The Spectral Domain Method in Electromagnetics

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Additional Resources

- Some basic references are provided at the end of these viewgraphs.
- You are welcome to visit a website that goes along with a course at the University of Houston that discusses (among other things) the spectral domain immittance (SDI) method.

ECE 6341: "Advanced Electromagnetic Waves"

http://courses.egr.uh.edu/ECE/ECE6341/

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Spectral Domain Immittance Method

What is it?

It is a powerful, and systematic method for analyzing sources and structures in layered media.

- Electric or magnetic dipole sources within layered media.
- Microstrip and printed antennas.
- Phased arrays, FSS structures, planar radomes.
- Geophysical problems.

The method was originally developed by Itoh and Menzel in the early 1980s (references are given at the end).

Spectral Domain Immittance Method

Where does the name come from?

S D I

- The electric and magnetic fields are modeled in the Fourier transform (spectral) domain.
- The horizontal (transverse) electric and magnetic fields are modeled as voltage and current on a transmission line model of the layered structure (called the transverse equivalent network or TEN).
- The TEN has transmission lines described by characteristic impedances (or admittances). The word "immitance" means either impedance or admittance.

Purpose of Short Course

- To cover the fundamentals of the Spectral Domain Immittance (SDI) method.
- To explain the physical principles of the method.
- To develop the mathematics of the method.
- To develop a systematic "recipe" for applying the method to analyzing different types of sources and structures in layered media.
- To illustrate the method with various practical examples.

Outline

- Physical derivation of method for planar electric surface currents.
- Examples involving planar surface currents:
  - Microstrip line
  - Microstrip patch current
- General derivation (Fourier transforming Maxwell's equations) that allows for all types of sources to be included in one general derivation.
- Examples:
  - Vertical dipole over the earth (Sommerfeld problem)
  - Slot antenna covered with radome layer (magnetic current)

Outline

- **Physical derivation of method for planar electric surface currents.**
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  - Microstrip line
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- Examples:
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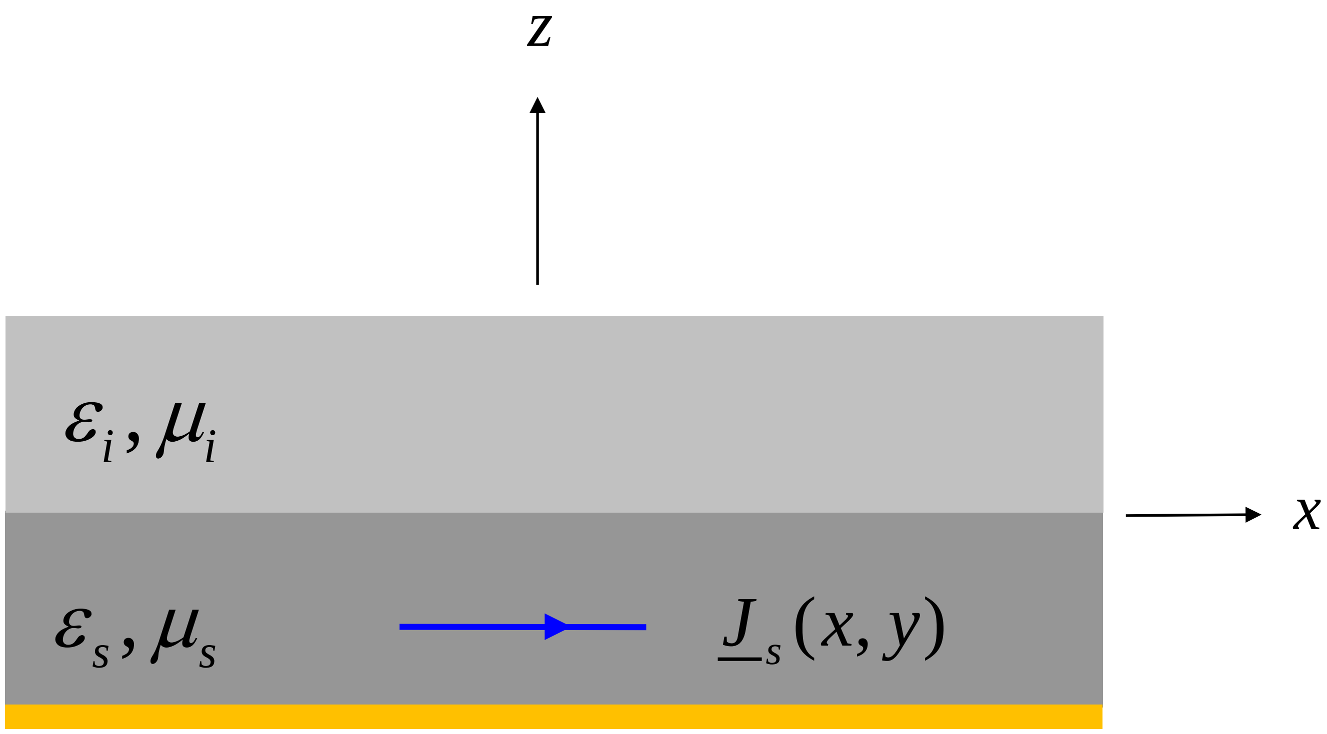
  - Slot antenna covered with radome layer (magnetic current)

Fourier Representation of Planar Currents (cont.)

Consider a planar electric surface current in a layered medium



Note: The subscript "*s*" denotes the source layer

Fourier Representation of Planar Currents (cont.)

A 2D spatial Fourier transform in (x,y) can be used to represent the finite-size current sheet as a collection (superposition) of *infinite phased current sheets.*

$$\underline{\tilde{J}}_s\left(k_x, k_y\right) \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{J}_s\left(x, y\right) e^{+j\left(k_x x + k_y y\right)} dx dy$$

$$\underline{J}_s\left(x, y\right) = \frac{1}{\left(2\pi\right)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{\tilde{J}}_s\left(k_x, k_y\right) e^{-j\left(k_x x + k_y y\right)} dk_x dk_y$$

Fourier Representation of Planar Currents (cont.)

$$\underline{J}_s(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{\tilde{J}}_s(k_x,k_y)\, e^{-j(k_x x + k_y y)}\, dk_x dk_y$$

$$\approx \sum_n \sum_m \underline{\tilde{J}}_s(k_{xm},k_{yn})\, e^{-j(k_{xm} x + k_{yn} y)}\, \Delta k_x \Delta k_y$$

$$= \sum_n \sum_m \underline{\tilde{J}}_s^{\,p}(x,y;k_{xm},k_{yn})$$

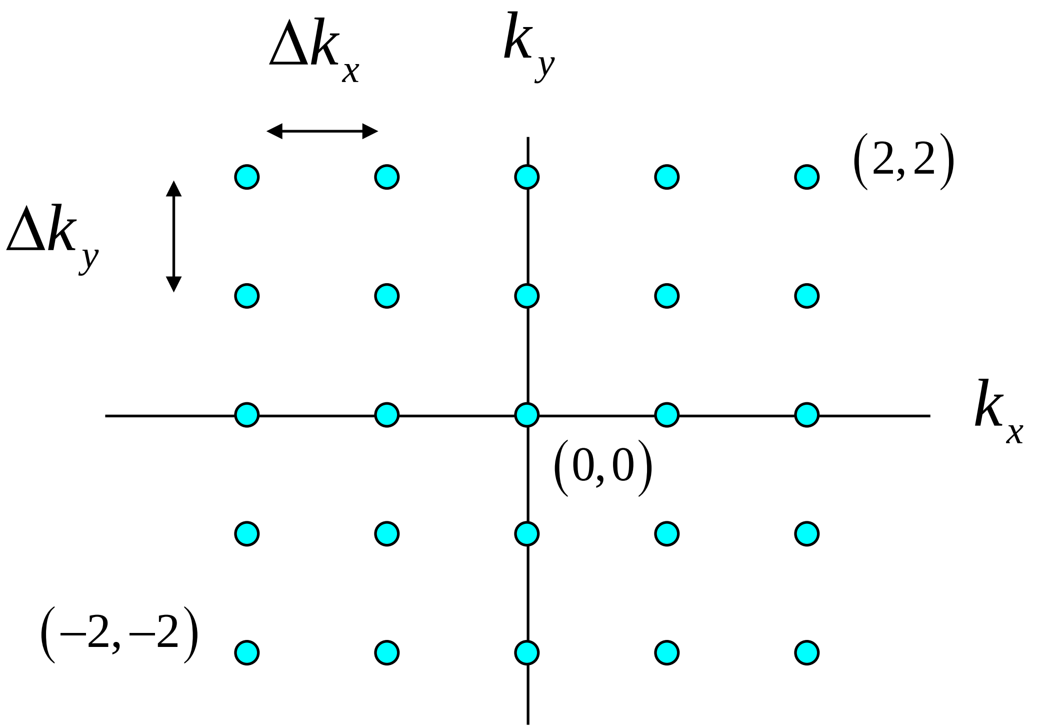
where

$$\underline{J}_s^{\,p}(x,y;k_{xm},k_{yn}) = \underline{J}_{s0}^{\,p}(k_{xm},k_{yn})\, e^{-j(k_{xm} x + k_{yn} y)}$$

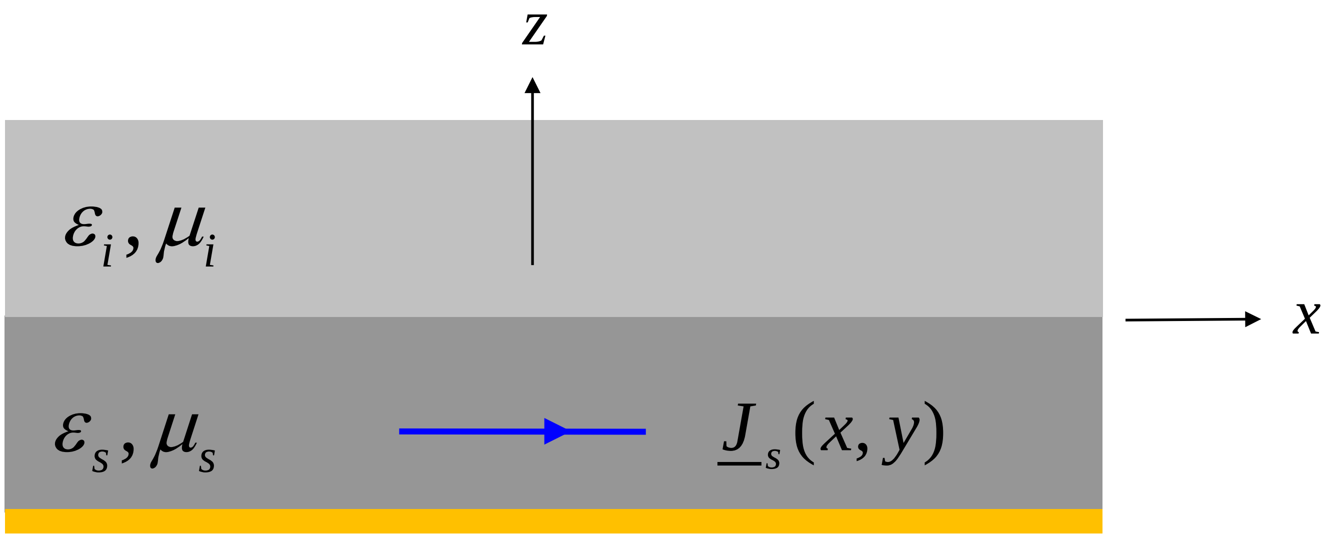
$$\underline{J}_{s0}^{\,p}(k_{xm},k_{yn}) = \frac{1}{(2\pi)^2}\, \underline{\tilde{J}}_s(k_{xm},k_{yn})\, \Delta k_x \Delta k_y$$

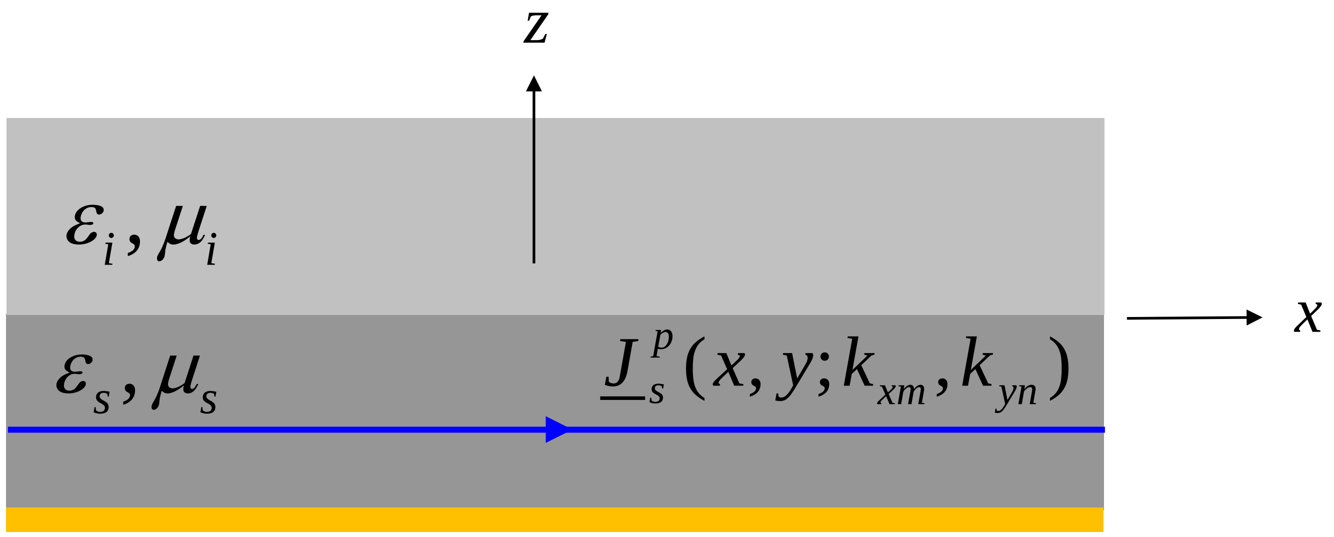
Fourier Representation of Planar Currents (cont.)

$$\underline{J}_{s0}^{p}\left(k_{xm},k_{yn}\right)=\frac{1}{\left(2\pi\right)^{2}}\underline{\tilde{J}}\left(k_{xm},k_{yn}\right)\Delta k_x\Delta k_y$$



Fourier Representation of Planar Currents (cont.)



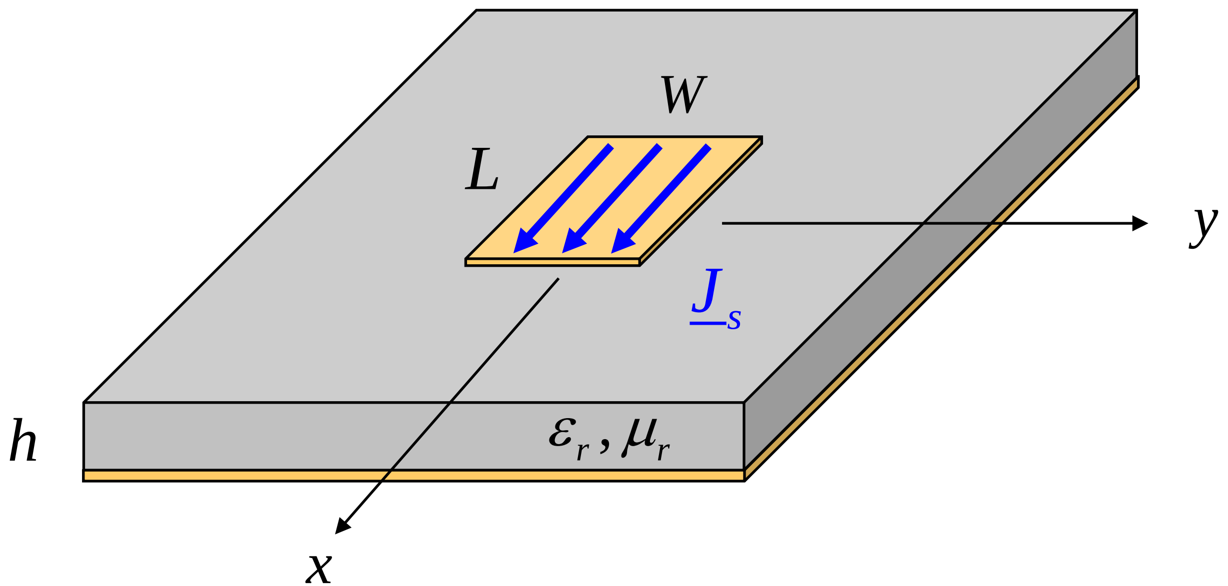


Fourier Representation of Planar Currents (cont.)

Example

A microstrip patch has a dominant (1,0) mode surface current that is described by

$$\underline{J}_s(x,y) = \underline{\hat{x}}\left[\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)\right]$$



Fourier Representation of Planar Currents (cont.)

Example (cont.)

$$\underline{J}_s(x, y) = \underline{\hat{x}}\left[\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)\right]$$

$$\underline{\tilde{J}}_s\left(k_x, k_y\right) = \underline{\hat{x}}\,\frac{1}{W}\int_{L/2}^{L/2}\cos\left(\frac{\pi y}{L}\right)e^{jk_x x}\,dx\int_{-W/2}^{W/2}e^{jk_y y}\,dy$$

$$\underline{\tilde{J}}_s\left(k_x, k_y\right) = \underline{\hat{x}}\,\frac{\pi L}{2}\left[\frac{\cos\left(k_x\dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2-\left(\dfrac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y\frac{W}{2}\right)$$

Fourier Representation of Planar Currents (cont.)

Example (cont.)

$$\underline{\tilde{J}}_s\left(k_x, k_y\right) = \underline{\hat{x}}\,\frac{\pi L}{2}\left[\frac{\cos\left(k_x \dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(\dfrac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y \frac{W}{2}\right)$$

$$\underline{J}_{s0}^{\,p}\left(k_{xm}, k_{yn}\right) = \frac{1}{(2\pi)^2}\,\underline{\tilde{J}}\left(k_{xm}, k_{yn}\right)\Delta k_x \Delta k_y$$

Hence

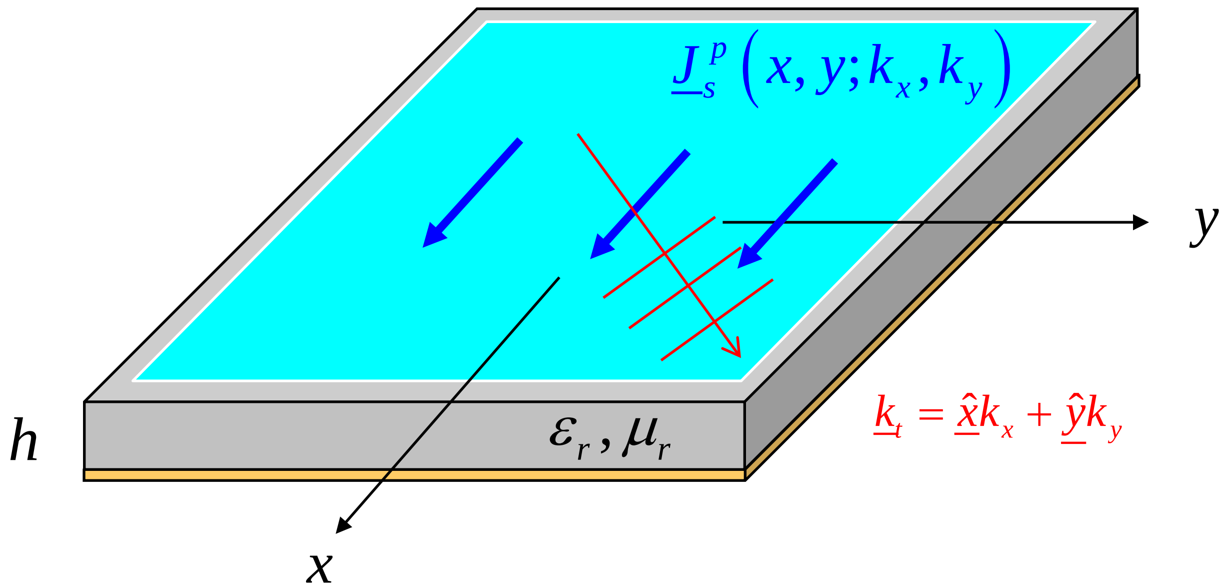
$$\underline{J}_{s0}^{\,p}\left(k_{xm}, k_{yn}\right) = \underline{\hat{x}}\,\frac{1}{(2\pi)^2}\left[\frac{\pi L}{2}\left[\frac{\cos\left(k_x \dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(\dfrac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y \frac{W}{2}\right)\right]\left(\Delta k_x \Delta k_y\right)$$

Fourier Representation of Planar Currents (cont.)

Example (cont.)

$$\underline{J}_s^p\left(x,y;k_{xm},k_{yn}\right)=\underline{J}_{s0}^p\left(k_{xm},k_{yn}\right)e^{-j\left(k_{xm}x+k_{yn}y\right)}$$

$$\underline{J}_{s0}^p\left(k_{xm},k_{yn}\right)=\hat{\underline{x}}\,\frac{1}{(2\pi)^2}\left[\frac{\pi L}{2}\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_xL}{2}\right)^2}\right]\operatorname{sinc}\left(k_y\frac{W}{2}\right)\right]\left(\Delta k_x\Delta k_y\right)$$

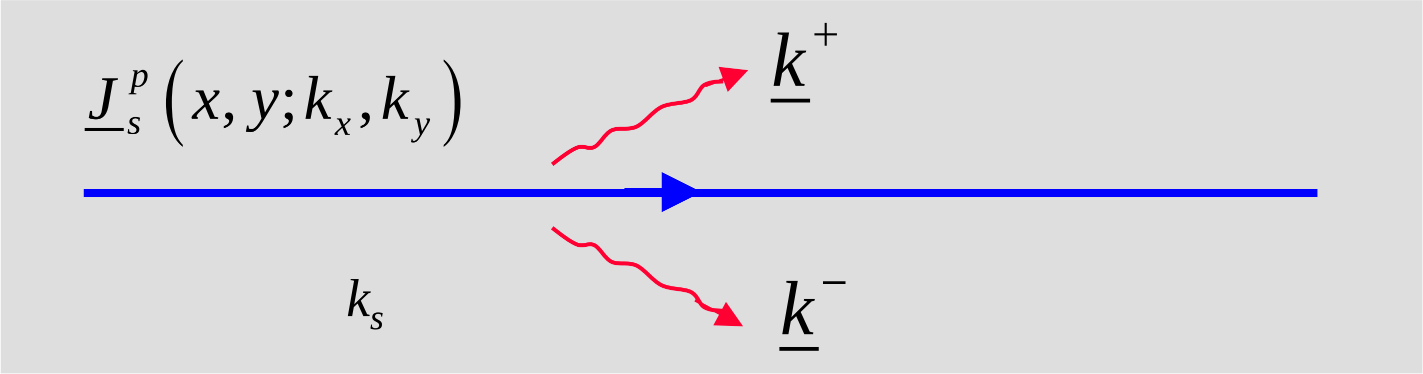


Fourier Representation of Planar Currents (cont.)

