ECE 6341

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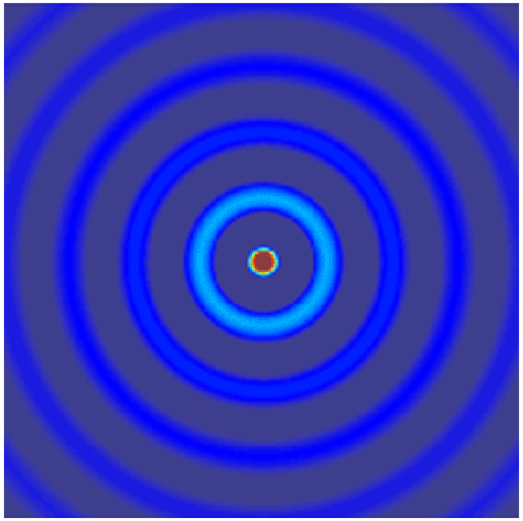

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Overview

In this set of notes we derive the SDI formulation using a more mathematical, but more general, approach (we directly Fourier transform Maxwell's equations).

- This allows for all possible types of sources (horizontal, vertical, electric, and magnetic) to be treated in one derivation.

General SDI Method

Start with Ampere's law:

$$\nabla \times \underline{H} = \underline{J}^{i} = j\omega\varepsilon \underline{E}$$

$$\nabla = \nabla_t + \underline{\hat{z}} \frac{\partial}{\partial z}$$

where $$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

Assume a 2D spatial transform:

$$\begin{aligned}\tilde{\nabla}_t &= \underline{\hat{x}}\left(-jk_x\right) + \underline{\hat{y}}\left(-jk_y\right) \\ &= -j\left(\underline{\hat{x}}\,k_x + \underline{\hat{y}}\,k_y\right) \\ &= -j\underline{k}_t \\ &= -jk_t\,\underline{\hat{u}}\end{aligned}$$

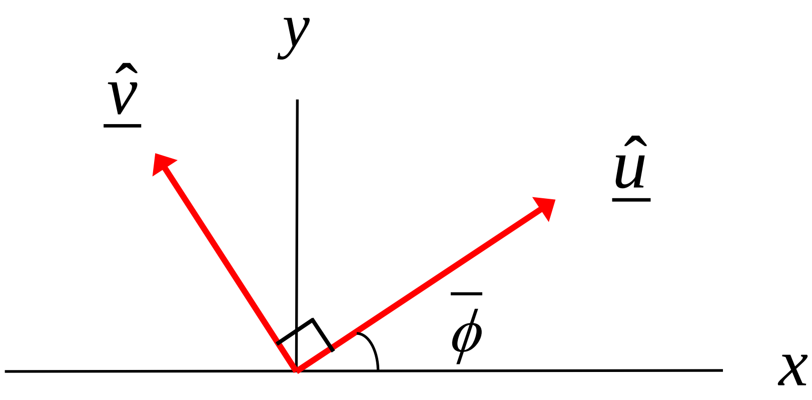
General SDI Method (cont.)

Hence we have
$$\left(-jk_t\,\underline{\hat{u}}+\underline{\hat{z}}\frac{\partial}{\partial z}\right)\times\underline{\tilde{H}}=\underline{\tilde{J}}^i+j\omega\varepsilon\underline{\tilde{E}}$$

Next, represent the field as
$$\underline{\tilde{H}}_t=\underline{\hat{u}}\,\tilde{H}_u+\underline{\hat{v}}\,\tilde{H}_v$$
$$=\underline{\hat{u}}\left(\underline{\tilde{H}}\cdot\underline{\hat{u}}\right)+\underline{\hat{v}}\left(\underline{\tilde{H}}\cdot\underline{\hat{v}}\right)$$

Note that

$$\underline{\hat{u}}\times\underline{\hat{v}}=\underline{\hat{z}}$$
$$\underline{\hat{z}}\times\underline{\hat{u}}=\underline{\hat{v}}$$
$$\underline{\hat{z}}\times\underline{\hat{v}}=-\underline{\hat{u}}$$



Take the $\underline{\hat{z}}$, $\underline{\hat{u}}$, $\underline{\hat{v}}$ components of the transformed Ampere's equation:

General SDI Method (cont.)

