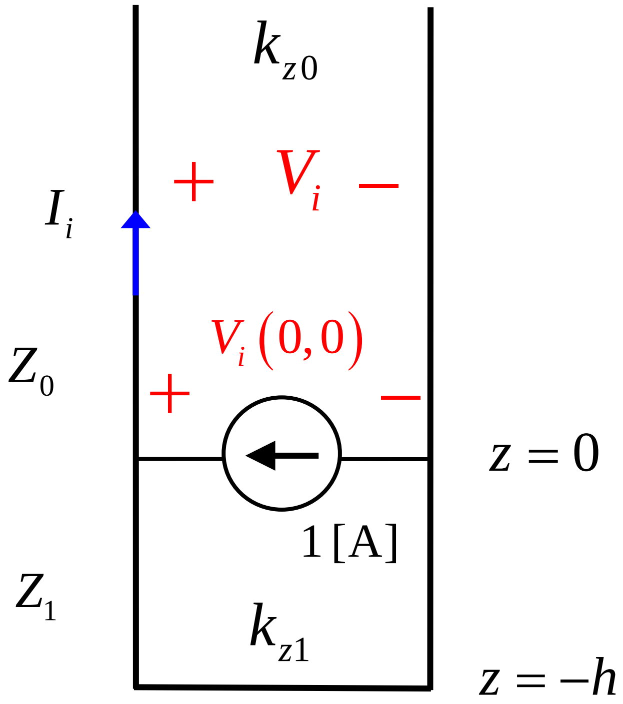
Microstrip Line (cont.)

At $z = 0$

$$V_i(0,0) = Z_{in}$$

$$= Y_{in}^{-1} = \left(Y_{in}^{+} + Y_{in}^{-}\right)^{-1}$$

$$= \left[Y_0 - jY_1 \cot\left(k_{z1} h\right)\right]^{-1}$$



Hence

$$V_i^{TM}(0,0) = \frac{1}{D_m(k_t)}$$

$$V_i^{TE}(0,0) = \frac{1}{D_e(k_t)}$$

$$D_m(k_t) = Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1} h\right)$$

$$D_e(k_t) = Y_0^{TE} - jY_1^{TE} \cot\left(k_{z1} h\right)$$

Microstrip Line (cont.)

Summary of the transcendental equation for the unknown wavenumber k_{x0}:

$$F\left(k_{x0}\right) \equiv \int_0^{\infty} J_0^2\left(\frac{k_y w}{2}\right)\frac{1}{k_t^2}\left[k_{x0}^2\, V_i^{TM}\left(0,0\right) + k_y^2\, V_i^{TE}\left(0,0\right)\right] dk_y = 0$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

$$V_i^{TM}\left(0,0\right) = \frac{1}{D_m\left(k_t\right)}$$

$$V_i^{TE}\left(0,0\right) = \frac{1}{D_e\left(k_t\right)}$$

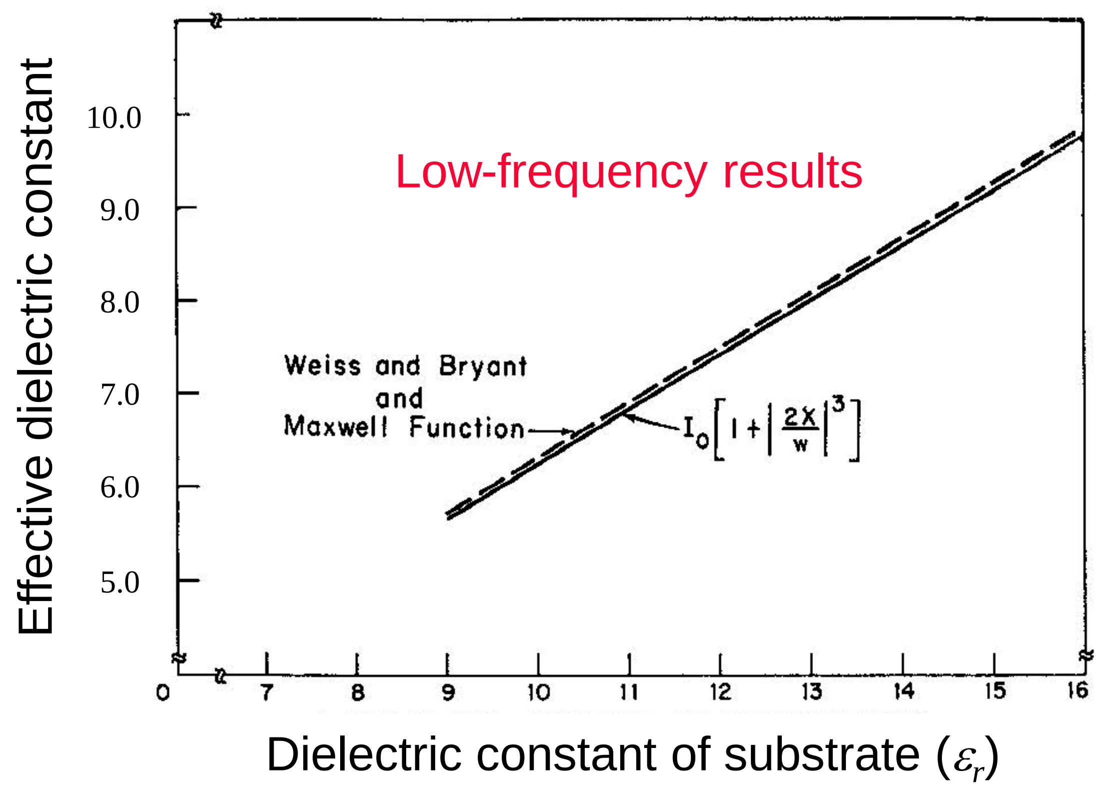
$$D_m\left(k_t\right) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$

$$D_e\left(k_t\right) = Y_0^{TE} - jY_1^{TE}\cot\left(k_{z1}h\right)$$

$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} \quad Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} \quad Z_0^{TE} = \frac{\omega\mu_0}{k_{z0}} \quad Z_1^{TE} = \frac{\omega\mu_1}{k_{z1}}$$