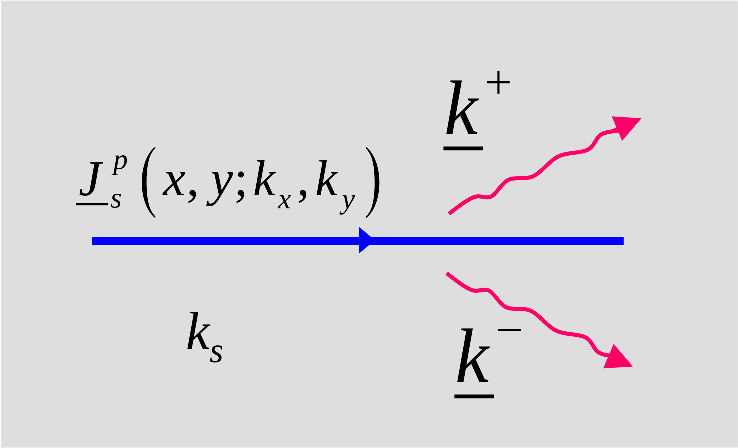
Consider an infinite phased current sheet having some (k_x, k_y):

$$\underline{J}_s^p\left(x, y; k_x, k_y\right) = \underline{J}_{s0}^p\left(k_x, k_y\right) e^{-j\left(k_x x + k_y y\right)}$$

This phased current sheet launches a pair of plane waves as shown below:



Fourier Representation of Planar Currents (cont.)



$$\underline{k}^{+} = \underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y + \underline{\hat{z}}\, k_{zs}$$

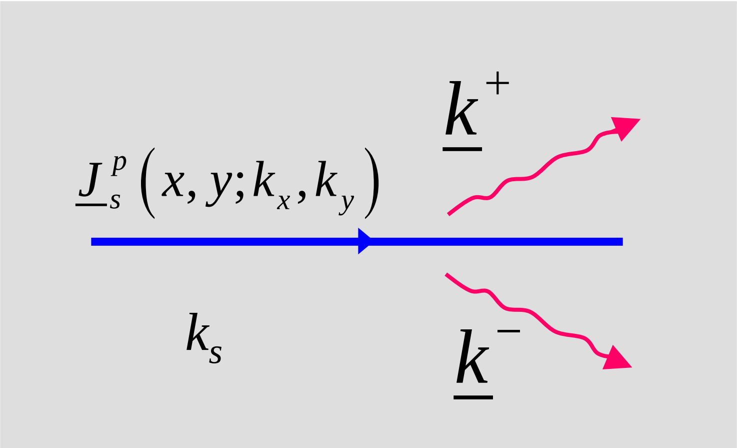
$$\underline{k}^{-} = \underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y - \underline{\hat{z}}\, k_{zs}$$

$$k_{zs} = \left(k_s^2 - k_t^2\right)^{\frac{1}{2}}$$

$$k_t^2 = k_x^2 + k_y^2$$

Note: We choose the physical choice where k_{zs} is either positive real or negative imaginary.

Fourier Representation of Planar Currents (cont.)



Goal: Decompose the current into two parts: one that excites only a pair of TM_z plane waves, and one that excites only a pair of TE_z plane waves.

TM_z and TE_z waves do not couple at boundaries.

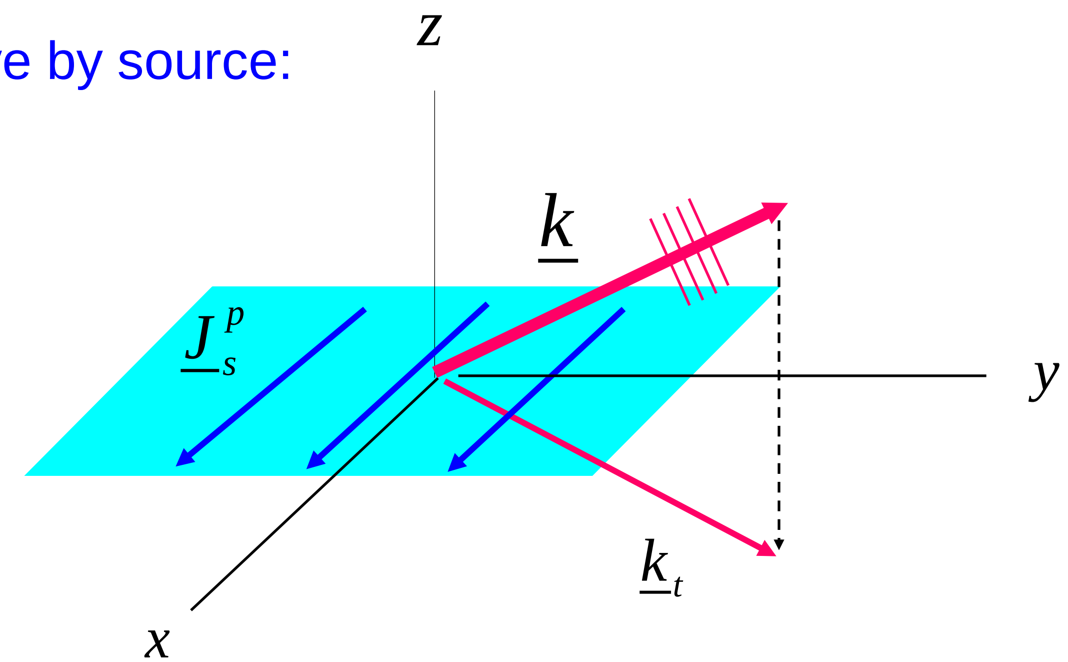
(u,v) Coordinates

Define:

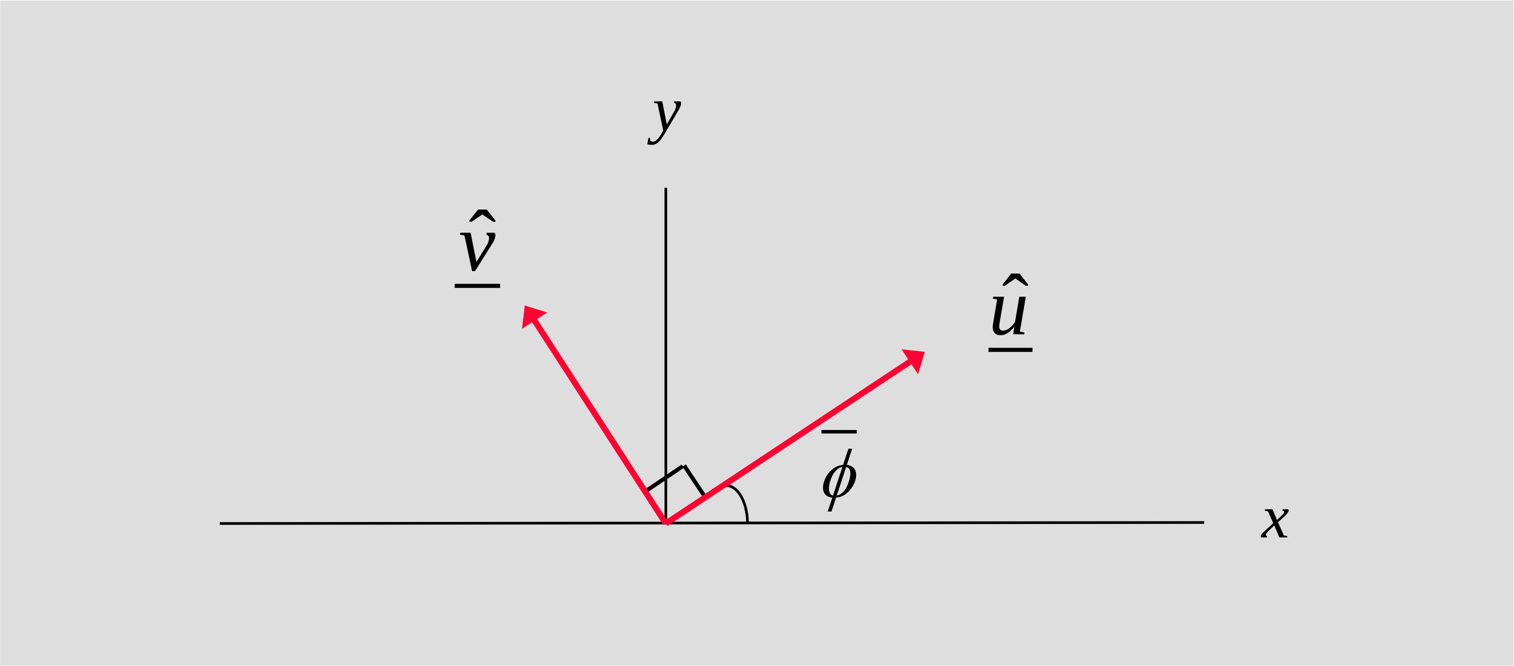
$$\underline{k}_t = \underline{\hat{x}}\, k_x + \underline{\hat{y}} k_y$$

$$\underline{\hat{u}} = \frac{\underline{k}_t}{k_t} \qquad \underline{\hat{v}} = \underline{\hat{z}} \times \underline{\hat{u}}$$

Launching of plane wave by source:

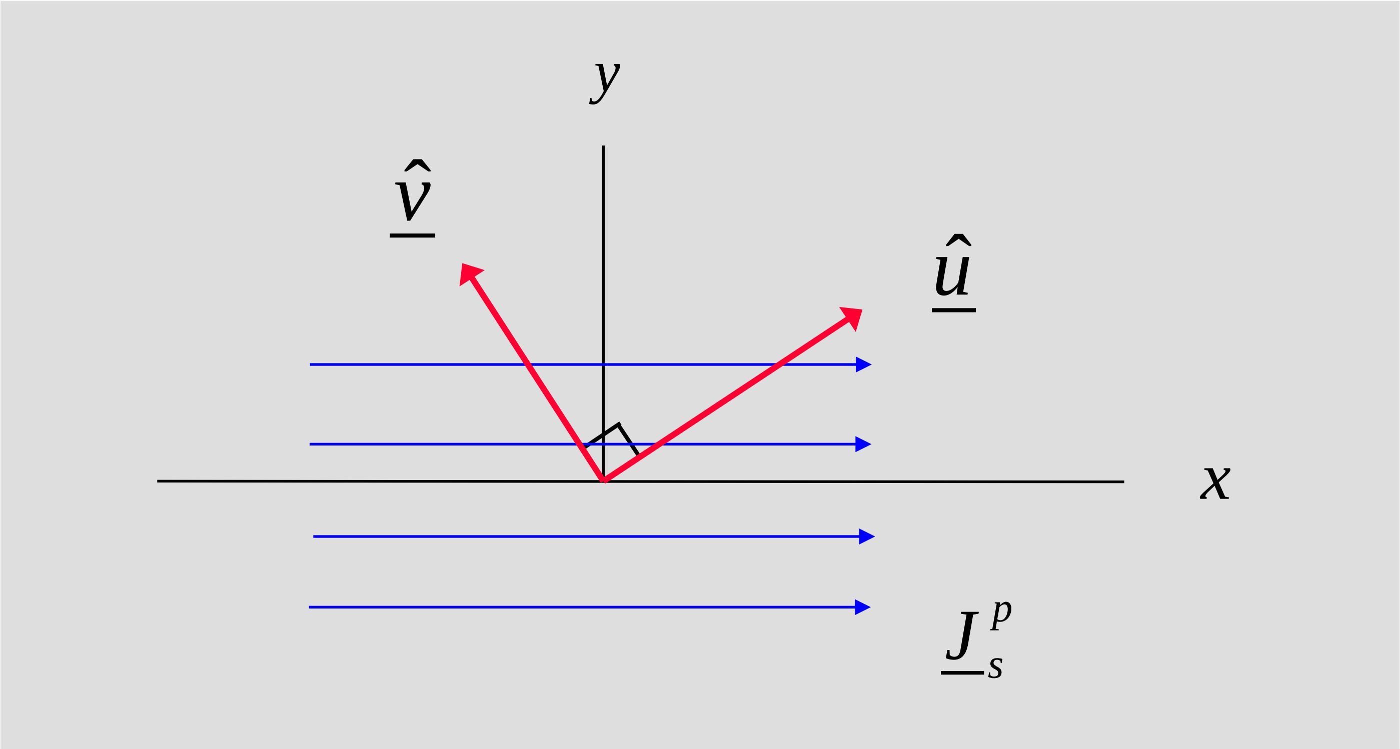


(u,v) Coordinates (cont.)



$$\cos\overline{\phi} = \frac{k_x}{k_t} \qquad \sin\overline{\phi} = \frac{k_y}{k_t}$$

(u,v) Coordinates (cont.)



$\underline{J}_{su}^p$: Launches a TM_z wave

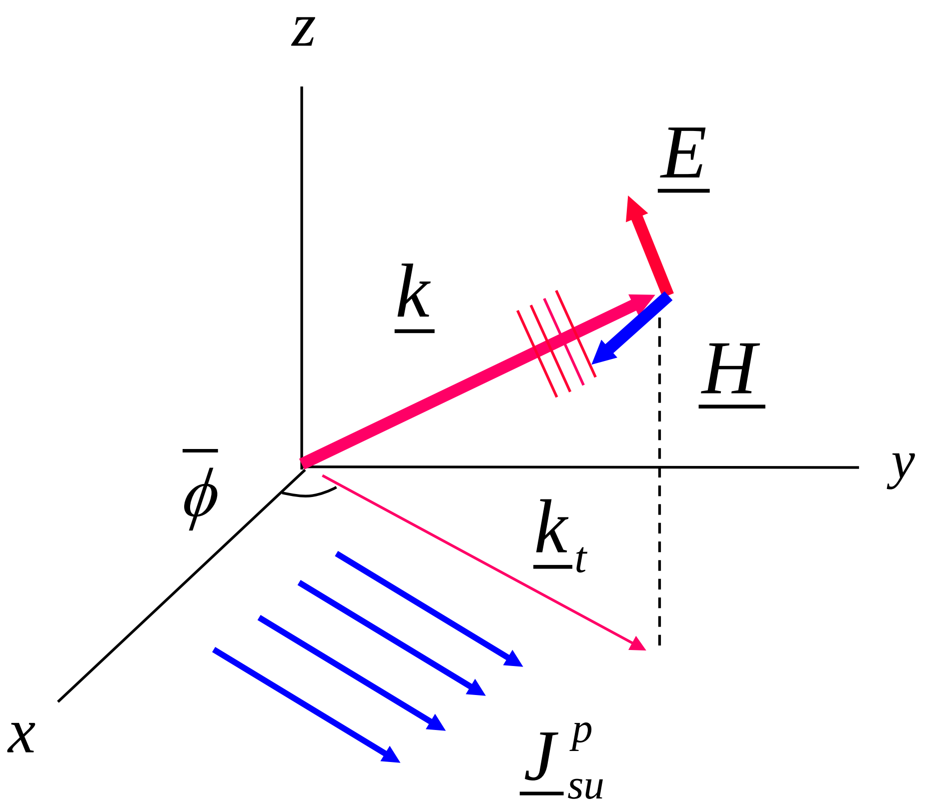
$\underline{J}_{sv}^p$: Launches a TE_z wave

TM$_z$-TE$_z$ Properties

TM$_z$

$$\underline{E}_t = \underline{\hat{u}}\, E_u$$

$$\underline{H}_t = \underline{\hat{v}}\, H_v$$

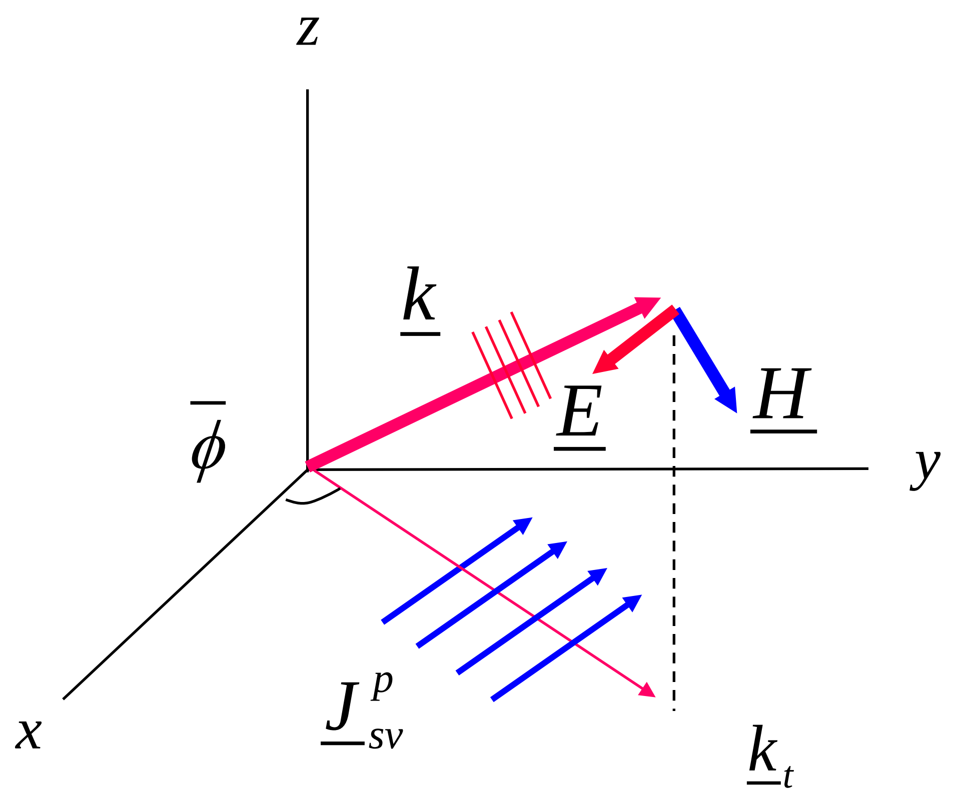


TM_z-TE_z Properties (cont.)

TE_z

$$\underline{E}_t = \underline{\hat{v}} E_v$$

$$\underline{H}_t = \underline{\hat{u}} H_u$$



TM_z-TE_z Properties (cont.)

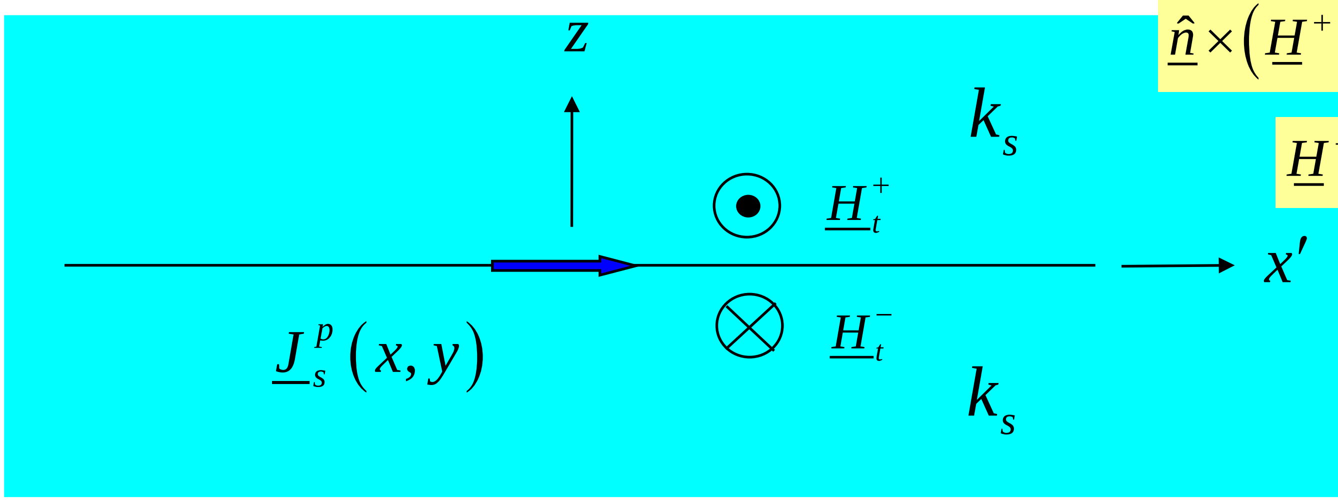
Proof of launching property:

Consider the TM_z case (u is labeled as x' for simplicity here).