$$\underline{\hat{z}}) \; -jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z$$

$$\underline{\hat{u}}) \quad -\frac{\partial \tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u$$

$$\underline{\hat{v}}) \qquad jk_t\tilde{H}_z + \frac{\partial \tilde{H}_u}{\partial z} = \tilde{J}_v^i + j\omega\varepsilon\tilde{E}_v$$

Examine the TM_z field: $\left(\tilde{E}_u, \tilde{H}_v, \tilde{E}_z\right)$ (Ignore $\underline{\hat{v}}$ equation)

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \qquad (1)$$

$$-\frac{\partial \tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u \qquad (2)$$

TM_z Fields

We wish to eliminate $\tilde{E}_z$ from Eq. (1). To do this, use Faraday's law:

$$\nabla \times \underline{E} = -\underline{M}^{i} - j\omega\mu\underline{H}$$

$$\Rightarrow \quad \left(-jk_t\,\underline{\hat{u}} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) \times \underline{\tilde{E}} = -\underline{\tilde{M}}^{i} - j\omega\mu\underline{\tilde{H}}$$

Take the $\underline{\hat{v}}$ component of the transformed Faraday's Law:

Recall:

$$\underline{\hat{u}} \times \underline{\hat{v}} = \underline{\hat{z}}$$

$$\underline{\hat{z}} \times \underline{\hat{u}} = \underline{\hat{v}}$$

$$\underline{\hat{z}} \times \underline{\hat{v}} = -\underline{\hat{u}}$$

$$jk_t\tilde{E}_z + \frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^{i} - j\omega\mu\tilde{H}_v \qquad (3)$$

TM_z Fields (cont.)

Substitute $\tilde{E}_z$ from (1) into (3) to eliminate $\tilde{E}_z$

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \qquad (1)$$

$$jk_t\tilde{E}_z + \frac{\partial\tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v \qquad (3)$$

$$\Rightarrow \quad jk_t\left[\frac{1}{j\omega\varepsilon}\left(-\tilde{J}_z^i - jk_t\tilde{H}_v\right)\right] + \frac{\partial\tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v$$

or

$$\frac{\partial\tilde{E}_u}{\partial z} + \frac{k_t^2}{j\omega\varepsilon}\tilde{H}_v + j\omega\mu\tilde{H}_v = -\tilde{M}_v^i + \frac{k_t}{\omega\varepsilon}\tilde{J}_z^i$$

TM$_z$ Fields (cont.)

Note that

$$\frac{k_t^2}{j\omega\varepsilon} + j\omega\mu = \frac{1}{j\omega\varepsilon}\left(k_t^2 - \omega^2\mu\varepsilon\right)$$

$$= \frac{1}{j\omega\varepsilon}\left(k_t^2 - k^2\right)$$

$$= \frac{-1}{j\omega\varepsilon}\left(k^2 - k_t^2\right)$$

$$= \frac{-k_z^2}{j\omega\varepsilon}$$

Hence

$$\frac{\partial \tilde{E}_u}{\partial z} - \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i \qquad (4)$$

TM$_z$ Fields (cont.)

Equations (2) and (4) are the final TM$_z$ field modeling equations:

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM$_z$ Fields (cont.)

Define TEN modeling equations:

$$V^{TM}(z) \equiv \tilde{E}_u\left(k_x, k_y, z\right)$$

$$I^{TM}(z) \equiv \tilde{H}_v\left(k_x, k_y, z\right)$$

We then have

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^{\,i} - j\omega\varepsilon \tilde{E}_u$$