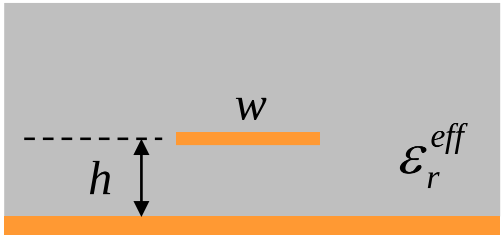
$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{1/2} \qquad k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{1/2}$$

Microstrip Line (cont.)



Parameters: $w/h = 0.40$

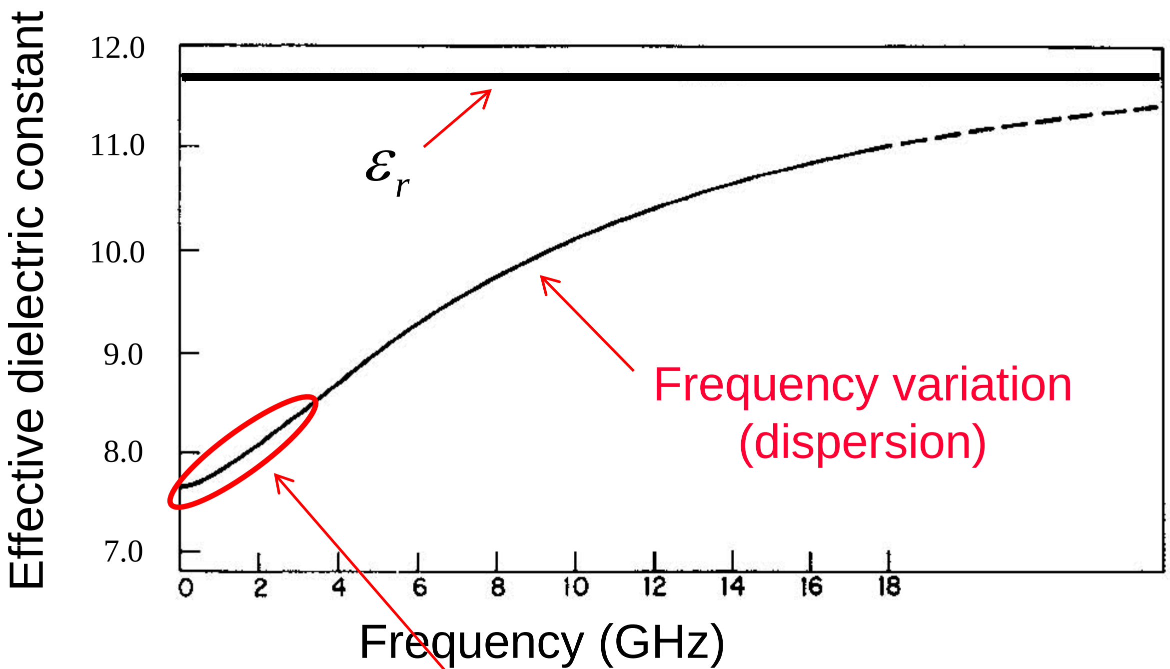
$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0}\right)^2$$



$$k_{x0} = k_0 \sqrt{\varepsilon_r^{eff}}$$

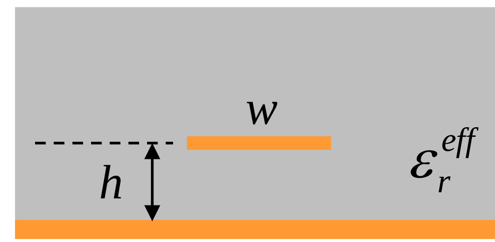
E. J. Denlinger, "A frequency-dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory and Techniques,* vol. 19, pp. 30-39, Jan. 1971.

Microstrip Line (cont.)



Parameters: $\varepsilon_r = 11.7$, $w/h = 0.96$, $h = 0.317$ cm

$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0} \right)^2$$



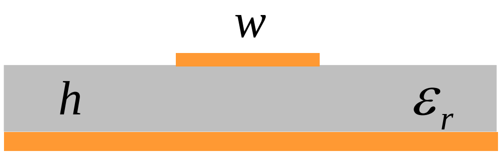
$$k_{x0} = k_0 \sqrt{\varepsilon_r^{eff}}$$

E. J. Denlinger, "A frequency-dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory and Techniques,* vol. 19, pp. 30-39, Jan. 1971.

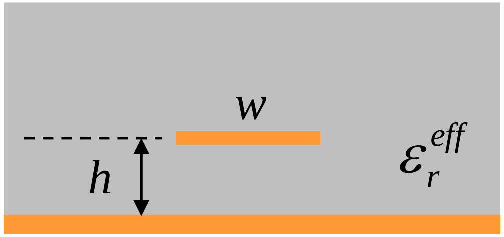
Microstrip Line (cont.)