An infinite medium is considered, since we are only looking at the launching property of the waves.

$$\underline{\hat{n}} \times \left(\underline{H}^{+}(0) - \underline{H}^{-}(0) \right) = \underline{J}_s$$

$$\underline{H}^{+}(0) = -\underline{H}^{-}(0)$$



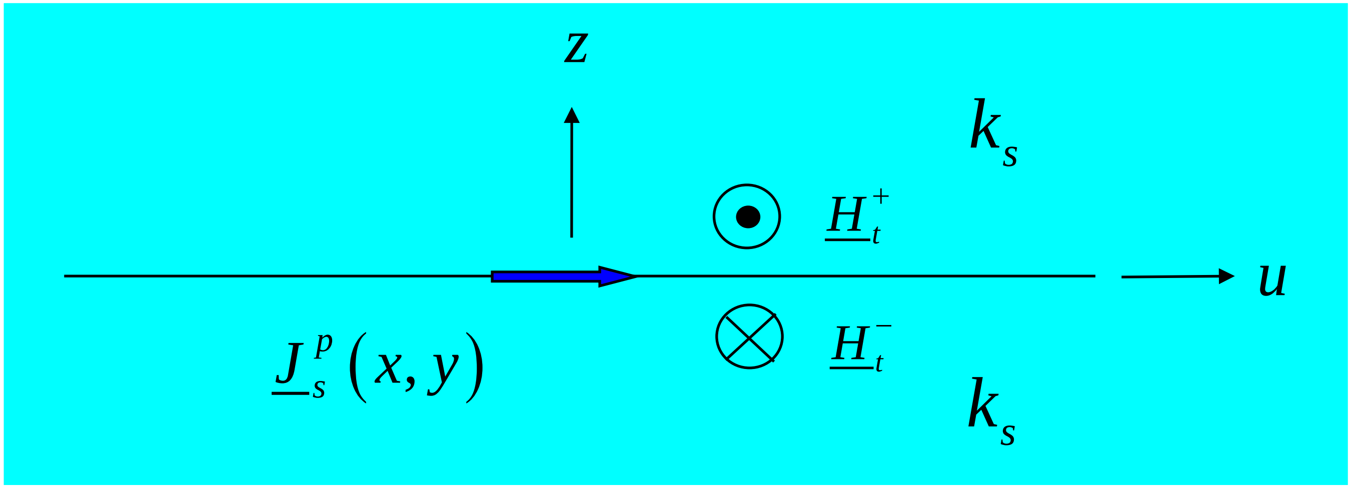
$$\underline{J}_s^{p}(x, y) = \underline{\hat{x}}' A e^{-jk_x x'}$$

Exact solution:

$$\underline{H}^{\pm}(x, y, z) = \pm \frac{1}{2} \left(-\underline{\hat{y}}' \right) (A) e^{-jk_x x'} e^{\mp jk_z z}$$

TM_z-TE_z Properties (cont.)

In the general coordinate system:



$$\underline{J}_s^p(x,y) = \underline{\hat{u}} A e^{-j(k_x x + k_y y)}$$

$$\underline{H}^{\pm}(x,y,z) = \pm\frac{1}{2}(-\underline{\hat{v}})(A)e^{-j(k_x x + k_y y)}e^{\mp jk_z z}$$

TM_z

TM$_z$-TE$_z$ Properties (cont.)

Two cases of interest:

$$\underline{J}_s^p(x,y) = \underline{\hat{u}}\, J_{su}^p(x,y) \quad \Rightarrow \quad \underline{H}_t^+ = \underline{\hat{v}}\, H_v^+ \quad (\text{TM}_z)$$

$$\underline{J}_s^p(x,y) = \underline{\hat{v}} J_{sv}^p(x,y) \quad \Rightarrow \quad \underline{H}_t^+ = \underline{\hat{u}}\, H_u^+ \quad (\text{TE}_z)$$

Decomposition of Current

Split general phased current into two parts:

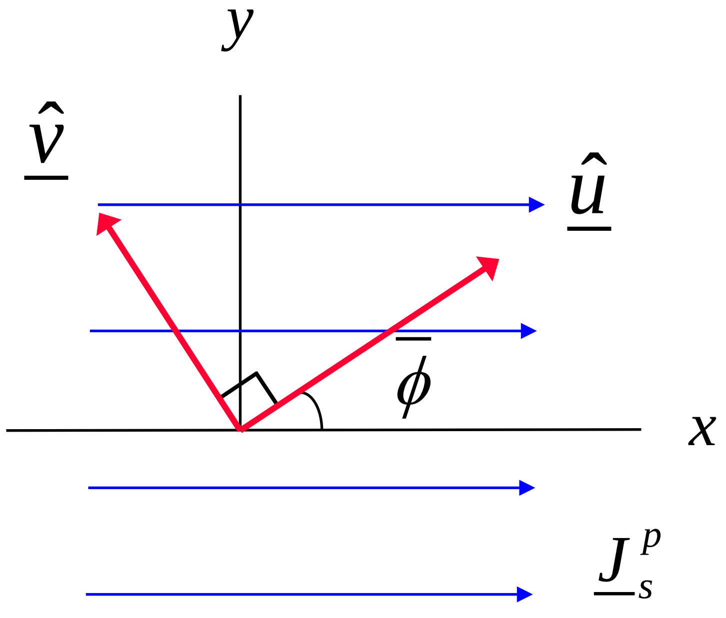
$$\underline{J}_s^p(x,y) = \underline{\hat{u}}\, J_{su}^p(x,y) + \underline{\hat{v}}\, J_{sv}^p(x,y)$$

Launches TM_z

Launches TE_z

$$J_{su}^p(x,y) = \underline{J}_s^p(x,y)\cdot\underline{\hat{u}}$$

$$J_{sv}^p(x,y) = \underline{J}_s^p(x,y)\cdot\underline{\hat{v}}$$

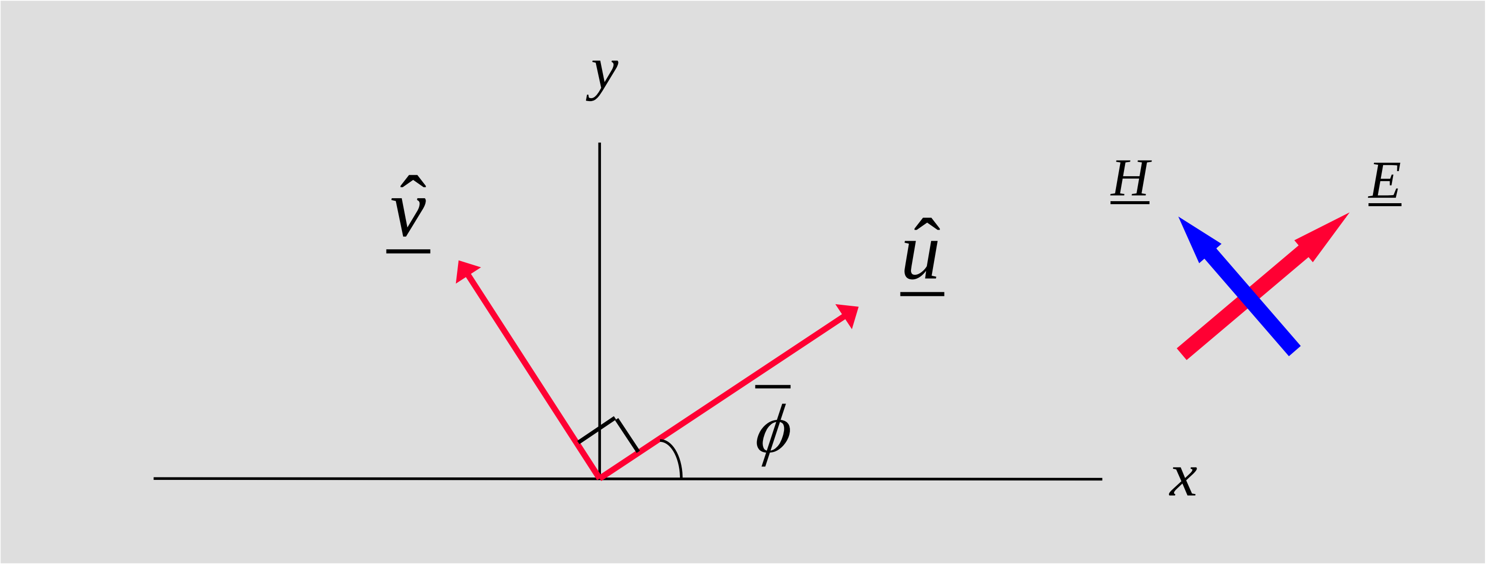


Transverse Fields

TM_z :

$$\underline{E}_t = \underline{\hat{u}}\, E_u(x,y,z) = \underline{\hat{u}}\, E_{u0}(z)\, e^{-j\left(k_x x + k_y y\right)}$$

$$\underline{H}_t = \underline{\hat{v}}\, H_v(x,y,z) = \underline{\hat{v}}\, H_{v0}(z)\, e^{-j\left(k_x x + k_y y\right)}$$

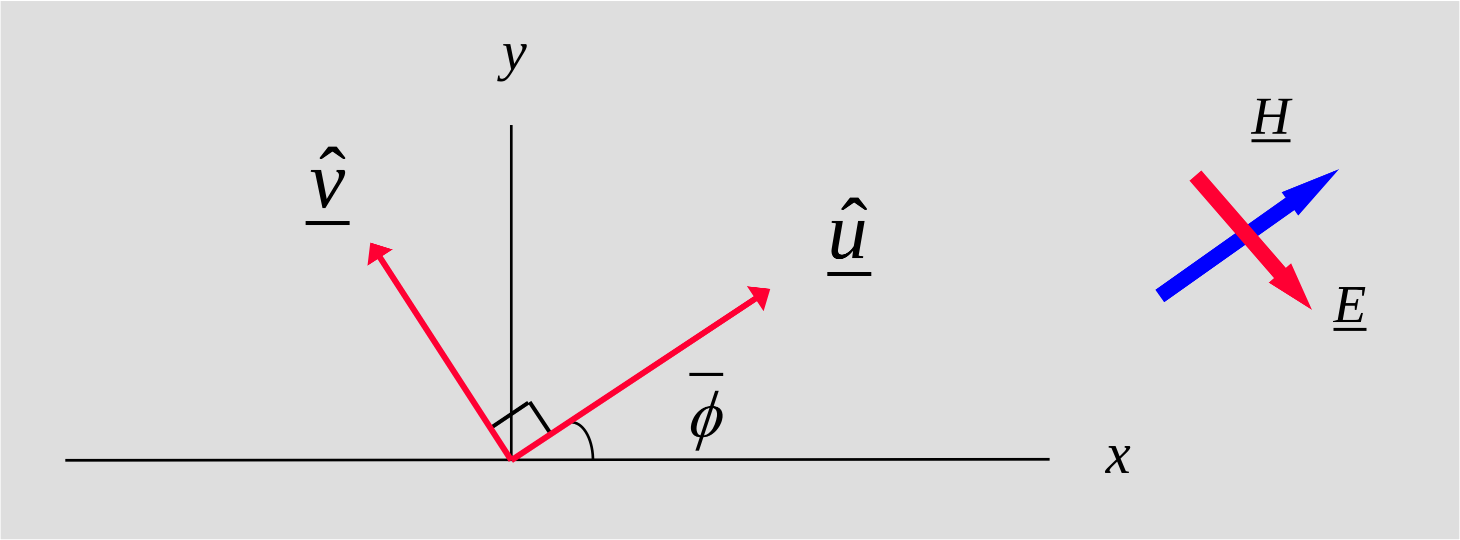


Transverse Fields (cont.)

TE_z :

$$\underline{E}_t = \underline{\hat{v}}\, E_v(x,y,z) = \underline{\hat{v}}\, E_{v0}(z)\, e^{-j(k_x x + k_y y)}$$

$$\underline{H}_t = \underline{\hat{u}}\, H_u(x,y,z) = \underline{\hat{u}}\, H_{u0}(z)\, e^{-j(k_x x + k_y y)}$$



Transverse Equivalent Network (TEN)

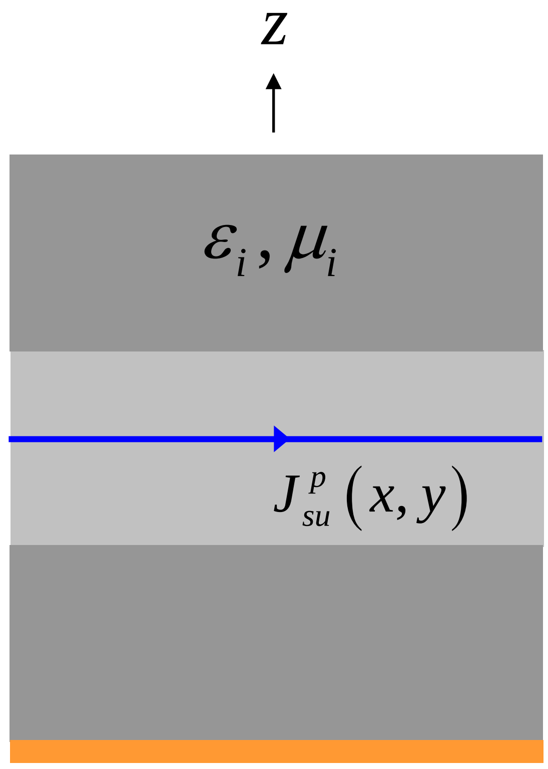
TEN modeling equations:

$$\text{TM}_z: \quad V^{TM}(z) = E_{u0}(z)$$
$$I^{TM}(z) = H_{v0}(z)$$

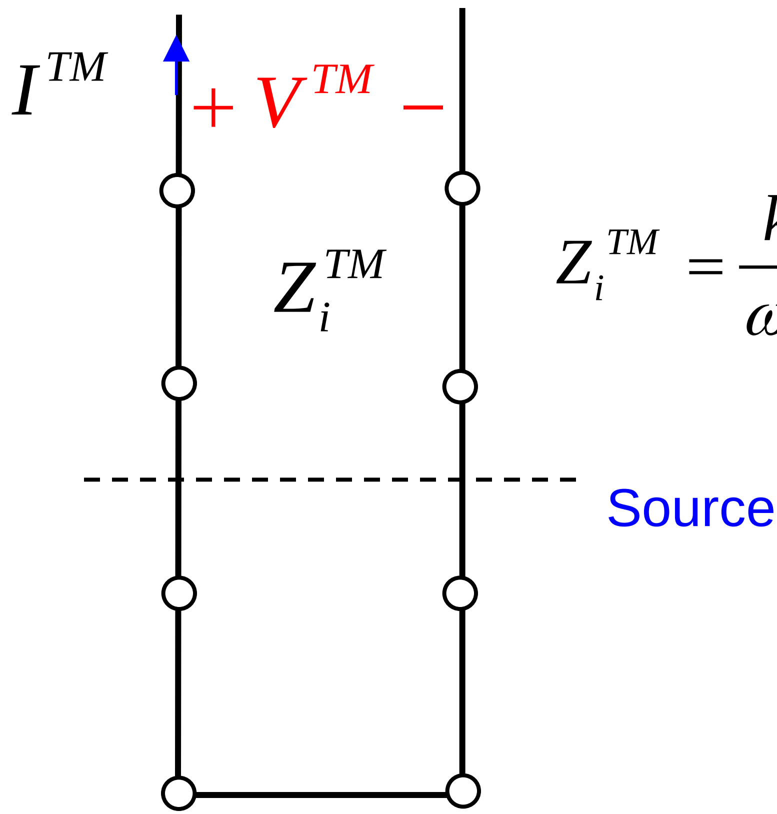
$$\text{TE}_z: \quad V^{TE}(z) = -E_{v0}(z)$$
$$I^{TE}(z) = H_{u0}(z)$$

TEN (cont.)

TM_z



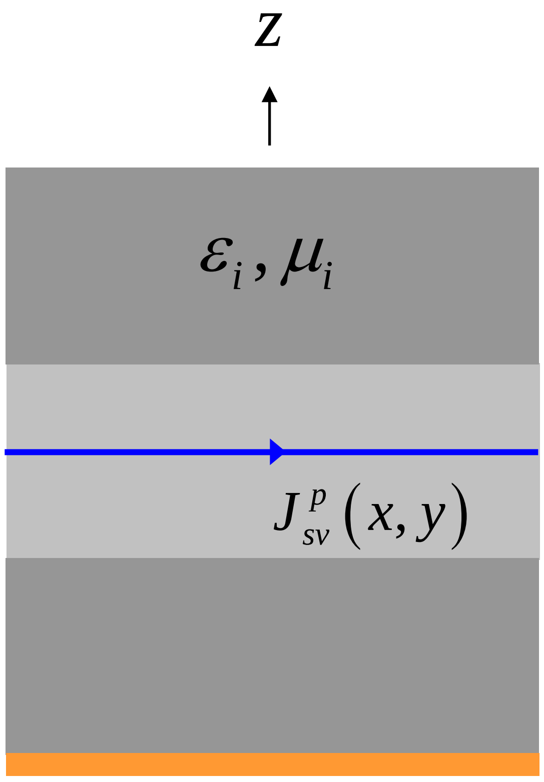
$$k_{zi} = \left(k_i^2 - k_x^2 - k_y^2\right)^{1/2}$$

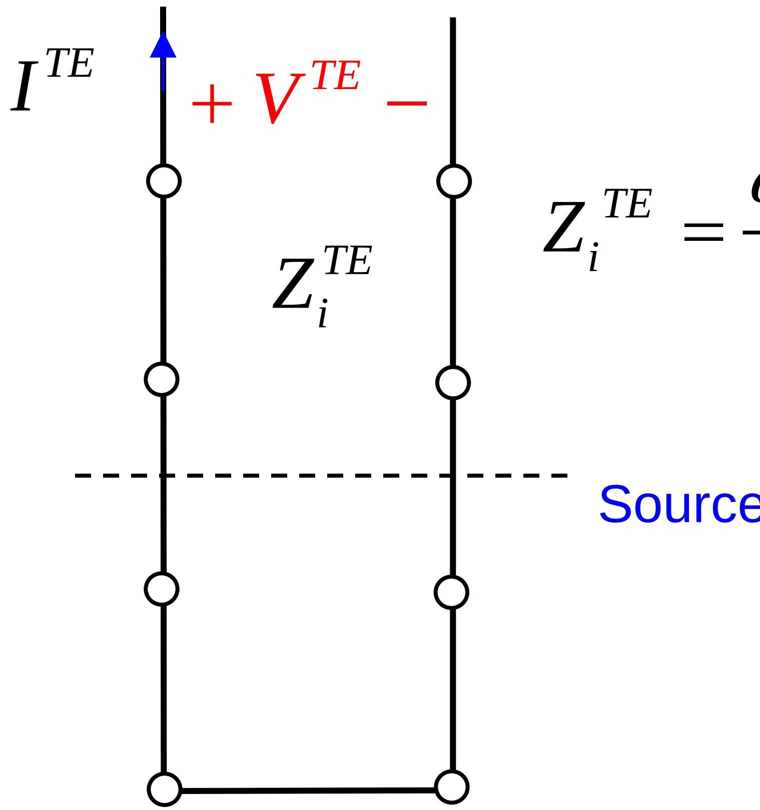


$$Z_i^{TM} = \frac{k_{zi}}{\omega \varepsilon_i}$$

TEN (cont.)

TE_z





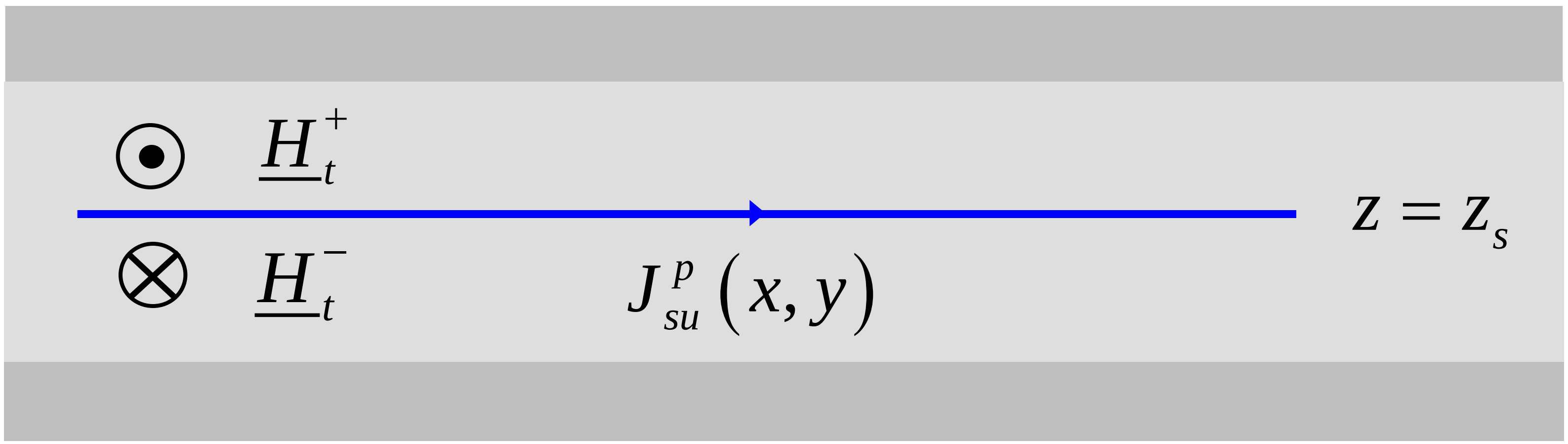
$$Z_i^{TE} = \frac{\omega \mu_i}{k_{zi}}$$

$$k_{zi} = \left(k_i^2 - k_x^2 - k_y^2\right)^{1/2}$$

Source Model

TM_z

Phased current sheet inside layered medium



$$\underline{\hat{z}} \times \left(\underline{H}_t^+ - \underline{H}_t^- \right) = \underline{J}_s(x,y) = \underline{\hat{u}}\, J_{su}^p(x,y)$$

$$\underline{\hat{z}} \times \left[\underline{\hat{v}} \left(H_v^+ - H_v^- \right) \right] = \underline{\hat{u}}\, J_{su}^p(x,y)$$

$$-\underline{\hat{u}} \left(H_v^+ - H_v^- \right) = \underline{\hat{u}}\, J_{su}^p(x,y)$$

Source Model (cont.)

$$H_v^+(x,y) - H_v^-(x,y) = -J_{su}^p(x,y)$$

so $$H_{v0}^+ - H_{v0}^- = -J_{su0}^p$$

Hence $$I^{TM}(z_s^+) - I^{TM}(z_s^-) = -J_{su0}^p$$

Also $$\underline{\hat{z}} \times \left(\underline{E}_t^+ - \underline{E}_t^-\right) = \underline{0}$$

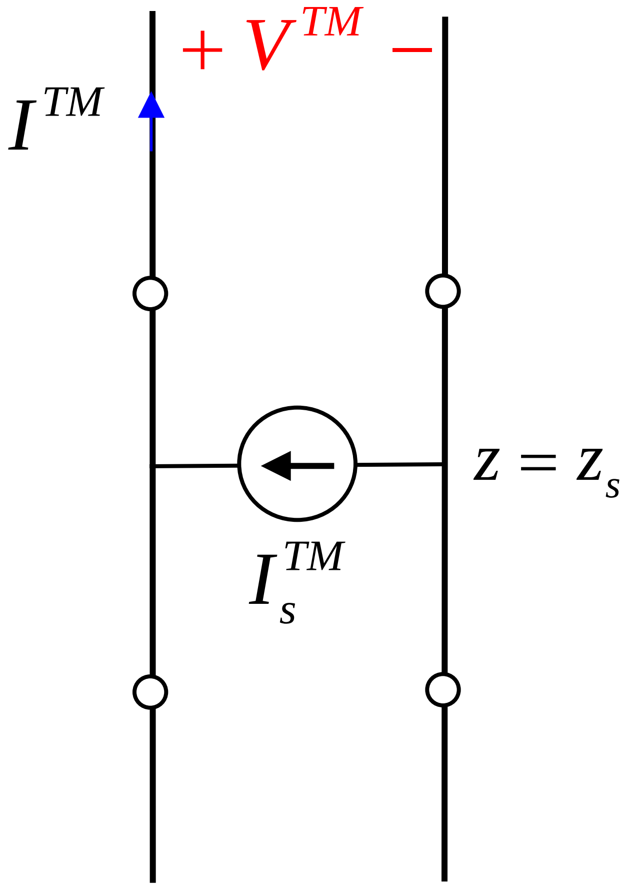
$$\Longrightarrow \left(\underline{\hat{z}} \times \underline{\hat{u}}\right)\left(E_u^+ - E_u^-\right) = \underline{0} \Longrightarrow E_u^+ = E_u^- \Longrightarrow E_{u0}^+ = E_{u0}^-$$

Hence $$V^{TM}(z_s^+) = V^{TM}(z_s^-)$$

Source Model (cont.)