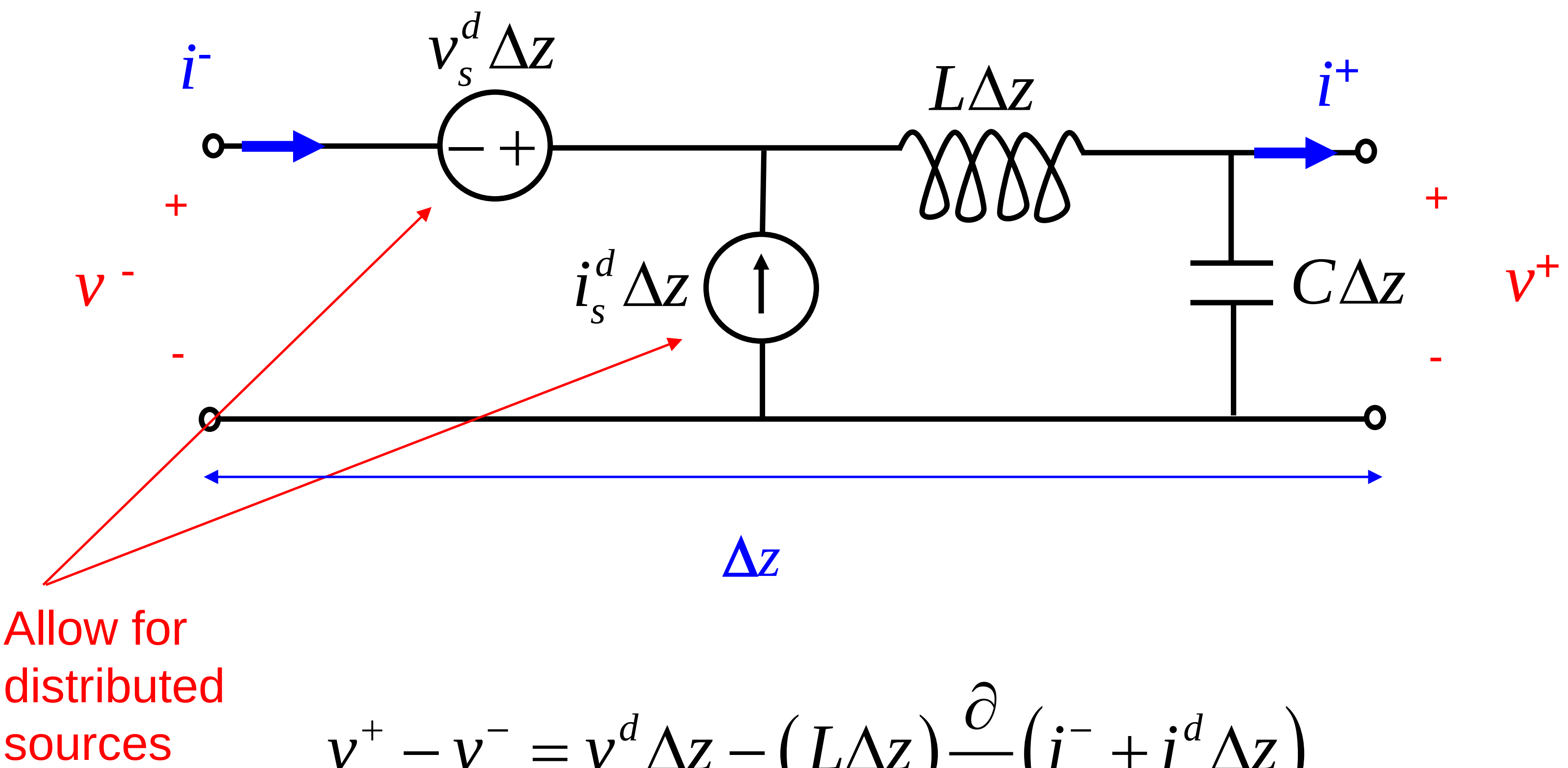
$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^{\,i} + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^{\,i} + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

$$\Downarrow$$

$$\frac{\partial I^{TM}}{\partial z} = -j\omega\varepsilon V^{TM} - \tilde{J}_u^{\,i}$$

$$\frac{\partial V^{TM}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{TM} + \left[-\tilde{M}_v^{\,i} + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^{\,i}\right]$$

Telegrapher's Equation



$$v^+ - v^- = v_s^d \Delta z - (L\Delta z)\frac{\partial}{\partial t}\left(i^- + i_s^d \Delta z\right)$$

$\Delta z \to 0$ so $$\frac{\partial v}{\partial z} = v_s^d - L\frac{\partial i}{\partial t}$$

Telegrapher's Equation (cont.)

Hence, in the phasor domain,

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$

Also,

$$i^+ - i^- = i_s^d \Delta z - \left(C \Delta z \right) \frac{\partial v^+}{\partial t}$$

$$\Delta z \to 0 \quad \text{so} \quad \frac{\partial i}{\partial z} = i_s^d - C \frac{\partial v}{\partial t}$$

Hence, we have

$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

Telegrapher's Equation (cont.)