Characteristic Impedance

1) Quasi-TEM Method



Original problem



Equivalent problem (TEM)

$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0} \right)^2$$

We calculate the characteristic impedance of the equivalent *homogeneous medium problem*.

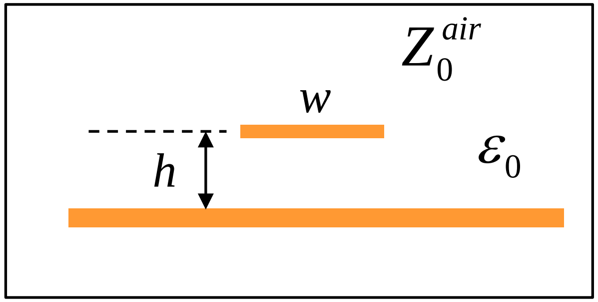
Microstrip Line (cont.)

Using the equivalent TEM problem:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{L_0}{C_0 \varepsilon_r^{eff}}}$$

(The zero subscript denotes the value when using an air substrate.)

$$\Rightarrow \quad Z_0 = Z_0^{air} \frac{1}{\sqrt{\varepsilon_r^{eff}}}$$



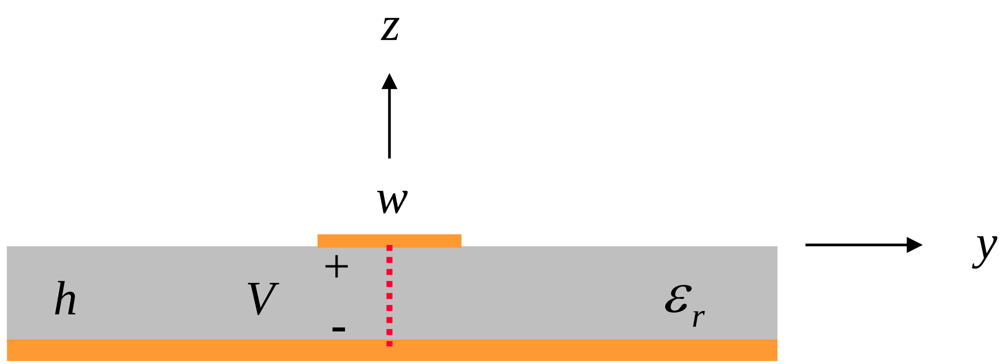
Simple CAD formulas may be used for the Z_0 of an air line.

$$Z_0^{air} = \begin{cases} 60 \ln\left(\dfrac{8h}{w} + \dfrac{w}{4h}\right); & \text{for } \dfrac{w}{h} \le 1 \\[2ex] \dfrac{120\pi}{\left(\dfrac{w}{h} + 1.393 + 0.667 \ln\left(\dfrac{w}{h} + 1.444\right)\right)}; & \text{for } \dfrac{w}{h} \ge 1 \end{cases}$$

Microstrip Line (cont.)

2) Voltage-Current Method

$$Z_0 = \frac{V(x)}{I(x)} = \frac{V(0)}{I(0)} = \frac{-1}{I_0}\int_{-h}^{0} E_z(0,0,z)\,dz$$



This is derived in the Appendix.

$$\tilde{E}_z(k_x,k_y,z) = \frac{-1}{\omega\varepsilon_0\varepsilon_r}(k_t)\,I^{TM}(z) = \frac{-1}{\omega\varepsilon_0\varepsilon_r}(k_t)\,I_i^{TM}(z)\left(-\tilde{J}_s\cdot\underline{\hat{u}}\right)$$

$$= \frac{-1}{\omega\varepsilon_0\varepsilon_r}(k_t)\,I_i^{TM}(z)\left(-\tilde{J}_{sx}\right)\cos\overline{\phi}$$

Microstrip Line (cont.)

$$Z_0 = \frac{-1}{I_0}\int_{-h}^{0}\left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}_z\left(k_x,k_y,z\right)e^{-j\left(k_x x+k_y y\right)}dk_x dk_y\right)_{\substack{x=0\\y=0}} dz$$

Set x and y to zero

$$Z_0 = \frac{-1}{I_0}\int_{-h}^{0}\left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}_z\left(k_x,k_y,z\right)dk_x dk_y\right)dz$$

Substitute for the transform of E_z

$$Z_0 = \frac{-1}{I_0}\int_{-h}^{0}\left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left(-\tilde{J}_{sx}\right)\left(\frac{k_x}{k_t}\right)dk_x dk_y\right)dz$$

Substitute for the transform of the surface current

$$Z_0 = \frac{-1}{I_0}\int_{-h}^{0}\left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left(-I_0 J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]\right)\left(\frac{k_x}{k_t}\right)dk_x dk_y\right)dz$$

Microstrip Line (cont.)

$$Z_0 = -\int_{-h}^{0}\left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega\varepsilon_0\varepsilon_r}(k_t)\,I_i^{TM}(z)\left(J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta(k_x-k_{x0})\right]\right)\left(\frac{k_x}{k_t}\right)dk_x dk_y\right)dz$$

Integrate in k_x

$$Z_0 = -\int_{-h}^{0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega\varepsilon_0\varepsilon_r}(k_t)\,I_i^{TM}(z)\,J_0\left(\frac{k_y w}{2}\right)\left(\frac{k_{x0}}{k_t}\right)dk_y\right)dz$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

Switch order of integration

$$Z_0 = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k_{x0}}{\omega\varepsilon_0\varepsilon_r}J_0\left(\frac{k_y w}{2}\right)\left(\int_{-h}^{0}I_i^{TM}(z)\,dz\right)dk_y$$

Use symmetry

$$Z_0 = -\frac{1}{\pi}\int_{0}^{\infty}\frac{k_{x0}}{\omega\varepsilon_0\varepsilon_r}J_0\left(\frac{k_y w}{2}\right)\left(\int_{-h}^{0}I_i^{TM}(z)\,dz\right)dk_y$$

Microstrip Line (cont.)