TM$_z$ Model:

$$I_s^{TM} = -\underline{J}_{s0}^{p} \cdot \underline{\hat{u}}$$

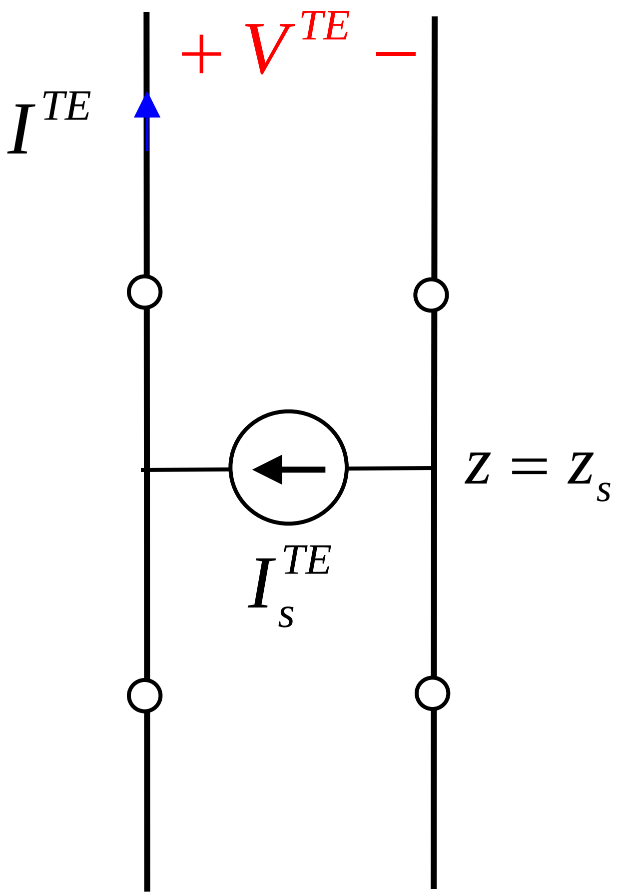


Source Model (cont.)

TE_z Model:

$$I_s^{TE} = \underline{J}_{s0}^{\,p} \cdot \underline{\hat{v}}$$

(derivation omitted)



Source Model (cont.)

The transverse fields can then be found from

$$E_u(x,y,z) = V^{TM}(z)\, e^{-j(k_x x + k_y y)}$$

$$H_v(x,y,z) = I^{TM}(z)\, e^{-j(k_x x + k_y y)}$$

$$-E_v(x,y,z) = V^{TE}(z)\, e^{-j(k_x x + k_y y)}$$

$$H_u(x,y,z) = I^{TE}(z)\, e^{-j(k_x x + k_y y)}$$

Source Model (cont.)

Introduce normalized voltage and current functions:
(following the notation of K. A. Michalski)

$$V_i^{TM/TE}(z) = V^{TM/TE}(z) \quad \text{when} \quad I_s^{TM/TE} = 1 \ [A]$$

$$I_i^{TM/TE}(z) = I^{TM/TE}(z) \quad \text{when} \quad I_s^{TM/TE} = 1 \ [A]$$

Then we can write:

$$V^{TM}(z) = V_i^{TM}(z)\left(-\underline{J}_{s0}^{\,p} \cdot \underline{\hat{u}}\right)$$

$$I^{TM}(z) = I_i^{TM}(z)\left(-\underline{J}_{s0}^{\,p} \cdot \underline{\hat{u}}\right)$$

$$V^{TE}(z) = V_i^{TE}(z)\left(\underline{J}_{s0}^{\,p} \cdot \underline{\hat{v}}\right)$$

$$I^{TE}(z) = I_i^{TE}(z)\left(\underline{J}_{s0}^{\,p} \cdot \underline{\hat{v}}\right)$$

Source Model (cont.)

The transverse fields are then expressed as

$$E_u(x,y,z) = V_i^{TM}(z)\left(-\underline{J}_{s0}^{p} \cdot \underline{\hat{u}}\right) e^{-j\left(k_x x + k_y y\right)}$$

$$H_v(x,y,z) = I_i^{TM}(z)\left(-\underline{J}_{s0}^{p} \cdot \underline{\hat{u}}\right) e^{-j\left(k_x x + k_y y\right)}$$

$$-E_v(x,y,z) = V_i^{TE}(z)\left(+\underline{J}_{s0}^{p} \cdot \underline{\hat{v}}\right) e^{-j\left(k_x x + k_y y\right)}$$

$$H_u(x,y,z) = I_i^{TE}(z)\left(+\underline{J}_{s0}^{p} \cdot \underline{\hat{v}}\right) e^{-j\left(k_x x + k_y y\right)}$$

Example

Find $E_x(x, y, z)$ for $z \geq 0$ due to a surface current at $z = 0$,

$$\underline{J}_s^p = \underline{\hat{x}}\, e^{-j\left(k_x x + k_y y\right)}$$

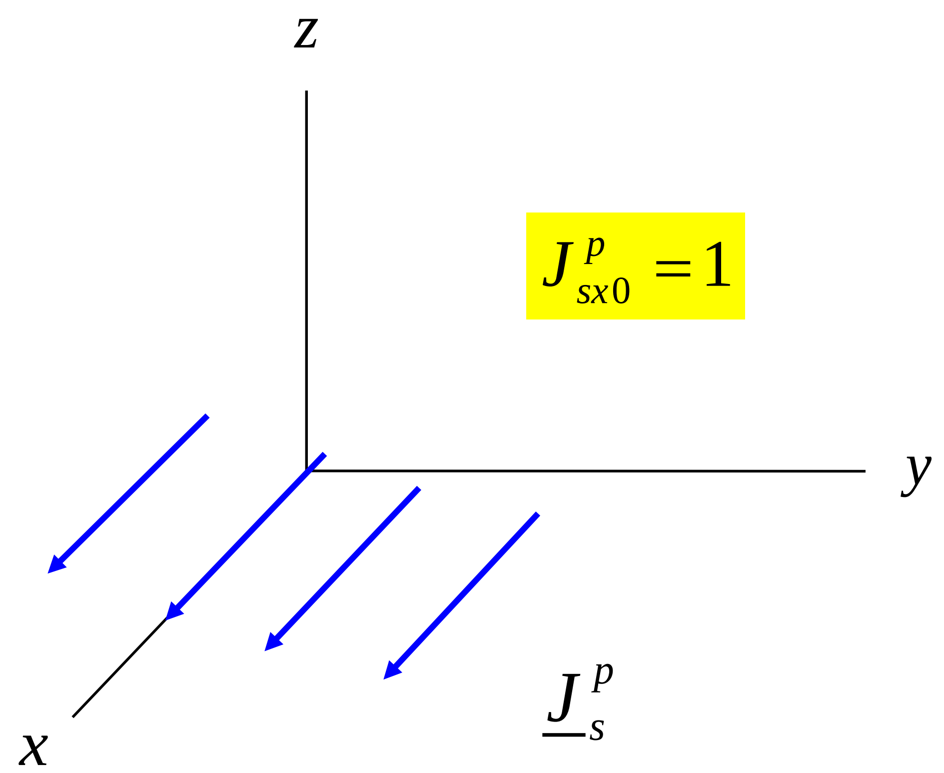
with

$$k_x = 2k_0$$

$$k_y = k_0$$

The current is radiating in free space.

$$k_t = \sqrt{5}k_0$$



Example (cont.)

Decompose current into two parts:

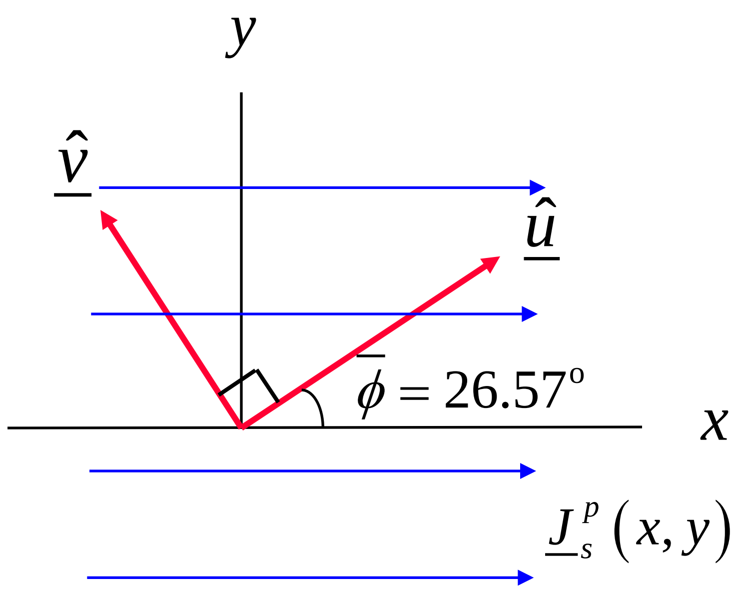
$$E_x = E_u\left(\underline{\hat{u}} \cdot \underline{\hat{x}}\right) + E_v\left(\underline{\hat{v}} \cdot \underline{\hat{x}}\right)$$

$$\tan\overline{\phi} = \frac{k_y}{k_x} = 0.5$$

$$\overline{\phi} = 26.57^{\circ}$$

$$\underline{\hat{u}} \cdot \underline{\hat{x}} = \cos\overline{\phi} = \frac{k_x}{k_t} = \frac{2}{\sqrt{5}}$$

$$\underline{\hat{v}} \cdot \underline{\hat{x}} = -\sin\overline{\phi} = -\frac{k_y}{k_t} = -\frac{1}{\sqrt{5}}$$



Example (cont.)

$$
\begin{aligned}
E_x &= E_u\left(\cos\overline{\phi}\right)+E_v\left(-\sin\overline{\phi}\right)\\
&= E_u\left(\frac{k_x}{k_t}\right)+E_v\left(-\frac{k_y}{k_t}\right)\\
&= \frac{1}{k_t}\left[k_x E_{u0}-k_y E_{v0}\right]e^{-j\left(k_x x+k_y y\right)}\\
&= \frac{1}{k_t}\left[k_x V^{TM}\left(z\right)+k_y V^{TE}\left(z\right)\right]e^{-j\left(k_x x+k_y y\right)}\\
&= \frac{1}{k_t}\left[k_x V_i^{TM}\left(z\right)\left(I_s^{TM}\right)+k_y V_i^{TE}\left(z\right)\left(I_s^{TE}\right)\right]e^{-j\left(k_x x+k_y y\right)}\\
&= \frac{1}{k_t}\left[k_x V_i^{TM}\left(z\right)\left(-\underline{J}_{s0}^{p}\cdot\underline{\hat{u}}\right)+k_y V_i^{TE}\left(z\right)\left(+\underline{J}_{s0}^{p}\cdot\underline{\hat{v}}\right)\right]e^{-j\left(k_x x+k_y y\right)}
\end{aligned}
$$

Example (cont.)

$$E_x = \frac{1}{k_t}\left[k_x V_i^{TM}(z)\left(-\underline{J}_{s0}^{\,p}\cdot\underline{\hat{u}}\right)+k_y V_i^{TE}(z)\left(+\underline{J}_{s0}^{\,p}\cdot\underline{\hat{v}}\right)\right]e^{-j\left(k_x x+k_y y\right)}$$

$$= \frac{1}{k_t}\left[k_x V_i^{TM}(z)\left(-J_{sx0}^{\,p}\right)\left(\underline{\hat{x}}\cdot\underline{\hat{u}}\right)+k_y V_i^{TE}(z)\left(+J_{sx0}^{\,p}\right)\left(x\cdot\underline{\hat{v}}\right)\right]e^{-j\left(k_x x+k_y y\right)}$$

$$= \frac{1}{k_t}\left[k_x V_i^{TM}(z)\left(-J_{sx0}^{\,p}\right)\left(\cos\overline{\phi}\right)+k_y V_i^{TE}(z)\left(+J_{sx0}^{\,p}\right)\left(-\sin\overline{\phi}\right)\right]e^{-j\left(k_x x+k_y y\right)}$$

$$= \frac{1}{k_t}\left[k_x V_i^{TM}(z)\left(-J_{sx0}^{\,p}\right)\left(\frac{k_x}{k_t}\right)+k_y V_i^{TE}(z)\left(+J_{sx0}^{\,p}\right)\left(-\frac{k_y}{k_t}\right)\right]e^{-j\left(k_x x+k_y y\right)}$$

Example (cont.)

Hence

$$E_x(x,y,z) = \frac{-1}{k_t^{\,2}}\left[k_x^2 V_i^{TM}(z) + k_y^2 V_i^{TE}(z)\right] J_{sx0}^{\,p}\, e^{-j\left(k_x x + k_y y\right)}$$

$$J_{sx0}^{\,p} = 1$$

$$k_x = 2k_0$$

$$k_y = k_0$$

$$k_t = \sqrt{5}k_0$$

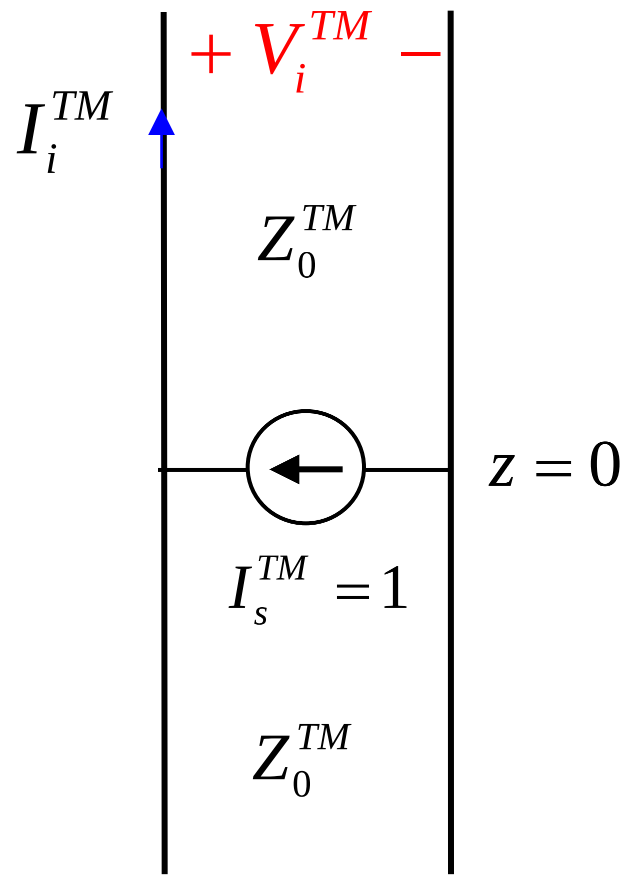
Example (cont.)

TM$_z$ Model

For $z > 0$:

$$V_i^{TM}(z) = \frac{Z_0^{TM}}{2} e^{-jk_{z0}z}$$

$$I_i^{TM}(z) = \frac{1}{2} e^{-jk_{z0}z}$$



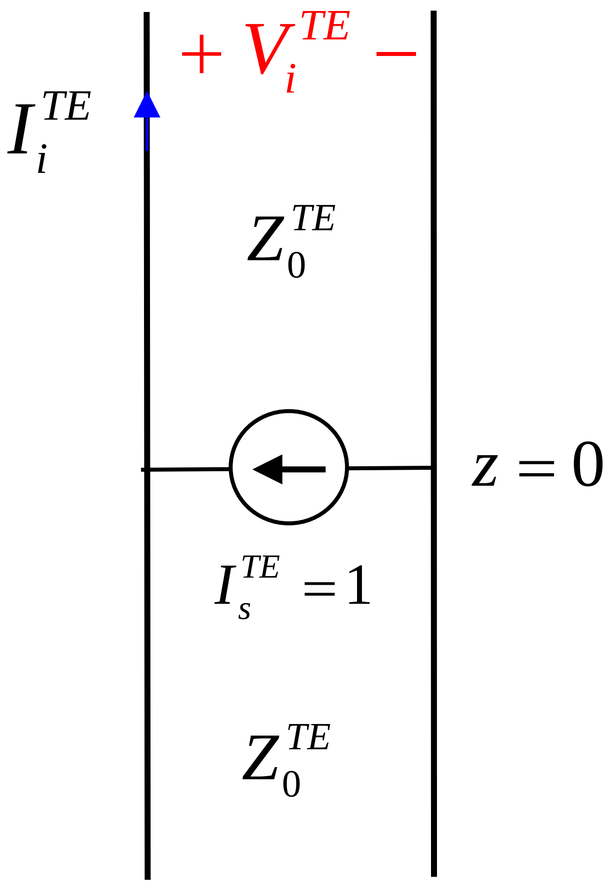
Example (cont.)

TE$_z$ Model

For $z > 0$:

$$V_i^{TE}(z) = \frac{Z_0^{TE}}{2} e^{-jk_{z0}z}$$

$$I_i^{TE}(z) = \frac{1}{2} e^{-jk_{z0}z}$$



Example (cont.)

Hence

$$E_x(x,y,z) = \frac{-1}{k_t^2}\left[k_x^2 V_i^{TM}(z) + k_y^2 V_i^{TE}(z)\right] J_{sx0}^{p}\, e^{-j\left(k_x x + k_y y\right)}$$

⇓

$$E_x(x,y,z) = \frac{-1}{k_t^2}\left[k_x^2\left(\frac{Z_0^{TM}}{2}\right) + k_y^2\left(\frac{Z_0^{TE}}{2}\right)\right] J_{sx0}^{p}\, e^{-j\left(k_x x + k_y y\right)}\, e^{-jk_{z0}z}$$

where $Z_0^{TM} = \dfrac{k_{z0}}{\omega\varepsilon_0}$ $\qquad Z_0^{TE} = \dfrac{\omega\mu_0}{k_{z0}}$

$$k_{z0} = \left(k_0^2 - k_x^2 - k_y^2\right)^{1/2}$$

Example (cont.)

Substituting in values,
we have:

$$k_x = 2k_0 \qquad k_y = k_0 \qquad k_t^2 = 5k_0^2$$

$$k_{z0} = -j2k_0 \qquad Z_0^{TM} = -j2\eta_0 \qquad Z_0^{TE} = j\frac{\eta_0}{2} \qquad J_{sx0}^{p} = 1$$

Hence

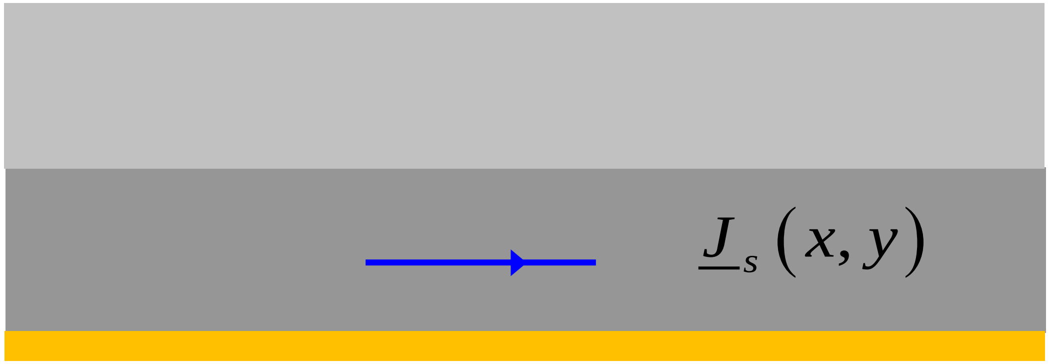
$$E_x(x,y,z) = \frac{-1}{5k_0^{\,2}}\left[4k_0^2(-j\eta_0) + k_0^2\left(\frac{j\eta_0}{4}\right)\right] e^{-j(2k_0x+k_0y)}\, e^{-2k_0z}$$

$$E_x(x,y,z) = \frac{j\eta_0}{5}\left[\frac{15}{16}\right] e^{-jk_0(2x+y)}\, e^{-2k_0z}$$

or

$$E_x(x,y,z) = \frac{j3}{16}\,\eta_0\, e^{-jk_0(2x+y)}\, e^{-2k_0z} \qquad (z \ge 0)$$

Finite Source



For a phased current sheet: $$\underline{J}_s^{\,p}(x,y)=\underline{J}_{s0}^{\,p}\,e^{-j\left(k_x x+k_y y\right)}$$

$$\underline{E}_t(x,y,z)=\underline{\hat{u}}\,V_i^{TM}(z)\left(-\underline{J}_{s0}^{\,p}\cdot\underline{\hat{u}}\right)e^{-j\left(k_x x+k_y y\right)}$$
$$+\underline{\hat{v}}\left(-V_i^{TE}(z)\right)\left(+\underline{J}_{s0}^{\,p}\cdot\underline{\hat{v}}\right)e^{-j\left(k_x x+k_y y\right)}$$

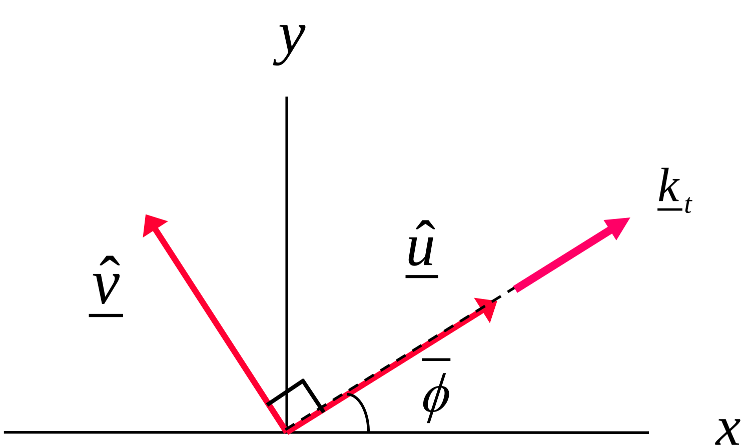
Recall that $$\underline{J}_{s0}^{\,p}=\frac{1}{(2\pi)^2}\underline{\tilde{J}}_s\left(k_x,k_y\right)dk_x dk_y$$

Finite Source (cont.)