Compare field equations for TM$_z$ fields with TL equations:

$$\frac{\partial I^{TM}}{\partial z} = -j\omega\varepsilon V^{TM} - \tilde{J}_u^i$$

$$\frac{\partial V^{TM}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{TM} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$

Telegrapher's Equation (cont.)

We then make the following identifications:

$$C = \varepsilon$$

$$\omega L = \frac{k_z^2}{\omega \varepsilon}$$

Hence

$$k_z^{TL} = \omega\sqrt{LC} = \omega\sqrt{\left(\frac{k_z^2}{\omega^2 \varepsilon}\right)\varepsilon} = k_z$$

$$Z_0^{TL} = \sqrt{\frac{L}{C}} = \sqrt{\left(\frac{k_z^2}{\omega^2 \varepsilon}\right)\frac{1}{\varepsilon}} = \frac{k_z}{\omega \varepsilon}$$

so

$$k_z^{TL} = k_z$$

$$Z_0^{TL} = \frac{k_z}{\omega \varepsilon}$$

Sources: TM$_z$

For the sources we have, for the TM$_z$ case:

$$I_s^{d^{TM}} = -\tilde{J}_u^i$$

$$V_s^{d^{TM}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

Sources: TM$_z$ (cont.)

Special case: planar horizontal surface current sources:

Assume

$$\underline{J}^{i}(x,y,z)=\underline{J}_{s}^{i}(x,y)\,\delta(z)$$

$$\underline{M}^{i}(x,y,z)=\underline{M}_{s}^{i}(x,y)\,\delta(z)$$

Then we have

$$I_s^{d^{TM}}=-\tilde{J}_u^{i}=-\tilde{J}_{su}^{i}\,\delta(z)$$

$$V_s^{d^{TM}}=-\tilde{M}_v^{i}=-\tilde{M}_{sv}^{i}\,\delta(z)$$

These correspond to lumped current and voltage sources:

$$I_s^{TM}=-\tilde{J}_{su}^{i}$$ *lumped* parallel current source

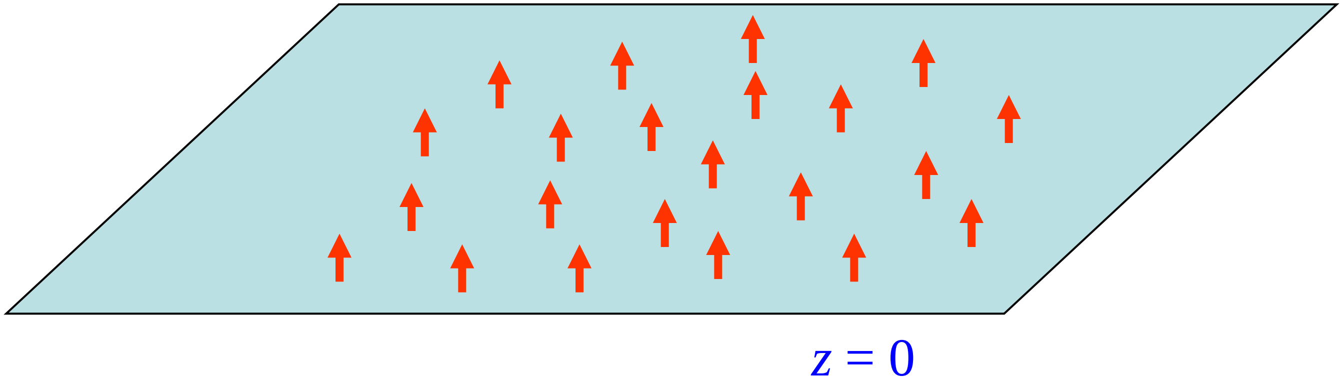
$$V_s^{TM}=-\tilde{M}_{sv}^{i}$$ *lumped* series voltage source

Sources: TM$_z$ (cont.)

For a vertical electric current:

Assume $$J_z^i\left(x,y,z\right) = J_{sz}^i(x,y)\,\delta(z)$$

"planar vertical current distribution"



Sources: TM$_z$ (cont.)

$$J_z^i(x,y,z) = J_{sz}^i(x,y)\,\delta(z)$$

$$V_s^{d^{TM}} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i(k_x,k_y) = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^i(k_x,k_y)\,\delta(z)$$

This corresponds to a *lumped* series voltage source:

$$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^i(k_x,k_y)$$

Sources: TM$_z$ (cont.)