The final result is then

$$Z_0 = -\frac{1}{\pi}\int_0^{\infty} \frac{k_{x0}}{\omega\varepsilon_0\varepsilon_r} J_0\left(\frac{k_y w}{2}\right) F\left(k_t\right) dk_y$$

where

$$k_t^2 = k_{x0}^2 + k_y^2$$

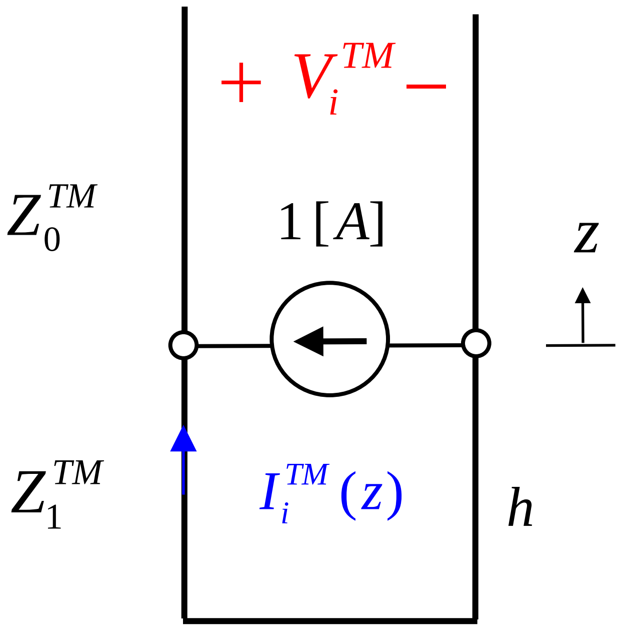
$$F\left(k_t\right) \equiv \int_{-h}^{0} I_i^{TM}\left(z\right) dz$$

The $F(k_t)$ function is calculated next.

Microstrip Line (cont.)

We first calculate the function

$$I_i^{TM}(z)$$



$$I_i^{TM}\left(0^-\right) = -\left(\frac{V(0)}{Z_{in}^-}\right) = -\left(\frac{Z_{in}}{Z_{in}^-}\right) = -\left(\frac{1/D_m(k_t)}{j\,Z_1^{TM}\tan\left(k_{z1}h\right)}\right)$$

Microstrip Line (cont.)

Because of the short circuit,

$$I_i^{TM}(z) = A\cos\left(k_{z1}(z+h)\right), \quad -h < z < 0$$

At $z = 0$: $I_i^{TM}(0^-) = A\cos(k_{z1}h)$

Therefore

$$A = \frac{I_i^{TM}(0^-)}{\cos(k_{z1}h)}$$

Hence

$$I_i^{TM}(z) = I_i^{TM}(0^-)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos(k_{z1}h)}\right) \qquad -h < z < 0$$

Microstrip Line (cont.)

Hence

$$I_i^{TM}(z) = -\left(\frac{1/D_m(k_t)}{j Z_1^{TM} \tan(k_{z1}h)}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos(k_{z1}h)}\right)$$

$$-h < z < 0$$

or

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j Z_1^{TM}}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\sin(k_{z1}h)}\right)$$

Microstrip Line (cont.)

Hence

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\sin\left(k_{z1}h\right)}\right)$$

where

$$D_m(k_t) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$

Microstrip Line (cont.)

We then have

$$F(k_t) \equiv \int_{-h}^{0} I_i^{TM}(z)\,dz$$

$$= \int_{-h}^{0} -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{\cos(k_{z1}(z+h))}{\sin(k_{z1}h)}\right)dz$$

$$= -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{1}{\sin(k_{z1}h)}\right)\int_{-h}^{0} \cos(k_{z1}(z+h))\,dz$$

$$= -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{1}{\sin(k_{z1}h)}\right)\left[\frac{1}{k_{z1}}\sin(k_{z1}(z+h))\right]_{-h}^{0}$$

$$= -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{1}{\sin(k_{z1}h)}\right)\left[\frac{1}{k_{z1}}\sin(k_{z1}h)\right]$$

Hence

$$F(k_t) = -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{1}{k_{z1}}\right)$$

Microstrip Line (cont.)

Summary of Voltage-Current Formula:

$$Z_0 = -\frac{1}{\pi}\int_0^{\infty} \frac{k_{x0}}{\omega \varepsilon_0 \varepsilon_r} J_0\left(\frac{k_y w}{2}\right) F\left(k_t\right) dk_y$$

where

$$F\left(k_t\right) = -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\, Z_1^{TM}}\right)\left(\frac{1}{k_{z1}}\right)$$

$$D_m\left(k_t\right) = Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1} h\right)$$