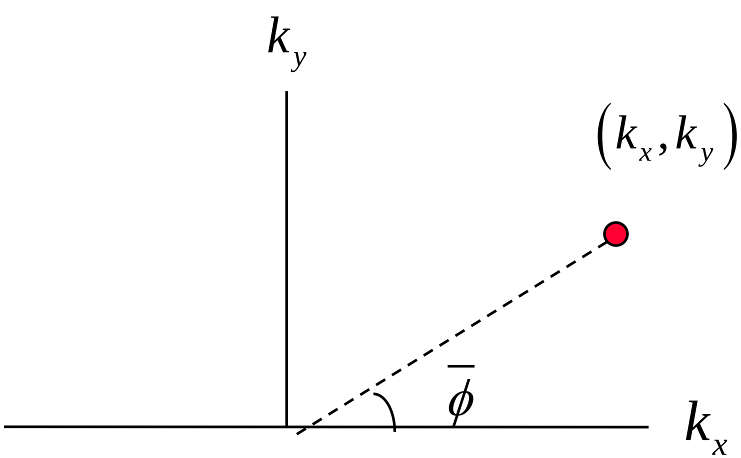
Hence

$$\underline{E}_t(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ \underline{\hat{u}} V_i^{TM}(z) \left[-\underline{\tilde{J}}_s \cdot \underline{\hat{u}} \right] - \underline{\hat{v}} V_i^{TE}(z) \left[\underline{\tilde{J}}_s \cdot \underline{\hat{v}} \right] \right\} \cdot e^{-j(k_x x + k_y y)} dk_x dk_y$$

Note: $\underline{\hat{u}} = \underline{\hat{u}}(k_x, k_y), \quad \underline{\hat{v}} = \underline{\hat{v}}(k_x, k_y)$



Spatial coordinates



Wavenumber plane

TEN Model for $\underline{\tilde{E}}_t$

We can also write (from the definition of inverse transform)

$$\underline{E}_t(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underline{\tilde{E}}_t(k_x,k_y,z)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

Comparing with the previous result, we have

$$\underline{\tilde{E}}_t(k_x,k_y,z) = \underline{\hat{u}}\, V_i^{TM}(z)\left[-\underline{\tilde{J}}_s \cdot \underline{\hat{u}}\right] - \underline{\hat{v}}\, V_i^{TE}(z)\left[\underline{\tilde{J}}_s \cdot \underline{\hat{v}}\right]$$

Similarly,

$$\underline{\tilde{H}}_t(k_x,k_y,z) = \underline{\hat{u}}\, I_i^{TE}(z)\left[\underline{\tilde{J}}_s \cdot \underline{\hat{v}}\right] + \underline{\hat{v}}\, I_i^{TM}(z)\left[-\underline{\tilde{J}}_s \cdot \underline{\hat{u}}\right]$$

This motivates the following identifications:

TEN Model (cont.)

$$V^{TM}(z) = \underline{\tilde{E}}_t\left(k_x, k_y, z\right) \cdot \underline{\hat{u}}$$

$$V^{TE}(z) = -\underline{\tilde{E}}_t\left(k_x, k_y, z\right) \cdot \underline{\hat{v}}$$

$$I^{TE}(z) = \underline{\tilde{H}}_t\left(k_x, k_y, z\right) \cdot \underline{\hat{u}}$$

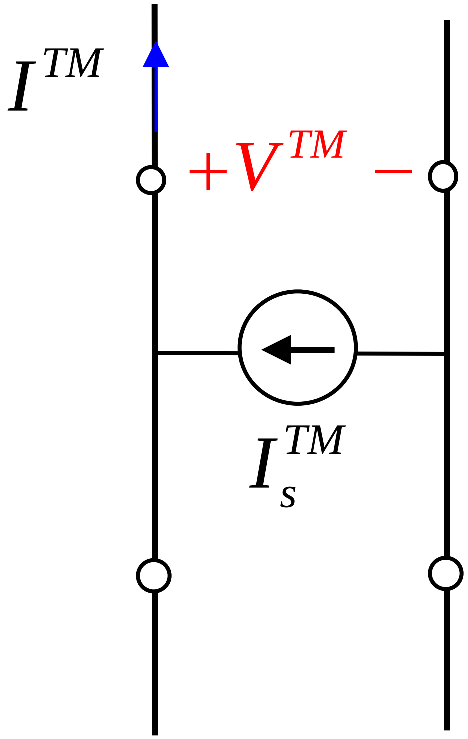
$$I^{TM}(z) = \underline{\tilde{H}}_t\left(k_x, k_y, z\right) \cdot \underline{\hat{v}}$$

$$I_s^{TM} = -\underline{\tilde{J}}_s\left(k_x, k_y\right) \cdot \underline{\hat{u}}$$

$$I_s^{TE} = +\underline{\tilde{J}}_s\left(k_x, k_y\right) \cdot \underline{\hat{v}}$$

TEN Model (cont.)

$$TM_z:$$



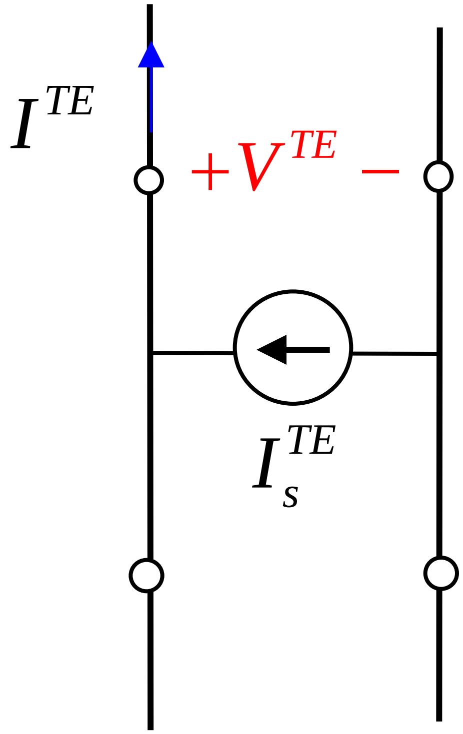
$$V^{TM} = \underline{\tilde{E}}_t \cdot \underline{\hat{u}}$$

$$I^{TM} = \underline{\tilde{H}}_t \cdot \underline{\hat{v}}$$

$$I_s^{TM} = -\underline{\tilde{J}}_s \cdot \underline{\hat{u}}$$

TEN Model (cont.)

TE_z :

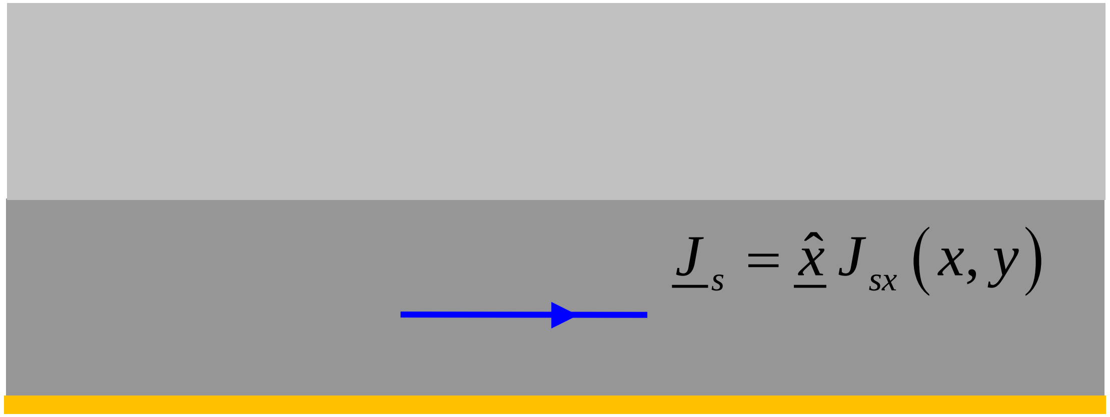


$$V^{TE} = -\underline{\tilde{E}}_t \cdot \underline{\hat{v}}$$

$$I^{TE} = \underline{\tilde{H}}_t \cdot \underline{\hat{u}}$$

$$I_s^{TE} = \underline{\tilde{J}}_s \cdot \underline{\hat{v}}$$

Example



Find $E_x(x, y, z)$

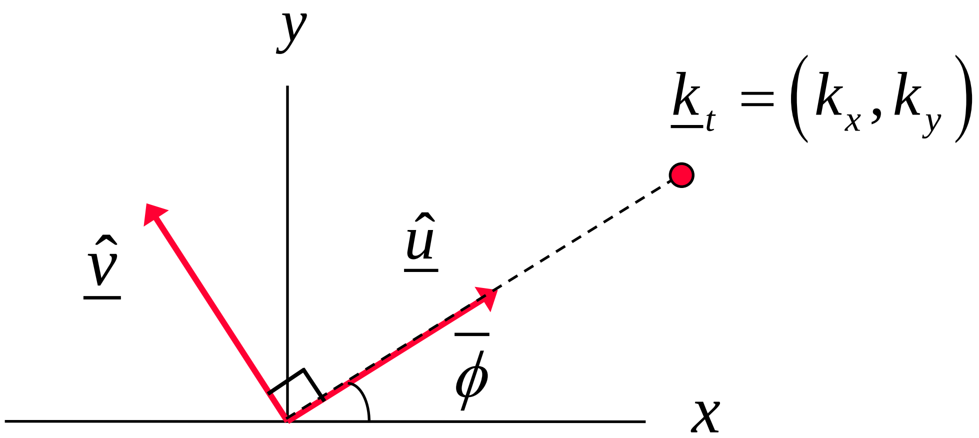
$$\tilde{E}_x(k_x, k_y, z) = \underline{\hat{x}} \cdot \left[\underline{\hat{u}}\left(\underline{\tilde{E}}_t \cdot \underline{\hat{u}} \right) + \underline{\hat{v}}\left(\underline{\tilde{E}}_t \cdot \underline{\hat{v}} \right) \right]$$

$$= \left(\underline{\hat{x}} \cdot \underline{\hat{u}} \right) V^{TM}(z) + \left(\underline{\hat{x}} \cdot \underline{\hat{v}} \right)\left(-V^{TE}(z) \right)$$

Example (cont.)

$$\underline{\hat{x}} \cdot \underline{\hat{u}} = \cos\overline{\phi} = \frac{k_x}{k_t}$$

$$\underline{\hat{x}} \cdot \underline{\hat{v}} = -\sin\overline{\phi} = -\frac{k_y}{k_t}$$



Hence

$$\tilde{E}_x(k_x, k_y, z) = \left(\frac{k_x}{k_t}\right) V^{TM}(z) + \left(\frac{k_y}{k_t}\right) V^{TE}(z)$$

$$= \left(\frac{k_x}{k_t}\right) V_i^{TM}(z)\left[-\underline{\tilde{J}}_s \cdot \underline{\hat{u}}\right] + \left(\frac{k_y}{k_t}\right) V_i^{TE}(z)\left[\underline{\tilde{J}}_s \cdot \underline{\hat{v}}\right]$$

Example (cont.)

$$\tilde{E}_x\left(k_x,k_y,z\right)=\left(\frac{k_x}{k_t}\right)V_i^{TM}\left(z\right)\left[-\tilde{J}_{sx}\left(\frac{k_x}{k_t}\right)\right]+\left(\frac{k_y}{k_t}\right)V_i^{TE}\left(z\right)\left[\tilde{J}_{sx}\left(-\frac{k_y}{k_t}\right)\right]$$

$$=-\frac{1}{k_t^2}\tilde{J}_{sx}\left[k_x^2V_i^{TM}\left(z\right)+k_y^2V_i^{TE}\left(z\right)\right]$$

$$E_x(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\tilde{J}_{sx}\left[k_x^2V_i^{TM}\left(z\right)+k_y^2V_i^{TE}\left(z\right)\right]$$

$$\cdot e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Dyadic Green's Function

$$\underline{\underline{G}}(x-x', y-y'; z, z') = \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{bmatrix}$$

where

$G_{ij} = E_i(x, y, z)$ due to the unit-amplitude electric dipole at (x', y', z')

$$\underline{J}(x, y, z) = \underline{\hat{j}}\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

From superposition:

$$\underline{E}(x, y; z, z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{\underline{G}}(x-x', y-y'; z, z') \cdot \underline{J}_s(x', y'; z')\, dx'\, dy'$$

where $\underline{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$ $\quad \underline{J}_s = \begin{bmatrix} J_{sx} \\ J_{sy} \\ J_{sz} \end{bmatrix}$

We assume here that the currents are located on a planar surface.

Dyadic Green's Function (cont.)

$$\underline{E}(x,y;z,z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{\underline{G}}(x-x',y-y';z,z') \cdot \underline{J}_s(x',y';z')\,dx'\,dy'$$

This is recognized as a 2D convolution:

$$\underline{E} = \underline{\underline{G}} * \underline{J}_s$$

Taking the 2D Fourier transform of both sides,

$$\underline{\tilde{E}} = \underline{\underline{\tilde{G}}} \cdot \underline{\tilde{J}}_s$$

where $\underline{\underline{\tilde{G}}} = \underline{\underline{\tilde{G}}}(k_x,k_y;z,z')$

Dyadic Green's Function (cont.)

$$\underline{\tilde{E}} = \underline{\underline{\tilde{G}}} \cdot \underline{\tilde{J}}_s$$

Assuming we wish the x component of the electric field due to an x-directed current $J_{sx}\,(x', y')$, we have

$$\tilde{E}_x = \tilde{G}_{xx}\,\tilde{J}_{sx}$$

In order to indentify $\tilde{G}_{xx}$, we use

$$E_x(x,y;z,z') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{k_t^2}\,\tilde{J}_{sx}\left(k_x,k_y\right)\left[k_x^2\,V_i^{TM}\left(z,z'\right)+k_y^2\,V_i^{TE}\left(z,z'\right)\right] \cdot e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Dyadic Green's Function (cont.)

$$E_x(x,y;z,z') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{-\frac{1}{k_t^2}\tilde{J}_{sx}\left[k_x^2 V_i^{TM}(z,z') + k_y^2 V_i^{TE}(z,z')\right]\right\}$$

$$\cdot e^{-j\left(k_x x + k_y y\right)} dk_x dk_y$$

$$\downarrow$$

$$\tilde{E}_x(k_x,k_y;z,z') = \left(-\frac{1}{k_t^2}\left[k_x^2 V_i^{TM}(z,z') + k_y^2 V_i^{TE}(z,z')\right]\right)\tilde{J}_{sx}\left(k_x,k_y\right)$$

Recall that $\tilde{E}_x = \tilde{G}_{xx}\,\tilde{J}_{sx}$

Hence $$\tilde{G}_{xx}\left(k_x,k_y;z,z'\right) = -\frac{1}{k_t^2}\left[k_x^2 V_i^{TM}(z,z') + k_y^2 V_i^{TE}(z,z')\right]$$

Dyadic Green's Function (cont.)

We the have

$$\underline{\underline{\tilde{G}}} = \begin{bmatrix} \tilde{G}_{xx} & \tilde{G}_{xy} & \tilde{G}_{xz} \\ \tilde{G}_{yx} & \tilde{G}_{yy} & \tilde{G}_{yz} \\ \tilde{G}_{zx} & \tilde{G}_{zy} & \tilde{G}_{zz} \end{bmatrix}$$

$$\tilde{G}_{xx}\left(k_x, k_y; z, z'\right) = -\frac{1}{k_t^2}\left[k_x^2 V_i^{TM}\left(z, z'\right) + k_y^2 V_i^{TE}\left(z, z'\right)\right]$$

The other eight components could be found in a similar way.

(The sources in the TEN that correspond to all possible sources are given on the next slide, and from these we can determine any component of the spectral-domain Green's function that we wish, for either an electric current or a magnetic current.)

Summary of Results for All Sources

These results are derived later.

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_u$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

$$I_s^{TM} = -\tilde{J}_{su}$$
$$V_s^{TM} = -\tilde{M}_{sv} + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}$$

$$V_s^{TE} = -\tilde{M}_{su}$$
$$I_s^{TE} = \tilde{J}_{sv} + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_{sz}$$

Definition of “vertical planar currents”:

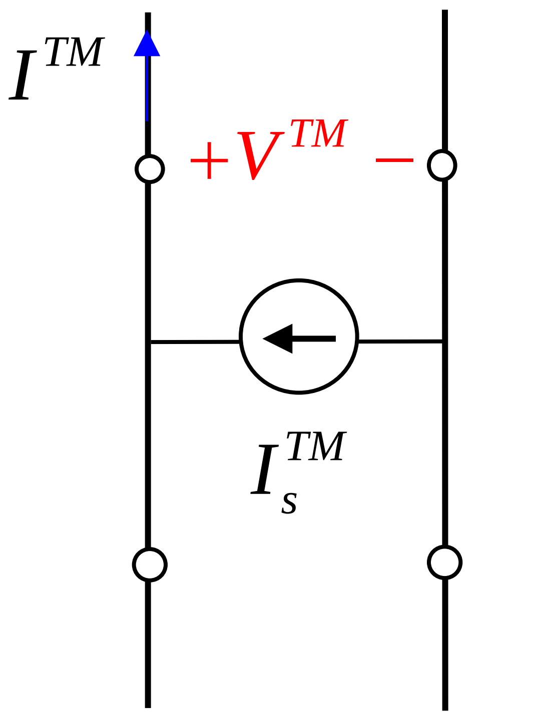
$$J_z(x,y,z) = J_{sz}(x,y)\,\delta(z)$$

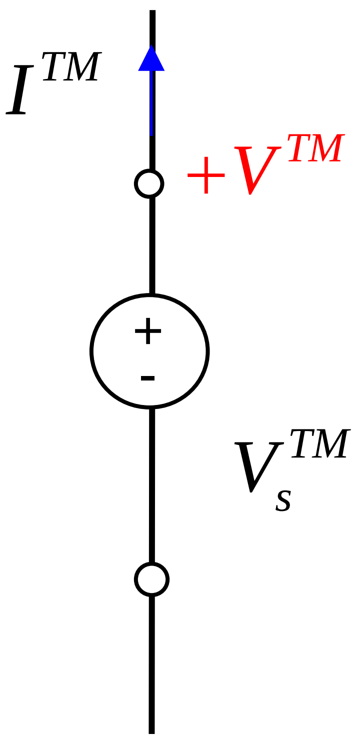
$$M_z(x,y,z) = M_{sz}(x,y)\,\delta(z)$$

Sources used in Modeling

TM_z :

V_i = voltage due to 1[A] parallel current source
I_i = current due to 1[A] parallel current source
V_v = voltage due to 1[V] series voltage source
I_v = current due to 1[V] series voltage source





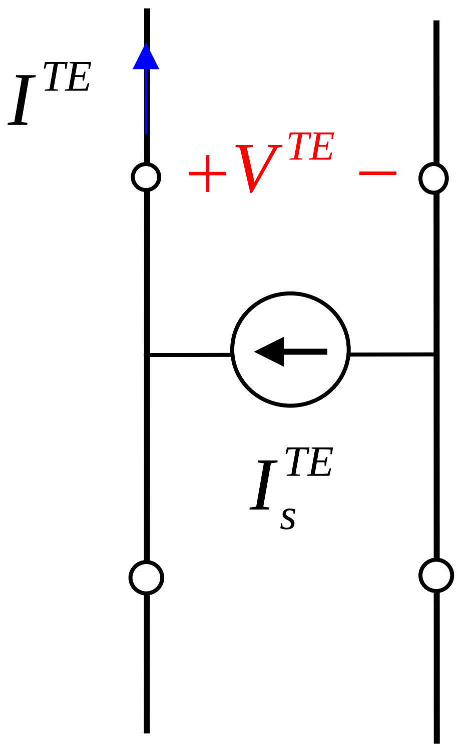
$$V^{TM}(z) = V_i^{TM}(z)\, I_s^{TM}$$
$$I^{TM}(z) = I_i^{TM}(z)\, I_s^{TM}$$

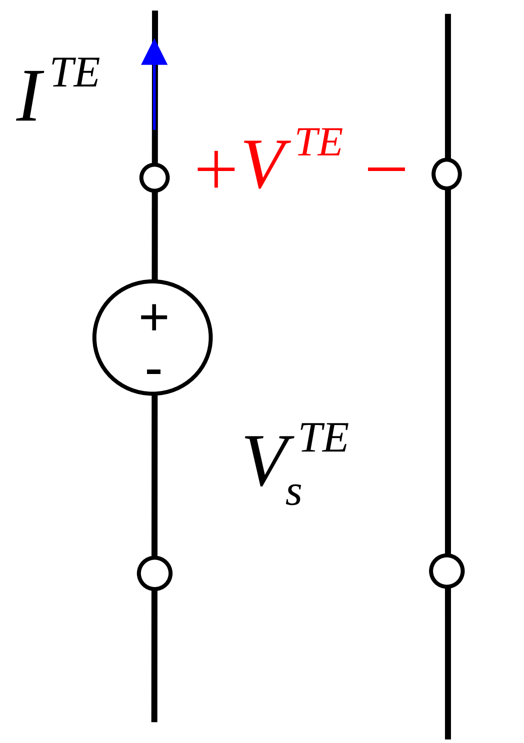
$$V^{TM}(z) = V_v^{TM}(z)\, V_s^{TM}$$
$$I^{TM}(z) = I_v^{TM}(z)\, V_s^{TM}$$

Sources used in Modeling (cont.)