Special case: vertical electric dipole

$$J_z^i(x,y,z) = \delta(x)\delta(y)\delta(z)$$

$$J_{sz}^i(x,y) = \delta(x)\delta(y)$$

$$\Rightarrow \quad \tilde{J}_{sz}^i(k_x,k_y) = 1$$

Hence

$$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)$$

TE_z Fields

Use duality:

$$\underline{E} \rightarrow \underline{H}$$

$$\underline{H} \rightarrow -\underline{E}$$

$$\underline{J}^{\,i} \rightarrow \underline{M}^{\,i}$$

$$\underline{M}^{\,i} \rightarrow -\underline{J}^{\,i}$$

$$\varepsilon \rightleftarrows \mu$$

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^{\,i} - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^{\,i} + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^{\,i} + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM_z

$$-\frac{\partial \tilde{E}_v}{\partial z} = -\tilde{M}_u^{\,i} - j\omega\mu \tilde{H}_u$$

$$\frac{\partial \tilde{H}_u}{\partial z} = +\tilde{J}_v^{\,i} + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^{\,i} - \left(\frac{k_z^2}{j\omega\mu}\right)\tilde{E}_v$$

TE_z

TE$_z$ (cont.)

Define:

$$V^{TE}(z) \equiv -\tilde{E}_v\left(k_x, k_y, z\right)$$
$$I^{TE}(z) \equiv \tilde{H}_u\left(k_x, k_y, z\right)$$

$$\frac{\partial V^{TE}}{\partial z} = -j\omega\mu I^{TE} - \tilde{M}_u^i$$
$$\frac{\partial I^{TE}}{\partial z} = \left(\frac{k_z^2}{j\omega\mu}\right)V^{TE} + \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

$$\frac{\partial V}{\partial z} = -j\omega LI + V_s^d$$
$$\frac{\partial I}{\partial z} = -j\omega CV + I_s^d$$

We then identify

$$L = \mu$$
$$\omega C = \frac{k_z^2}{\omega\mu}$$

$\Rightarrow$

$$k_z^{TL} = k_z$$
$$Z_0^{TE} = \frac{\omega\mu}{k_z}$$

TE_z (cont.)

For the sources, we have

$$V_s^{d\,TE} = -\tilde{M}_u^i$$

$$I_s^{d\,TE} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of planar horizontal surface currents:

$$V_s^{TE} = -\tilde{M}_{su}^i$$ *lumped* series voltage source

$$I_s^{TE} = +\tilde{J}_{sv}^i$$ *lumped* parallel current source

Sources: TE$_z$ (cont.)

For a vertical magnetic current:

Assume $$M_z^i(x,y,z) = M_{sz}^i(x,y)\,\delta(z)$$

This corresponds to a *lumped* parallel current source:

Then we have $$I_s^{TM} = \left(\frac{k_t}{\omega u}\right)\tilde{M}_{sz}^i\left(k_x,k_y\right)$$

Sources: TE$_z$ (cont.)

Special case: vertical magnetic dipole

$$M_z^i(x,y,z) = \delta(x)\delta(y)\delta(z)$$

$$M_{sz}^i(x,y) = \delta(x)\delta(y)$$

$$\Rightarrow \quad \tilde{M}_{sz}^i(k_x,k_y) = 1$$

Hence $$I_s^{TM} = \left(\frac{k_t}{\omega u}\right)$$

Summary

Results for 3D (volumetric) sources

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

Horizontal

$$I_s^{d\,TM} = -\tilde{J}_u^i$$
$$V_s^{d\,TM} = -\tilde{M}_v^i$$

$$I_s^{d\,TE} = +\tilde{J}_v^i$$
$$V_s^{d\,TE} = -\tilde{M}_u^i$$

Vertical

$$V_s^{d\,TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

$$I_s^{d\,TE} = \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Distributed sources:
either parallel current sources
or series voltage sources