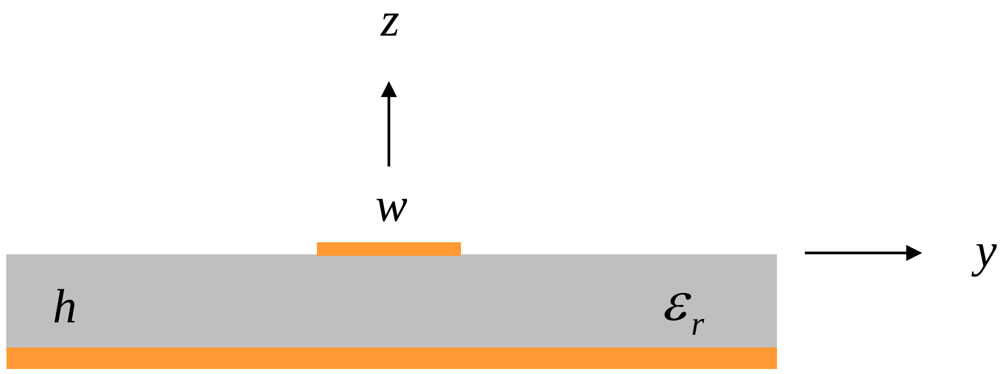
$$Z_0^{TM} = \frac{k_{z0}}{\omega \varepsilon_0} \qquad Z_1^{TM} = \frac{k_{z1}}{\omega \varepsilon_1} \qquad k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2} \qquad k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

Microstrip Line (cont.)

3) Power-Current Method

$$Z_0 = \frac{2P_x(0)}{|I(0)|^2} = \frac{2}{|I(0)|^2} \int_{-\infty}^{\infty} \int_{-h}^{\infty} S_x(0, y, z)\, dz\, dy$$

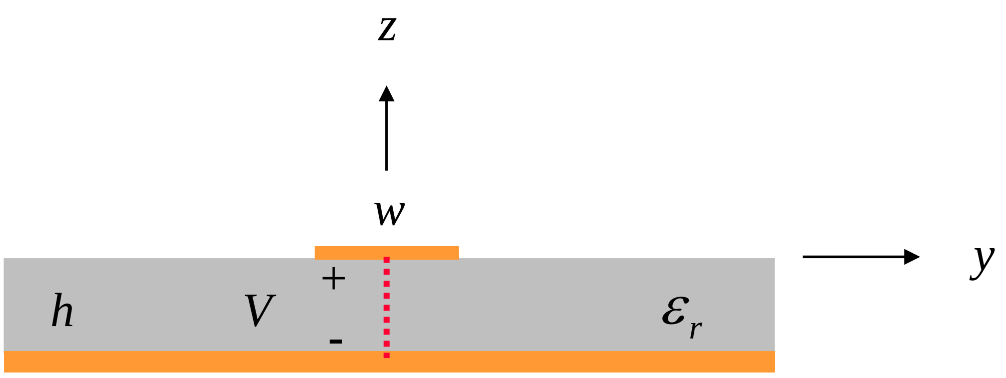


Note: It is possible to perform the spatial integrations for the power flow in closed form (details are omitted).

Microstrip Line (cont.)

4) Power-Voltage Method

$$Z_0 = \frac{|V(0)|^2}{2P_x(0)} = \frac{|V(0)|^2}{2\int_{-\infty}^{\infty}\int_{-h}^{\infty} S_x(0,y,z)\,dz\,dy}$$



Note: It is possible to perform the spatial integrations for the power flow in closed form (details are omitted).

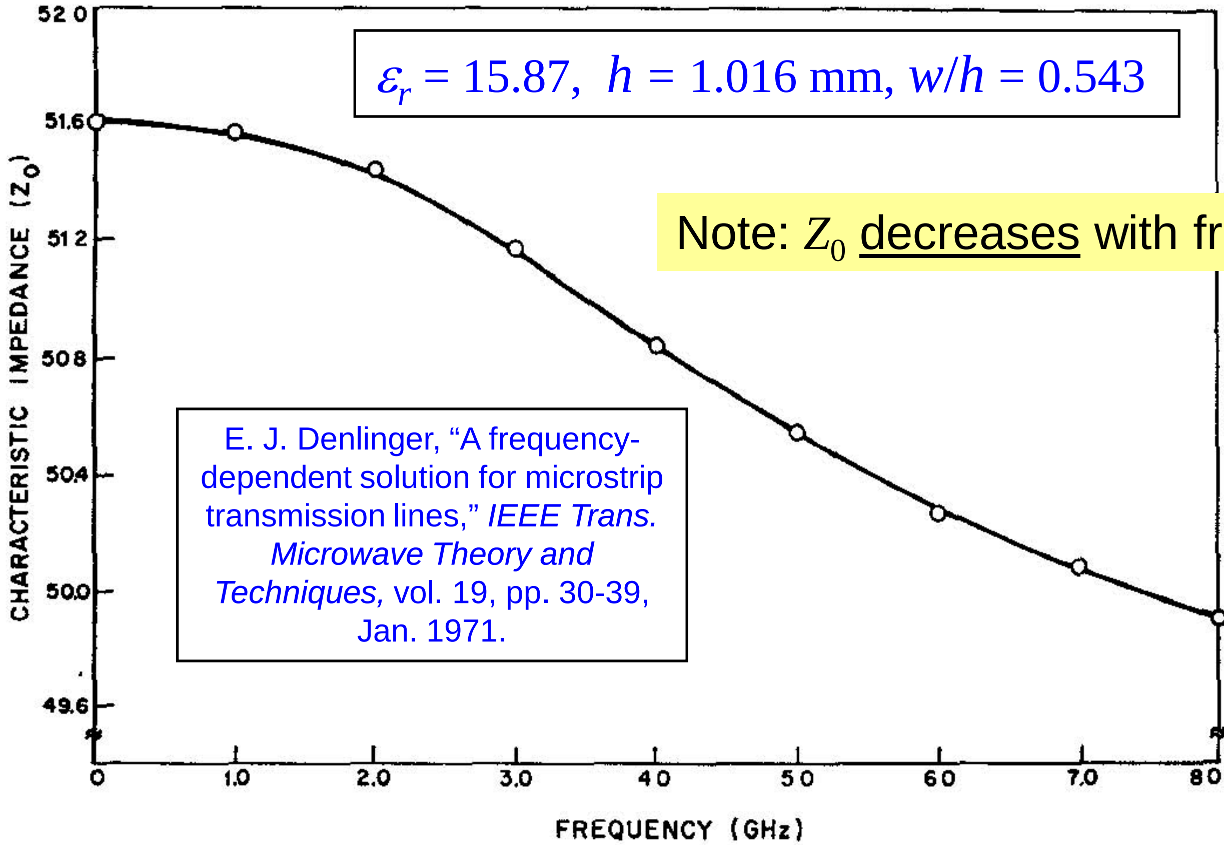
Microstrip Line (cont.)