TE_z :

V_i = voltage due to 1[A] parallel current source
I_i = current due to 1[A] parallel current source
V_v = voltage due to 1[V] series voltage source
I_v = current due to 1[V] series voltage source





$$V^{TE}(z) = V_i^{TE}(z)\, I_s^{TE}$$

$$I^{TE}(z) = I_i^{TE}(z)\, I_s^{TE}$$

$$V^{TM}(z) = V_v^{TE}(z)\, V_s^{TE}$$

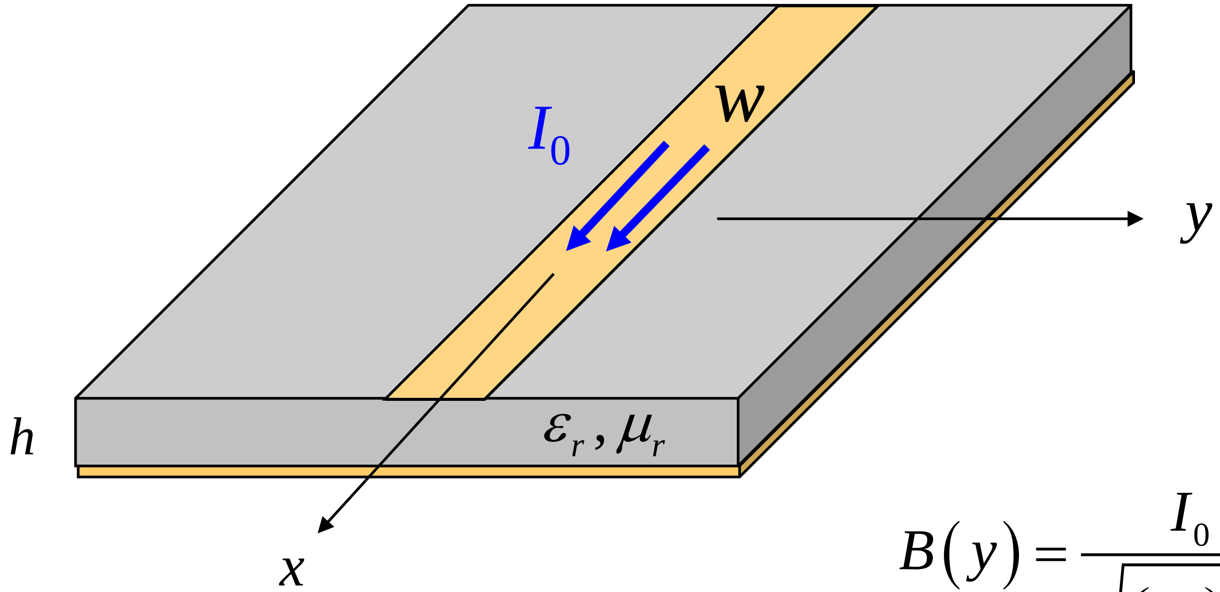
$$I^{TE}(z) = I_v^{TE}(z)\, V_s^{TE}$$

Outline

- Physical derivation of method for planar electric surface currents.
- Examples involving planar surface currents:
  - Microstrip line
  - Microstrip patch current
- General derivation (Fourier transforming Maxwell's equations) that allows for all types of sources to be included in one general derivation.
- Examples:
  - Vertical dipole over the earth (Sommerfeld problem)
  - Slot antenna covered with radome layer (magnetic current)

Microstrip Line

Microstrip Line



$$B(y) = \frac{I_0 / \pi}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}}$$

Dominant quasi-TEM mode:

$$J_{sx}(x, y) = B(y)\, e^{-jk_{x0}x}$$

We assume a purely x-directed current and a real wavenumber k_{x0}.

Microstrip Line (cont.)

Fourier transform of current:

$$\tilde{J}_{sx}\left(k_x, k_y\right) = I_0 \int_{-w/2}^{w/2} \frac{1/\pi}{\sqrt{\left(\dfrac{w}{2}\right)^2 - y^2}} e^{jk_y y}\, dy \int_{-\infty}^{\infty} e^{-jk_{x0}x} e^{jk_x x}\, dx$$

$$\tilde{J}_{sx}\left(k_x, k_y\right) = I_0\, J_0\left(\frac{k_y w}{2}\right) \int_{-\infty}^{\infty} e^{-jk_{x0}x} e^{jk_x x}\, dx$$

$$\tilde{J}_{sx}\left(k_x, k_y\right) = I_0\, J_0\left(\frac{k_y w}{2}\right) \left[2\pi\delta\left(k_x - k_{x0}\right)\right]$$

Microstrip Line (cont.)

$$\tilde{J}_{sx}\left(k_x,k_y\right)=I_0\,J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]$$

$$E_x(x,y;0,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,\tilde{J}_{sx}\left[k_x^2\,V_i^{TM}\left(0,0\right)+k_y^2\,V_i^{TE}\left(0,0\right)\right]$$
$$\cdot\, e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

$z = 0,\ z' = 0$

Hence we have:

$$E_x(x,y;0,0)=\frac{I_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,J_0\left(\frac{k_y w}{2}\right)\left[2\pi\delta\left(k_x-k_{x0}\right)\right]\left[k_x^2\,V_i^{TM}\left(0,0\right)+k_y^2\,V_i^{TE}\left(0,0\right)\right]$$
$$\cdot\, e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Microstrip Line (cont.)

$$E_x(x,y;0,0)=\frac{I_0}{(2\pi)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,J_0\left(\frac{k_y w}{2}\right)\left[\delta\left(k_x-k_{x0}\right)\right]\left[k_x^2 V_i^{TM}\left(0,0\right)+k_y^2 V_i^{TE}\left(0,0\right)\right]$$

$$\cdot e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Integrating over the δ-function, we have:

$$E_x(x,y;0,0)=\frac{I_0}{(2\pi)}e^{-jk_{x0}x}\int_{-\infty}^{\infty}-\frac{1}{k_t^2}\,J_0\left(\frac{k_y w}{2}\right)\left[k_{x0}^2 V_i^{TM}\left(0,0\right)+k_y^2 V_i^{TE}\left(0,0\right)\right]e^{-j\left(k_y y\right)}dk_y$$

where we now have $k_t^2=k_{x0}^2+k_y^2$

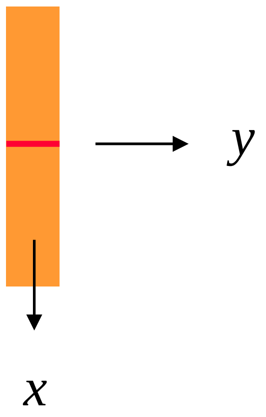
$$k_{z0}=\sqrt{k_0^2-k_t^2}=\sqrt{\left(k_0^2-k_{x0}^2\right)-k_y^2}\qquad k_{z1}=\sqrt{k_1^2-k_t^2}=\sqrt{\left(k_1^2-k_{x0}^2\right)-k_y^2}$$

Microstrip Line (cont.)

Enforce EFIE using Galerkin's method:

$$\int_{-w/2}^{w/2} E_x(0,y;0,0)\,T(y)\,dy = 0$$

The EFIE is enforced on the red line.



where $T(y) = B(y)$ (testing function = basis function)

Recall that

$$E_x(x,y;0,0) = \frac{I_0}{(2\pi)} e^{-jk_{x0}x} \int_{-\infty}^{\infty} -\frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] e^{-jk_y y} dk_y$$

Substituting into the EFIE integral, we have

$$\frac{I_0}{(2\pi)} \int_{-\infty}^{\infty} -\frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) \tilde{T}(-k_y) \left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Microstrip Line (cont.)

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) \tilde{T}\left(-k_y\right)\left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Since the testing function is the same as the basis function,

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0\left(\frac{k_y w}{2}\right) J_0\left(\frac{-k_y w}{2}\right)\left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Since the Bessel function is an even function,

$$\int_{-\infty}^{\infty} \frac{1}{k_t^2} J_0^2\left(\frac{k_y w}{2}\right)\left[k_{x0}^2 V_i^{TM}(0,0) + k_y^2 V_i^{TE}(0,0)\right] dk_y = 0$$

Microstrip Line (cont.)

Using symmetry, we have

$$\int_0^\infty J_0^2\left(\frac{k_y w}{2}\right)\left[k_{x0}^2 V_i^{TM}(0,0)+k_y^2 V_i^{TE}(0,0)\right]dk_y = 0$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

This is a transcendental equation of the following form:

$$F\left(k_{x0}\right) = 0$$

Note: $k_0 < k_{x0} < k_1$

Microstrip Line (cont.)

Branch points:

$$k_{z0}^2 = k_0^2 - k_t^2 = k_0^2 - \left(k_{x0}^2 + k_y^2\right)$$

Hence

$$k_{z0} = \left(\left(k_0^2 - k_{x0}^2\right) - k_y^2\right)^{1/2}$$

$$= -j\left(k_y^2 - \left(k_0^2 - k_{x0}^2\right)\right)^{1/2}$$

$$= -j\left(k_y^2 + \left(k_{x0}^2 - k_0^2\right)\right)^{1/2}$$

$$= -j\left(k_y - j\sqrt{k_{x0}^2 - k_0^2}\right)^{1/2}\left(k_y + j\sqrt{k_{x0}^2 - k_0^2}\right)^{1/2}$$

Note: The wavenumber k_{z0} causes branch points to arise.

$$\Rightarrow \quad k_{yb} = \pm j\sqrt{k_{x0}^2 - k_0^2}$$

Microstrip Line (cont.)

Poles ($k_y = k_{yp}$):

$$k_{tp}^2 = k_{x0}^2 + k_{yp}^2 = k_{\mathrm{TM}_0}^2$$

$$\Rightarrow \quad k_{yp}^2 = k_{\mathrm{TM}_0}^2 - k_{x0}^2$$

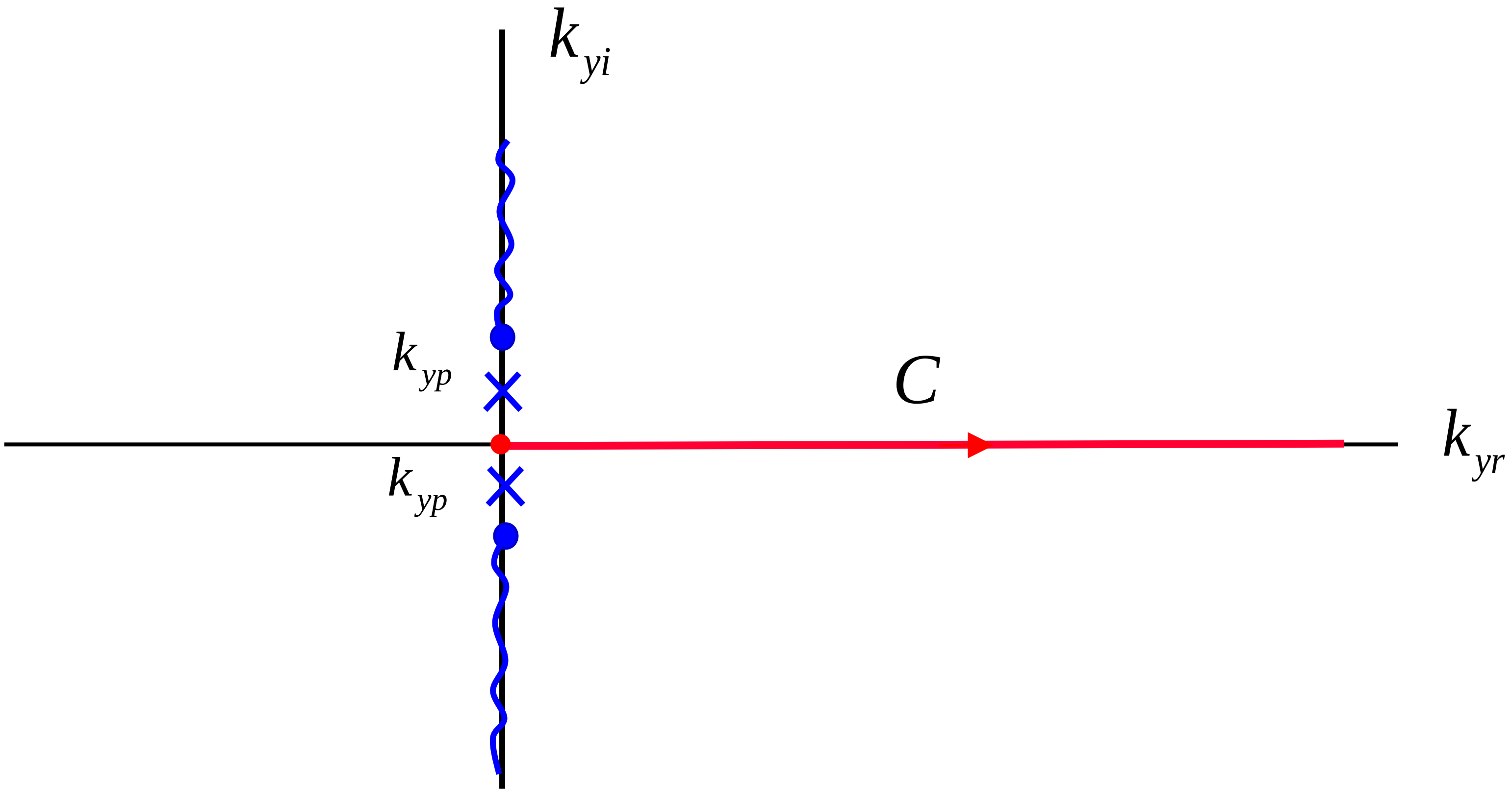
or

$$k_{yp} = \pm j\sqrt{k_{x0}^2 - k_{\mathrm{TM}_0}^2}$$

Microstrip Line (cont.)

Branch points: $k_{yb} = \pm j\sqrt{k_{x0}^2 - k_0^2}$

Poles: $k_{yp} = \pm j\sqrt{k_{x0}^2 - k_{\mathrm{TM}_0}^2}$



Microstrip Line (cont.)

Note on wavenumber k_{x0}

$$k_{yp} = \pm j\left(k_{x0}^2 - k_{\mathrm{TM}_0}^2\right)^{1/2}$$

For a real wavenumber k_{x0}, we must have that $k_{x0} > k_{\mathrm{TM}_0}$

Otherwise, there would be poles on the real axis, and this would correspond to leakage into the TM_0 surface-wave mode of the grounded substrate.

The mode would then be a leaky mode with a complex wavenumber k_{x0}, which contradicts the assumption that the pole is on the real axis.

Hence

$$k_0 < k_{\mathrm{TM}_0} < k_{x0} < k_1$$

Microstrip Line (cont.)

If we wanted to use multiple basis functions, we could consider the following choices:

Fourier-Maxwell Basis Function Expansion:

$$J_{sz}(x,y) = e^{-jk_{x0}x} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \left[\sum_{m=0}^{M-1} a_m \cos\left(\frac{2m\pi y}{w}\right) \right]$$

$$J_{sx}(x,y) = e^{-jk_{x0}x} \sqrt{\left(\frac{w}{2}\right)^2 - y^2} \left[\sum_{n=1}^{N} b_n \sin\left(\frac{(2n-1)\pi y}{w}\right) \right]$$

Chebyshev-Maxwell Basis Function Expansion:

$$J_{sz}(x,y) = e^{-jk_{x0}x} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - y^2}} \left[\sum_{m=0}^{M-1} a_m T_{2m}\left(\frac{2y}{w}\right) \right] \left(\frac{2(1+\delta_{m0})}{\pi w}\right)$$

$$J_{sx}(x,y) = e^{-jk_{x0}x} \sqrt{\left(\frac{w}{2}\right)^2 - y^2} \left[\sum_{n=1}^{N} b_n U_{2n-1}\left(\frac{2y}{w}\right) \right] \left(\frac{j4}{\pi w}\right)$$

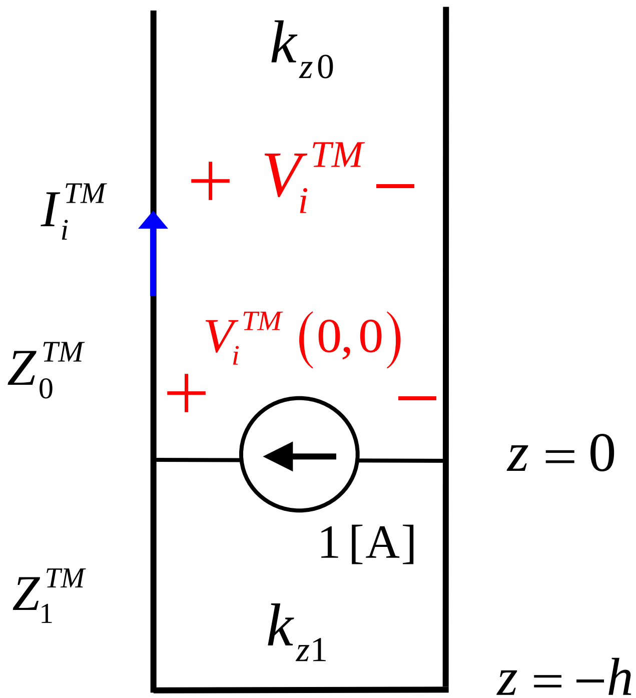
Microstrip Line (cont.)

We next proceed to calculate the Michalski voltage functions explicitly.

Microstrip Line (cont.)

TM_z :

$V_i^{TM}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

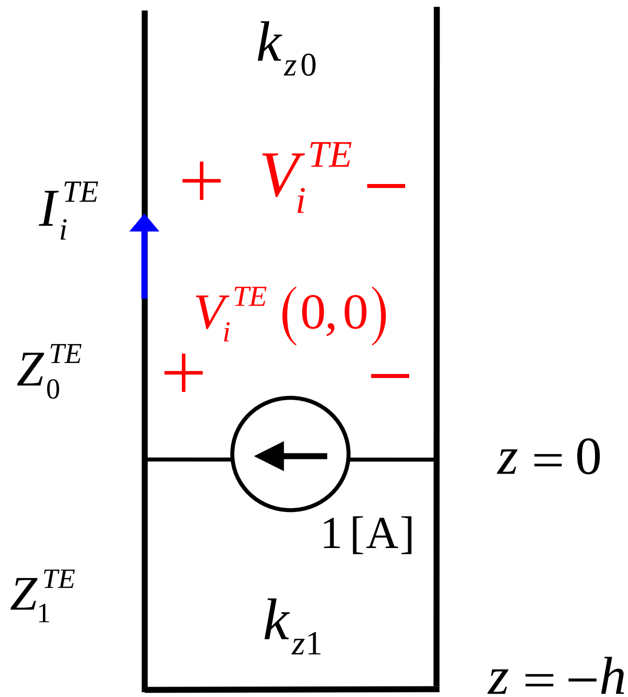
$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} = \eta_0\left(\frac{k_{z0}}{k_0}\right)$$

$$Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} = \frac{\eta_0}{\varepsilon_r}\left(\frac{k_{z1}}{k_0}\right)$$

Microstrip Line (cont.)

TE_z :

$V_i^{TE}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{z0}} = \frac{\eta_0}{\left(k_{z0}/k_0\right)} \qquad Z_1^{TE} = \frac{\omega\mu_1}{k_{z1}} = \frac{\eta_0\mu_r}{\left(k_{z1}/k_0\right)}$$

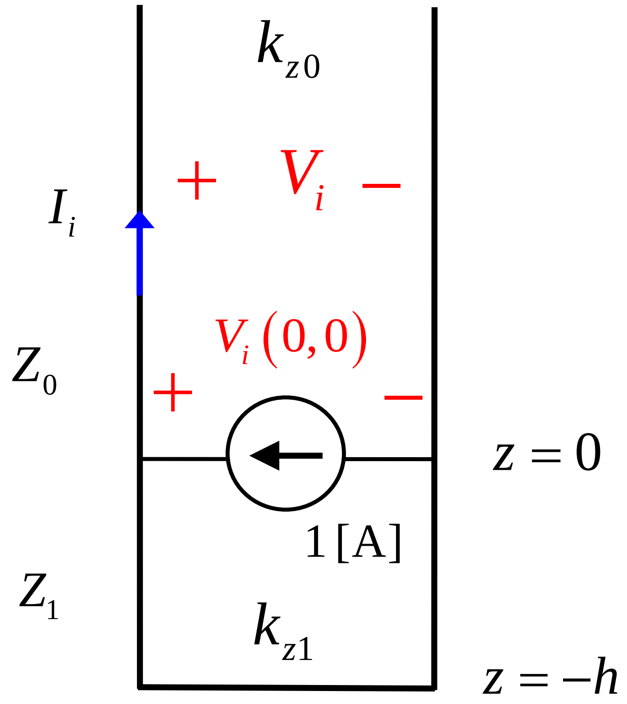
Microstrip Line (cont.)