Summary

Results for 2D (planar) sources

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

Horizontal

$$I_s^{TM} = -\tilde{J}_{su}^{\,i}$$
$$V_s^{TM} = -\tilde{M}_{sv}^{\,i}$$

$$I_s^{TE} = +\tilde{J}_{sv}^{\,i}$$
$$V_s^{TE} = -\tilde{M}_{su}^{\,i}$$

Vertical

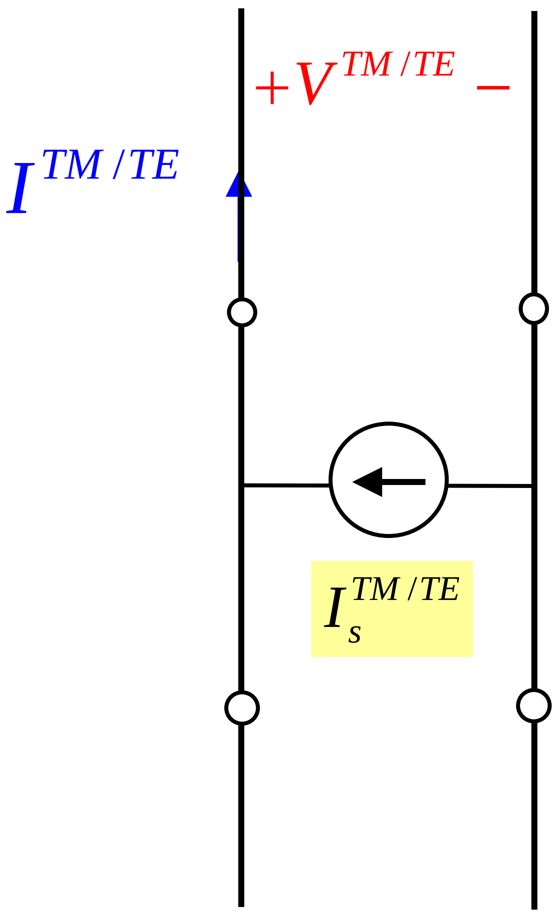
$$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^{\,i}$$

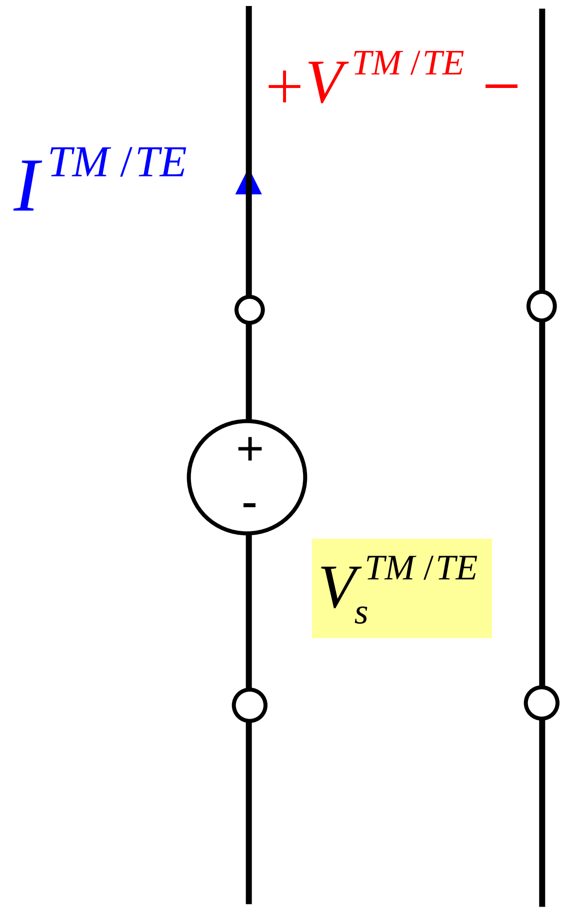
$$I_s^{TE} = \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_{sz}^{\,i}$$

Lumped sources:
either parallel current sources
or series voltage sources

Summary (cont.)

TEN Models





$$V^{TM/TE}(z) = V_i^{TM/TE}(z)\, I_s^{TM/TE}$$
$$I^{TM/TE}(z) = I_i^{TM/TE}(z)\, I_s^{TM/TE}$$

$$V^{TM/TE}(z) = V_v^{TM/TE}(z)\, V_s^{TM/TE}$$
$$I^{TM/TE}(z) = I_v^{TM/TE}(z)\, V_s^{TM/TE}$$

Summary (cont.)

Michalski functions