Comparison of methods:

- At low frequency all three methods agree well.
- As frequency increases, the VI, PI, and PV methods give a Z_0 that increases with frequency.
- The Quasi-TEM method gives a Z_0 that decreases with frequency.
- The PI method is usually regarded as being the best one for high frequency (agrees better with measurements).

Microstrip Line (cont.)

Quasi-TEM Method



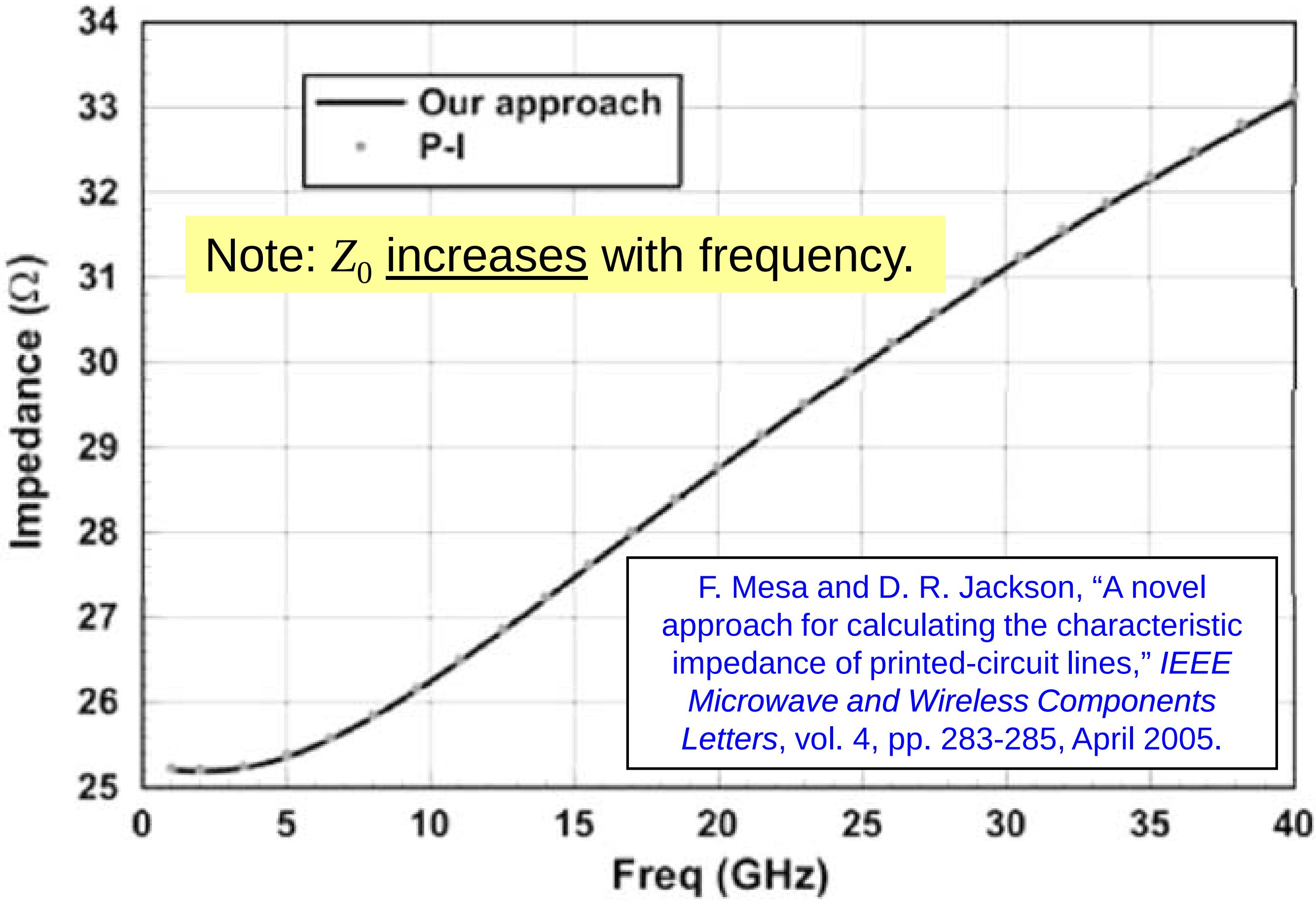
$\varepsilon_r = 15.87$, $h = 1.016$ mm, $w/h = 0.543$

Note: Z_0 decreases with frequency.