At $z = 0$

$$V_i(0,0) = Z_{in}$$

$$= Y_{in}^{-1} = \left(Y_{in}^{+} + Y_{in}^{-}\right)^{-1}$$

$$= \left[Y_0 - jY_1 \cot\left(k_{z1}h\right)\right]^{-1}$$



Hence

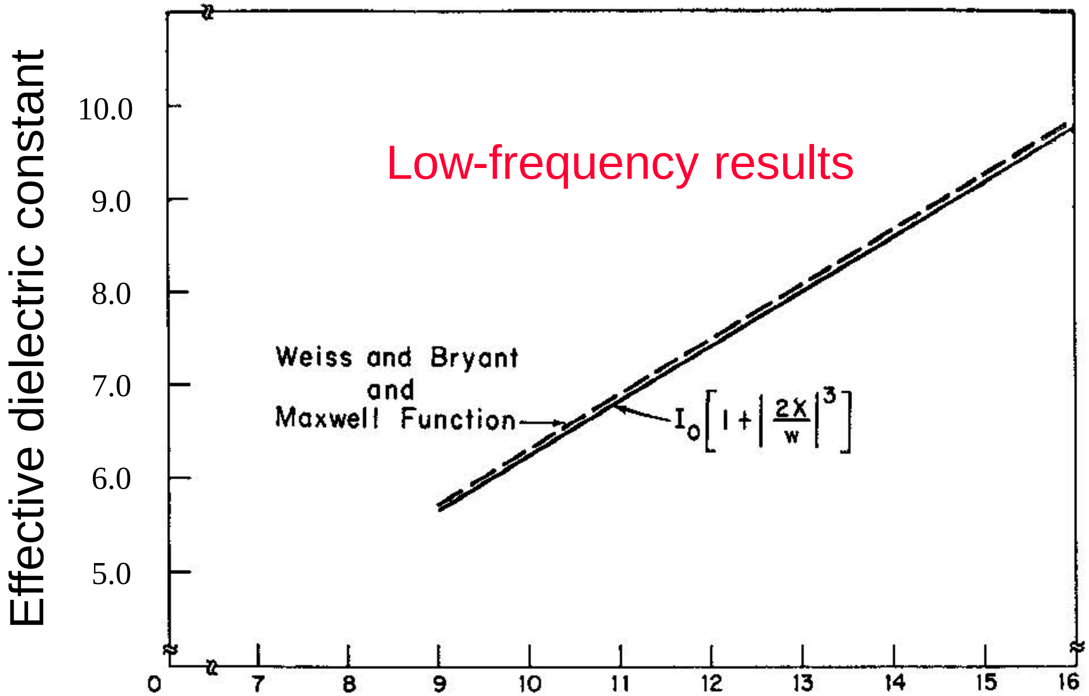
$$V_i^{TM}(0,0) = \frac{1}{D_m\left(k_x, k_y\right)}$$

$$V_i^{TE}(0,0) = \frac{1}{D_e\left(k_x, k_y\right)}$$

$$D_m\left(k_x, k_y\right) = Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right)$$

$$D_e\left(k_x, k_y\right) = Y_0^{TE} - jY_1^{TE} \cot\left(k_{z1}h\right)$$

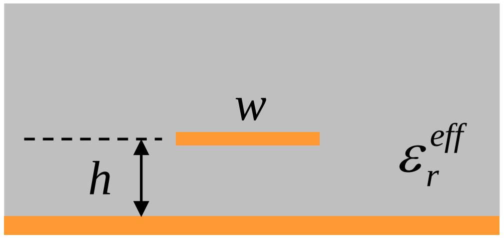
Microstrip Line (cont.)



Dielectric constant of substrate (ε_r)

Parameters: $w/h = 0.40$

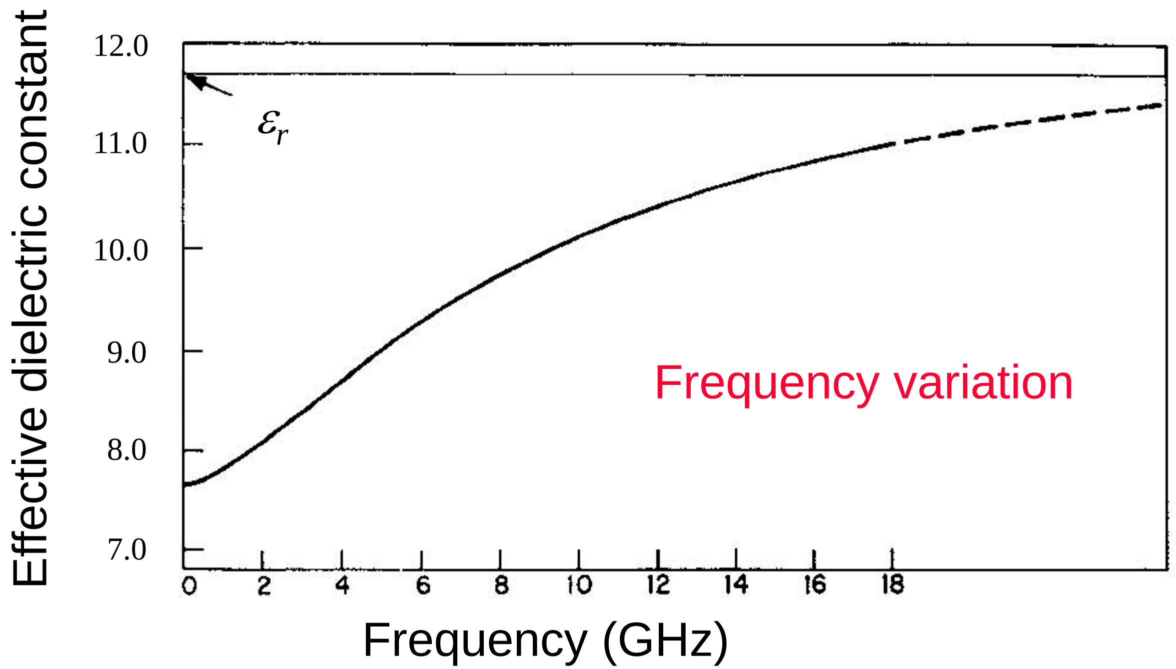
$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0}\right)^2$$



$$k_{x0} = k_0\sqrt{\varepsilon_r^{eff}}$$

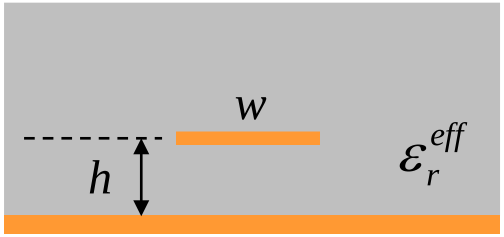
E. J. Denlinger, "A frequency-dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory and Techniques,* vol. 19, pp. 30-39, Jan. 1971.

Microstrip Line (cont.)



Parameters: $\varepsilon_r = 11.7$, $w/h = 0.96$, $h = 0.317$ cm

$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0}\right)^2$$



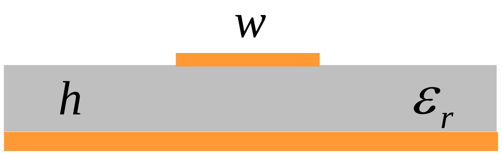
$$k_{x0} = k_0 \sqrt{\varepsilon_r^{eff}}$$

E. J. Denlinger, "A frequency-dependent solution for microstrip transmission lines," *IEEE Trans. Microwave Theory and Techniques,* vol. 19, pp. 30-39, Jan. 1971.

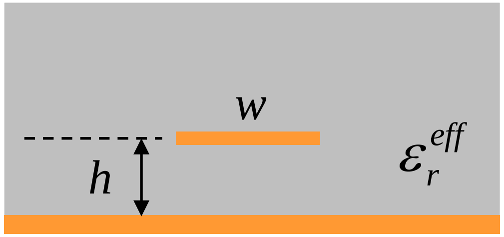
Microstrip Line (cont.)

Characteristic Impedance

1) Quasi-TEM Method



Original problem



Equivalent problem (TEM)

$$\varepsilon_r^{eff} = \left(\frac{k_{x0}}{k_0} \right)^2$$

We calculate the characteristic impedance of the equivalent *homogeneous medium problem*.

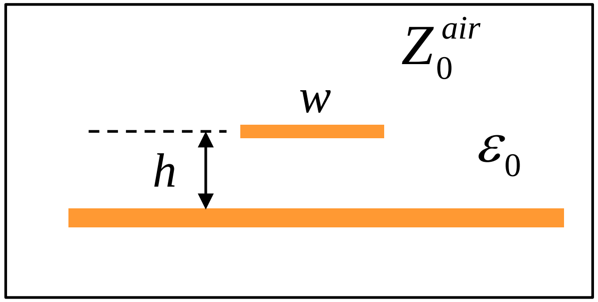
Microstrip Line (cont.)

Using the equivalent TEM problem:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{L_0}{C_0 \varepsilon_r^{eff}}}$$

(The zero subscript denotes the value when using an air substrate.)

$$\Rightarrow \quad Z_0 = Z_0^{air} \frac{1}{\sqrt{\varepsilon_r^{eff}}}$$



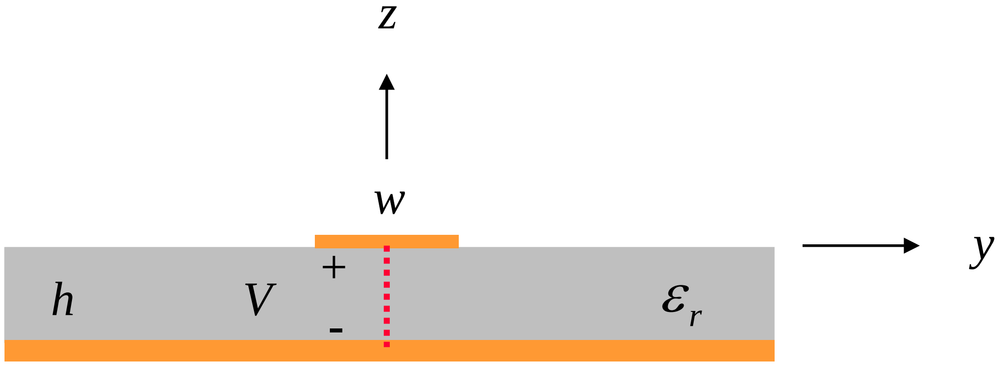
Simple CAD formulas may be used for the Z_0 of an air line.

$$Z_0^{air} = \begin{cases} 60 \ln\left(\dfrac{8h}{w} + \dfrac{w}{4h}\right) ; & \text{for } \dfrac{w}{h} \le 1 \\ \dfrac{120\pi}{\left(\dfrac{w}{h} + 1.393 + 0.667 \ln\left(\dfrac{w}{h} + 1.444\right)\right)} ; & \text{for } \dfrac{w}{h} \ge 1 \end{cases}$$

Microstrip Line (cont.)

2) Voltage-Current Method

$$Z_0 = \frac{V(x)}{I(x)} = \frac{V(0)}{I(0)} = \frac{-1}{I_0} \int_{-h}^{0} E_z(0,0,z)\,dz$$



$$\tilde{E}_z(k_x, k_y, z) = \frac{-1}{\omega\varepsilon_0}(k_t)\, I^{TM}(z)$$ (derived later)

$$= \frac{-1}{\omega\varepsilon_0}(k_t)\, I_i^{TM}(z)\left(-\tilde{J}_s \cdot \underline{\hat{u}}\right)$$

$$= \frac{-1}{\omega\varepsilon_0}(k_t)\, I_i^{TM}(z)\left(-\tilde{J}_{sx}\right)\cos\overline{\phi}$$

Microstrip Line (cont.)

$$Z_0 = \frac{-1}{I_0} \int_{-h}^{0} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_z\left(k_x, k_y, z\right) e^{-j\left(k_x x + k_y y\right)} dk_x dk_y \right)_{\substack{x=0 \\ y=0}} dz$$

$$Z_0 = \frac{-1}{I_0} \int_{-h}^{0} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_z\left(k_x, k_y, z\right) dk_x dk_y \right) dz$$

$$Z_0 = \frac{-1}{I_0} \int_{-h}^{0} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{\omega\varepsilon_0} (k_t) I_i^{TM}(z) \left(-\tilde{J}_{sx}\right) \left(\frac{k_x}{k_t} \right) dk_x dk_y \right) dz$$

$$Z_0 = \frac{-1}{\cancel{I_0}} \int_{-h}^{0} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{\omega\varepsilon_0} (k_t) I_i^{TM}(z) \left(-\cancel{I_0}\, J_0\left(\frac{k_y w}{2} \right) \left[2\pi\delta\left(k_x - k_{x0}\right) \right] \right) \left(\frac{k_x}{k_t} \right) dk_x dk_y \right) dz$$

Microstrip Line (cont.)

$$Z_0 = -\int_{-h}^{0} \left(\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\omega\varepsilon_0} (k_t) I_i^{TM}(z) \left(J_0\left(\frac{k_y w}{2}\right) \left[2\pi\delta(k_x - k_{x0}) \right] \right) \left(\frac{k_x}{k_t} \right) dk_x dk_y \right) dz$$

$$Z_0 = -\int_{-h}^{0} \left(\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{1}{\omega\varepsilon_0} (k_t) I_i^{TM}(z) J_0\left(\frac{k_y w}{2}\right) \left(\frac{k_{x0}}{k_t} \right) dk_y \right) dz$$

$$k_t^2 = k_{x0}^2 + k_y^2$$

$$Z_0 = -\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{k_{x0}}{\omega\varepsilon_0} J_0\left(\frac{k_y w}{2}\right) \left(\int_{-h}^{0} I_i^{TM}(z) dz \right) dk_y$$

$$Z_0 = -\frac{1}{\pi} \int_{0}^{\infty} \frac{k_{x0}}{\omega\varepsilon_0} J_0\left(\frac{k_y w}{2}\right) \left(\int_{-h}^{0} I_i^{TM}(z) dz \right) dk_y$$

Microstrip Line (cont.)

The final result is

$$Z_0 = -\frac{1}{\pi}\int_0^{\infty} \frac{k_{x0}}{\omega\varepsilon_0} J_0\left(\frac{k_y w}{2}\right) F\left(k_t\right) dk_y$$

where

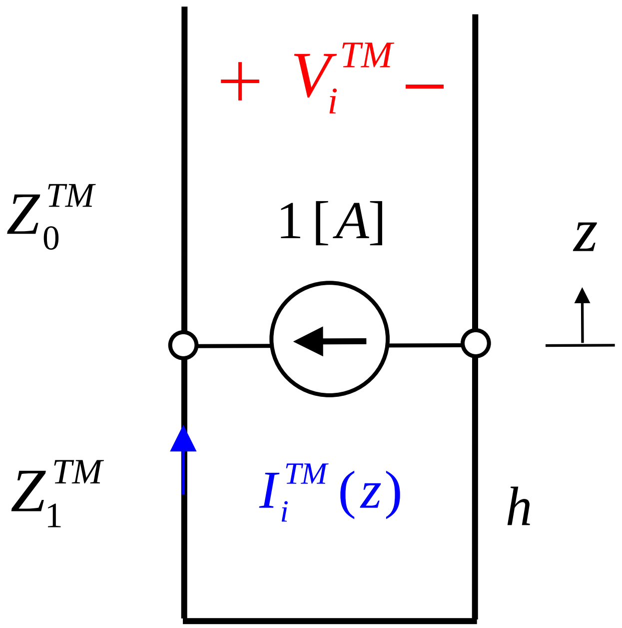
$$k_t^2 = k_{x0}^2 + k_y^2$$

$$F\left(k_t\right) \equiv \int_{-h}^{0} I_i^{TM}\left(z\right) dz$$

Microstrip Line (cont.)

We next calculate the function

$$I_i^{TM}(z)$$



$$I_i^{TM}\left(z=0^-\right) = -\left(\frac{V(0)}{Z_{in}^-}\right) = -\left(\frac{Z_{in}}{Z_{in}^-}\right) = -\left(\frac{1/D_m(k_t)}{j\,Z_1^{TM}\tan(k_{z1}h)}\right)$$

Microstrip Line (cont.)

Because of the short circuit,

$$I_i^{TM}(z) = A\cos\left(k_{z1}(z+h)\right), \quad -h < z < 0$$

At $z = 0$: $I_i^{TM}\left(0^-\right) = A\cos\left(k_{z1}h\right)$

Therefore

$$A = \frac{I_i^{TM}\left(0^-\right)}{\cos\left(k_{z1}h\right)}$$

Hence

$$I_i^{TM}(z) = I_i^{TM}\left(0^-\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos\left(k_{z1}h\right)}\right) \qquad -h < z < 0$$

Microstrip Line (cont.)

Hence

$$I_i^{TM}(z) = -\left(\frac{1/D_m(k_t)}{jZ_1^{TM}\tan(k_{z1}h)}\right)\left(\frac{\cos(k_{z1}(z+h))}{\cos(k_{z1}h)}\right)$$

$$-h < z < 0$$

or

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{jZ_1^{TM}}\right)\left(\frac{\cos(k_{z1}(z+h))}{\sin(k_{z1}h)}\right)$$

Microstrip Line (cont.)

Hence

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\sin\left(k_{z1}h\right)}\right)$$

where

$$D_m\left(k_x, k_y\right) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$

Microstrip Line (cont.)

We then have

$$F\left(k_t\right) \equiv \int_{-h}^{0} I_i^{TM}\left(z\right) dz$$

$$= \int_{-h}^{0} -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\,Z_1^{\,TM}}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\sin\left(k_{z1}h\right)}\right) dz$$

$$= -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\,Z_1^{\,TM}}\right)\left(\frac{1}{\sin\left(k_{z1}h\right)}\right)\int_{-h}^{0} \cos\left(k_{z1}(z+h)\right) dz$$

$$= -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\,Z_1^{\,TM}}\right)\left(\frac{1}{\sin\left(k_{z1}h\right)}\right)\left[\frac{1}{k_{z1}}\sin\left(k_{z1}(z+h)\right)\right]_{-h}^{0}$$

$$= -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\,Z_1^{\,TM}}\right)\left(\frac{1}{\sin\left(k_{z1}h\right)}\right)\left[\frac{1}{k_{z1}}\sin\left(k_{z1}h\right)\right]$$

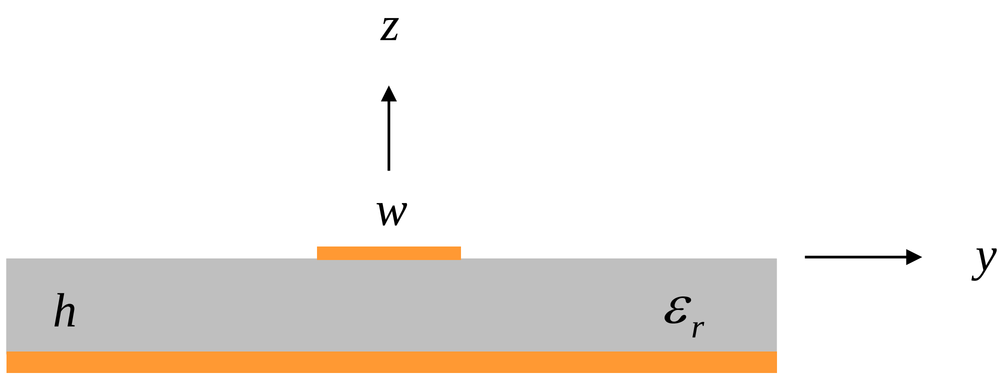
Hence

$$F\left(k_t\right) = -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j\,Z_1^{\,TM}}\right)\left(\frac{1}{k_{z1}}\right)$$

Microstrip Line (cont.)

3) Power-Current Method

$$Z_0 = \frac{2P_z(0)}{|I(0)|^2} = \frac{2}{|I(0)|^2} \int_{-\infty}^{\infty} \int_{-h}^{\infty} S_z(0, y, z)\, dz\, dy$$

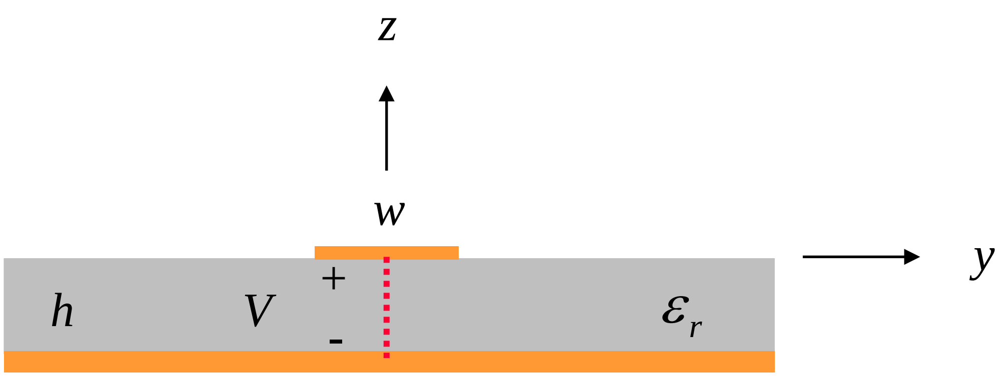


Note: It is possible to perform the spatial integrations for the power flow in closed form (details are omitted).

Microstrip Line (cont.)

4) Power-Voltage Method

$$Z_0 = \frac{|V(0)|^2}{2P_z(0)} = \frac{|V(0)|^2}{2\int_{-\infty}^{\infty}\int_{-h}^{\infty} S_z(0,y,z)\,dz\,dy}$$



Note: It is possible to perform the spatial integrations for the power flow in closed form (details are omitted).

Microstrip Line (cont.)

Comparison of methods:

- At low frequency all three methods agree well.
- As frequency increases, the VI, PI, and PV methods give a Z_0 that increases with frequency.
- The Quasi-TEM method gives a Z_0 that decreases with frequency.
- The PI method is usually regarded as being the best one for high frequency (agrees better with measurements).

Microstrip Line (cont.)