E. J. Denlinger, "A frequency-dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory and Techniques,* vol. 19, pp. 30-39, Jan. 1971.

Fig. 9. Characteristic impedance versus frequency (same parameters as in Fig. 8).

Microstrip Line (cont.)

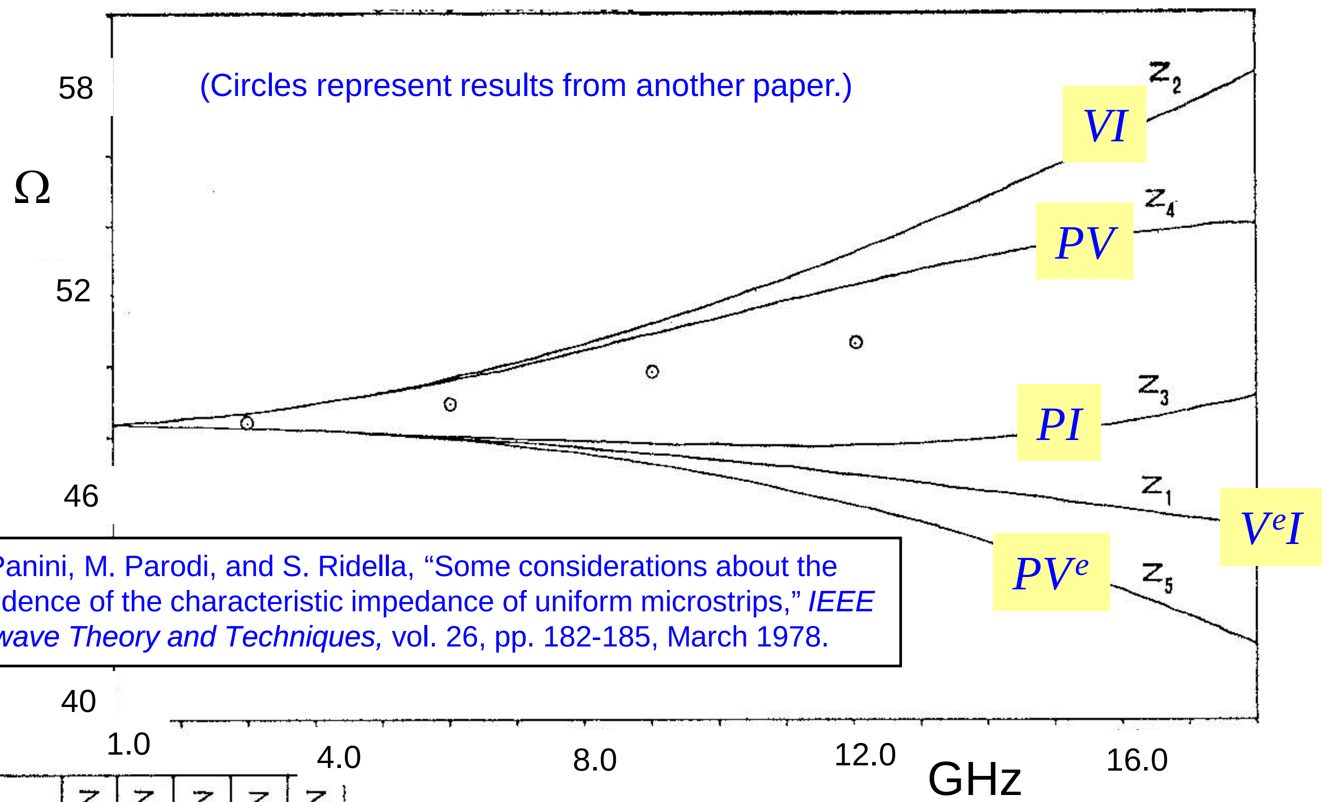


Note: Z_0 increases with frequency.

F. Mesa and D. R. Jackson, "A novel approach for calculating the characteristic impedance of printed-circuit lines," *IEEE Microwave and Wireless Components Letters*, vol. 4, pp. 283-285, April 2005.

Fig. 3. Results for the characteristic impedance of a microstrip with $w = 3$ mm, $h = 1$ mm, $\varepsilon_r = 10.2$.

Microstrip Line (cont.)



B Bianco, L. Panini, M. Parodi, and S. Ridella, "Some considerations about the frequency dependence of the characteristic impedance of uniform microstrips," *IEEE Trans. Microwave Theory and Techniques,* vol. 26, pp. 182-185, March 1978.

	Z_1	Z_2	Z_3	Z_4	Z_5
Limit value (Ω) for $f \to 0$	48.35	48.35	48.35	48.35	48.35
Limit value (Ω) for $f \to \infty$	40.06	80.00	61.48	104.41	26.10

$\varepsilon_r = 10,\ h = 0.635$ mm, $w/h = 1$

V^e = effective voltage (average taken over different paths).

Appendix: Transform of Maxwell Function

In this appendix we evaluate the transform of the Maxwell function.

$$\tilde{B}\left(k_y\right) = \int_{-w/2}^{w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \right) e^{jk_y y}\, dx$$

From symmetry:

$$\tilde{B}\left(k_y\right) = \int_{-w/2}^{w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \right) \cos\left(k_y y\right) dy = 2\int_{0}^{w/2} \left(\frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \right) \cos\left(k_y y\right) dy$$

Appendix (cont.)

Next, use the transformation

$$y = \frac{w}{2}\sin\theta$$

$$dy = \frac{w}{2}\cos\theta \, d\theta$$

so that

$$\tilde{B}(k_y) = \frac{2}{\pi}\int_0^{\pi/2} \left(\frac{\cos\left(\dfrac{k_y w}{2}\sin\theta\right)}{\dfrac{w}{2}\cos\theta} \right) \frac{w}{2}\cos\theta \, d\theta$$

Appendix (cont.)

We then have

$$\tilde{B}(k_y) = \frac{2}{\pi}\int_0^{\pi/2} \cos\left(\frac{k_y w}{2}\sin\theta\right) d\theta = \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{k_y w}{2}\sin\theta\right)$$

Next, use the following integral identify for the Bessel function:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)\, d\theta$$

so that

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta)\, d\theta$$

Appendix (cont.)

Hence, we have

$$\tilde{B}\left(k_y\right) = J_0\left(\frac{k_y w}{2}\right)$$

Appendix: Vertical Electric Field

Find $E_z\,(x,y,z)$ inside the substrate for $-h < z < 0$.

$$\nabla \times \underline{H} = j\omega\varepsilon_1 \underline{E}$$

$$E_z = \frac{1}{j\omega\varepsilon_1}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_1}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

$$= \frac{1}{\omega\varepsilon_0\varepsilon_r}\left(-k_x\tilde{H}_y + k_y\tilde{H}_x\right)$$

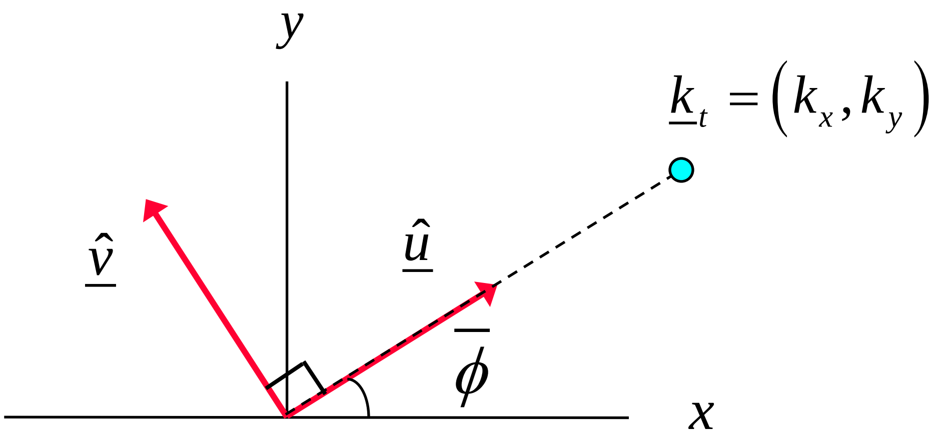
Appendix (cont.)

$$\tilde{H}_x = \underline{\hat{x}} \cdot \left(\underline{\hat{u}} \tilde{H}_u + \underline{\hat{v}} \tilde{H}_v \right)$$

$$= \tilde{H}_u \left(\underline{\hat{u}} \cdot \underline{\hat{x}} \right) + \tilde{H}_v \left(\underline{\hat{v}} \cdot \underline{\hat{x}} \right)$$

$$= \tilde{H}_u \left(\cos \overline{\phi} \right) + \tilde{H}_v \left(-\sin \overline{\phi} \right)$$

$$= \tilde{H}_u \left(\frac{k_x}{k_t} \right) + \tilde{H}_v \left(-\frac{k_y}{k_t} \right)$$



$$\tilde{H}_y = \underline{\hat{y}} \cdot \left(\underline{\hat{u}} \tilde{H}_u + \underline{\hat{v}} \tilde{H}_v \right)$$

$$= \tilde{H}_u \left(\underline{\hat{u}} \cdot \underline{\hat{y}} \right) + \tilde{H}_v \left(\underline{\hat{v}} \cdot \underline{\hat{y}} \right)$$

$$= \tilde{H}_u \left(\sin \overline{\phi} \right) + \tilde{H}_v \left(\cos \overline{\phi} \right)$$

$$= \tilde{H}_u \left(\frac{k_y}{k_t} \right) + \tilde{H}_v \left(\frac{k_x}{k_t} \right)$$

Appendix (cont.)

Hence

cancels

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0\varepsilon_r}\left(-k_x\left[\tilde{H}_u\left(\frac{k_y}{k_t}\right)+\tilde{H}_v\left(\frac{k_x}{k_t}\right)\right]+k_y\left[\tilde{H}_u\left(\frac{k_x}{k_t}\right)+\tilde{H}_v\left(-\frac{k_y}{k_t}\right)\right]\right)$$

or

$$\tilde{E}_z = \frac{1}{\omega\,\varepsilon_0\,\varepsilon_r}\left(\tilde{H}_v\left[-k_x^{\,2}-k_y^{\,2}\right]\frac{1}{k_t}\right)$$

or

$$\tilde{E}_z = \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\tilde{H}_v\right)$$

Appendix (cont.)

or

$$\tilde{E}_z\left(k_x,k_y,z\right)=\frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I^{TM}\left(z\right)$$

For a horizontal electric current source, we then have:

$$\begin{aligned}\tilde{E}_z\left(k_x,k_y,z\right)&=-\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left[-\underline{\tilde{J}}_s\cdot\underline{\hat{u}}\right]\\&=-\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left[-\tilde{J}_{sx}\left(\frac{k_x}{k_t}\right)\right]\end{aligned}$$

The result is then

$$\tilde{E}_z\left(k_x,k_y,z\right)=\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_x\right)\tilde{J}_{sx}I_i^{TM}\left(z\right)$$