Quasi-TEM Method

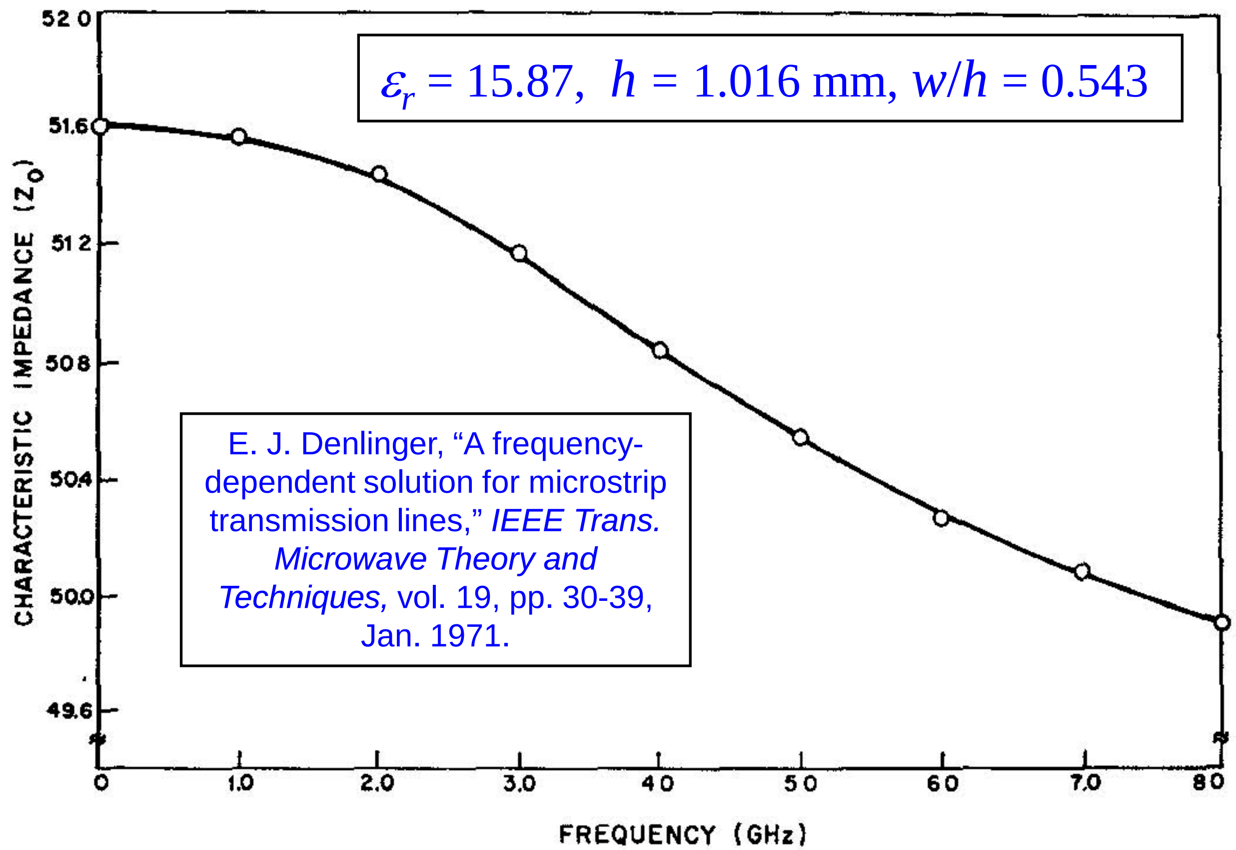
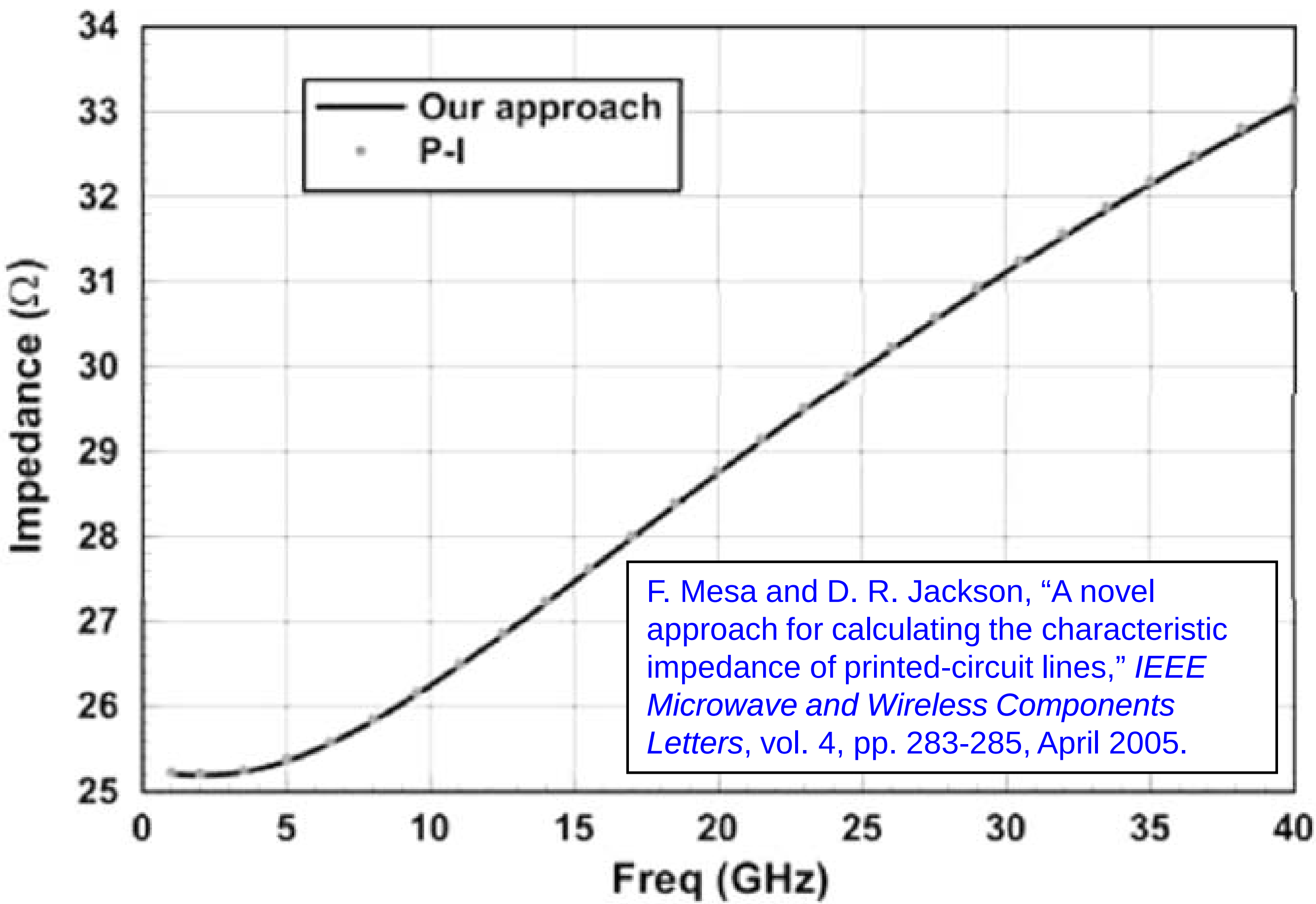


Fig. 9. Characteristic impedance versus frequency (same parameters as in Fig. 8).

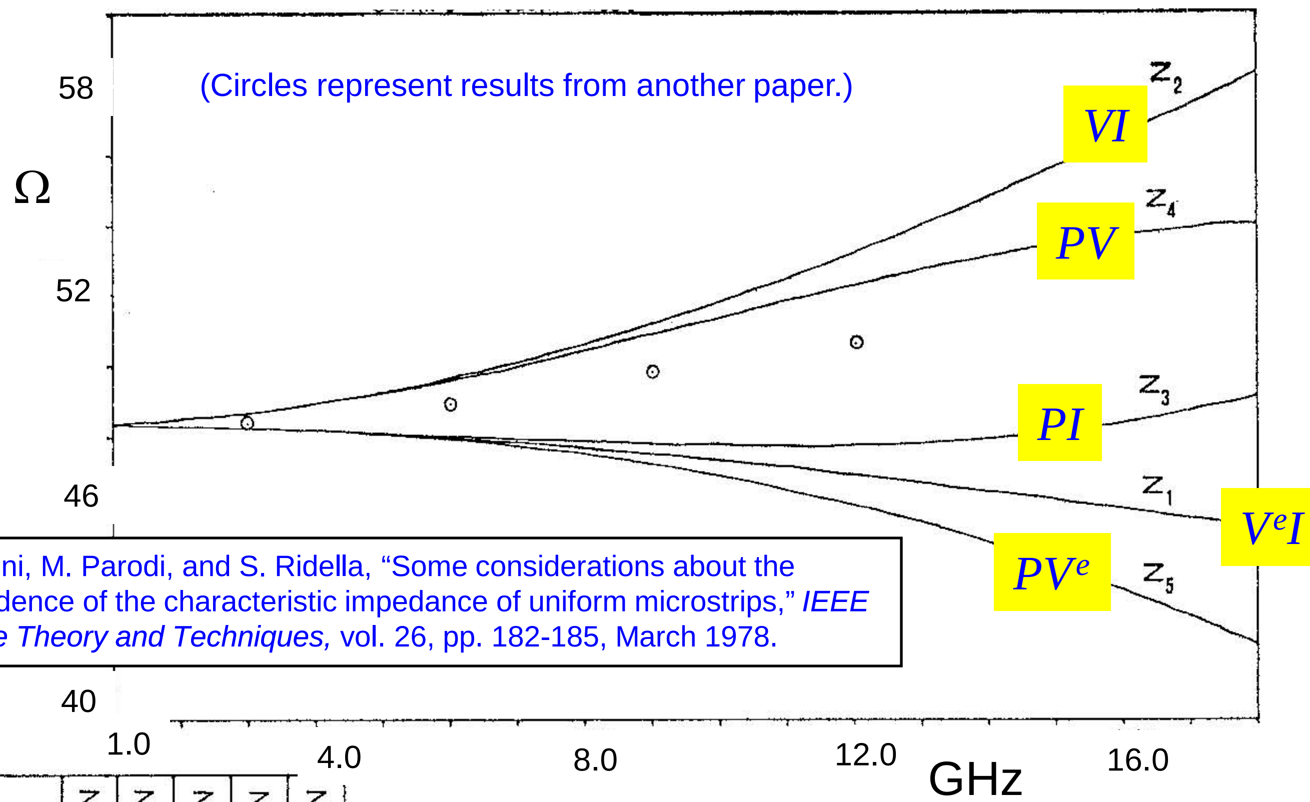
Microstrip Line (cont.)



F. Mesa and D. R. Jackson, “A novel approach for calculating the characteristic impedance of printed-circuit lines,” *IEEE Microwave and Wireless Components Letters*, vol. 4, pp. 283-285, April 2005.

Fig. 3. Results for the characteristic impedance of a microstrip with $w = 3$ mm, $h = 1$ mm, $\varepsilon_r = 10.2$.

Microstrip Line (cont.)



B Bianco, L. Panini, M. Parodi, and S. Ridella, "Some considerations about the frequency dependence of the characteristic impedance of uniform microstrips," *IEEE Trans. Microwave Theory and Techniques,* vol. 26, pp. 182-185, March 1978.

	Z_1	Z_2	Z_3	Z_4	Z_5
Limit value (Ω) for $f \to 0$	48.35	48.35	48.35	48.35	48.35
Limit value (Ω) for $f \to \infty$	40.06	80.00	61.48	104.41	26.10

$\varepsilon_r = 10, \; h = 0.635 \text{ mm}, \; w/h = 1$

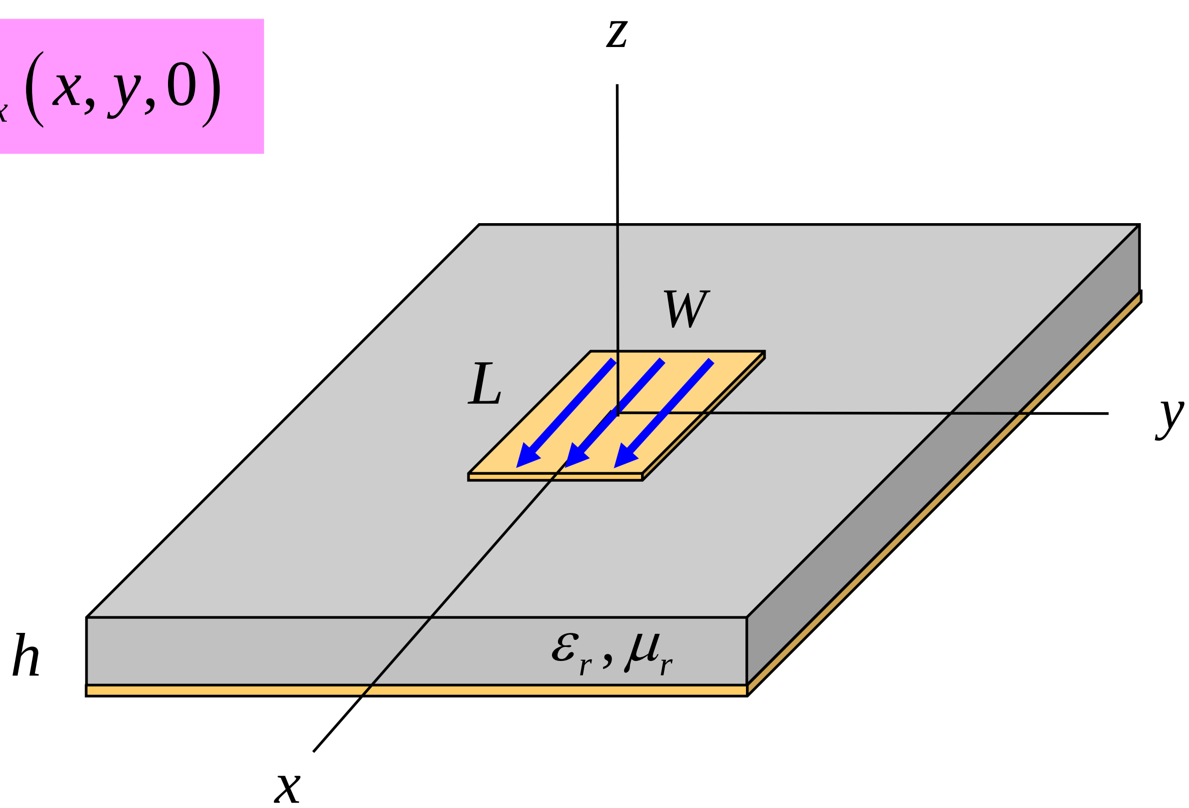
V^e = effective voltage (average taken over different paths).

Outline

- ❖ Physical derivation of method for planar electric surface currents.
- ❖ Examples involving planar surface currents:
  - ❑ Microstrip line
  - ❑ Microstrip patch current
- ❖ General derivation (Fourier transforming Maxwell's equations) that allows for all types of sources to be included in one general derivation.
- ❖ Examples:
  - ❑ Vertical dipole over the earth (Sommerfeld problem)
  - ❑ Slot antenna covered with radome layer (magnetic current)

Patch Antenna Example

Find $E_x(x, y, 0)$



Dominant (1,0) mode:

$$J_{sx}(x, y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Patch Antenna (cont.)

Recall that

$$\tilde{E}_x = \tilde{G}_{xx}\,\tilde{J}_{sx}$$

$$\tilde{G}_{xx}\left(k_x, k_y; z, z'\right) = -\frac{1}{k_t^2}\left[k_x^2\, V_i^{TM}\left(z, z'\right) + k_y^2\, V_i^{TE}\left(z, z'\right)\right]$$

In this problem

$$z' = 0$$

$$z = 0$$

Patch Antenna (cont.)

$$J_{sx}(x,y)=\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

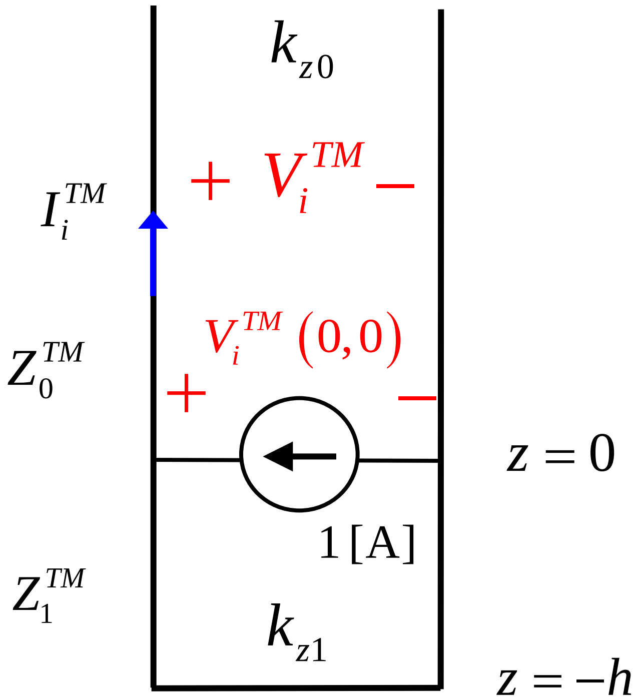
$$\tilde{J}_{sx}\left(k_x,k_y\right)=\frac{1}{W}\int_{L/2}^{L/2}\cos\left(\frac{\pi y}{L}\right)e^{jk_x x}\,dx\int_{-W/2}^{W/2}e^{jk_y y}\,dy$$

$$\tilde{J}_{sx}\left(k_x,k_y\right)=\frac{\pi L}{2}\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y\frac{W}{2}\right)$$

Microstrip Line (cont.)

TM_z :

$V_i^{TM}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

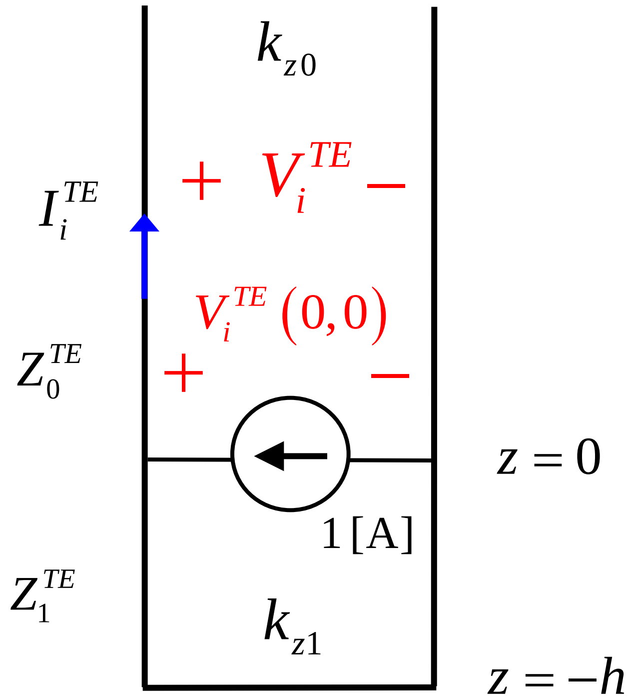
$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} = \eta_0\left(\frac{k_{z0}}{k_0}\right) \qquad Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} = \frac{\eta_0}{\varepsilon_r}\left(\frac{k_{z1}}{k_0}\right)$$

Microstrip Line (cont.)

TE_z :

$V_i^{TE}(0,0)$



$$k_{z0} = \left(k_0^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$k_{z1} = \left(k_1^2 - k_{x0}^2 - k_y^2\right)^{\frac{1}{2}}$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{z0}} = \frac{\eta_0}{\left(k_{z0} / k_0\right)}$$

$$Z_1^{TE} = \frac{\omega\mu_1}{k_{z1}} = \frac{\eta_0\mu_r}{\left(k_{z1} / k_0\right)}$$

Patch Antenna (cont.)

At $z = 0$:

$$V_i(0,0) = Z_{in}$$
$$= Y_{in}^{-1} = \left(Y_{in}^{+} + Y_{in}^{-}\right)^{-1}$$
$$= \left[Y_0 - jY_1 \cot\left(k_{z1}h\right)\right]^{-1}$$

Hence

$$V_i^{TM}(0,0) = \frac{1}{D_m\left(k_x, k_y\right)}$$

$$V_i^{TE}(0,0) = \frac{1}{D_e\left(k_x, k_y\right)}$$

$$D_m\left(k_x, k_y\right) = Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right)$$

$$D_e\left(k_x, k_y\right) = Y_0^{TE} - jY_1^{TE} \cot\left(k_{z1}h\right)$$

Patch Antenna (cont.)

$$D_m\left(k_x,k_y\right)=Y_0^{TM}-jY_1^{TM}\cot\left(k_{z1}h\right)$$

$$D_e\left(k_x,k_y\right)=Y_0^{TE}-jY_1^{TE}\cot\left(k_{z1}h\right)$$

Note: $D_{e,m}\left(k_x,k_y\right)=D_{e,m}\left(k_t\right)$

$$\tilde{G}_{xx}\left(k_x,k_y;z,z'\right)=-\frac{1}{k_t^2}\left[k_x^2\,V_i^{TM}\left(z,z'\right)+k_y^2\,V_i^{TE}\left(z,z'\right)\right]$$

Hence, we have

$$\tilde{E}_x\left(k_x,k_y,0\right)=\tilde{J}_{sx}\left(k_x,k_y\right)\left[\left(-\frac{1}{k_t^2}\right)\left[\frac{k_x^2}{D_m\left(k_t\right)}+\frac{k_y^2}{D_e\left(k_t\right)}\right]\right]$$

Patch Antenna (cont.)

Taking the inverse Fourier transform, we have

$$E_x(x,y,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} -\frac{1}{k_t^2} \tilde{J}_{sx}(k_x, k_y) \cdot \left[\frac{k_x^2}{D_m(k_t)} + \frac{k_y^2}{D_e(k_t)} \right] e^{-j(k_x x + k_y y)} \, dk_x dk_y$$

Example: Vertical Field

Find E_z (x,y,z) inside the substrate for the same patch antenna $(-h < z < 0)$.

$$\nabla \times \underline{H} = j\omega\varepsilon_1 \underline{E}$$

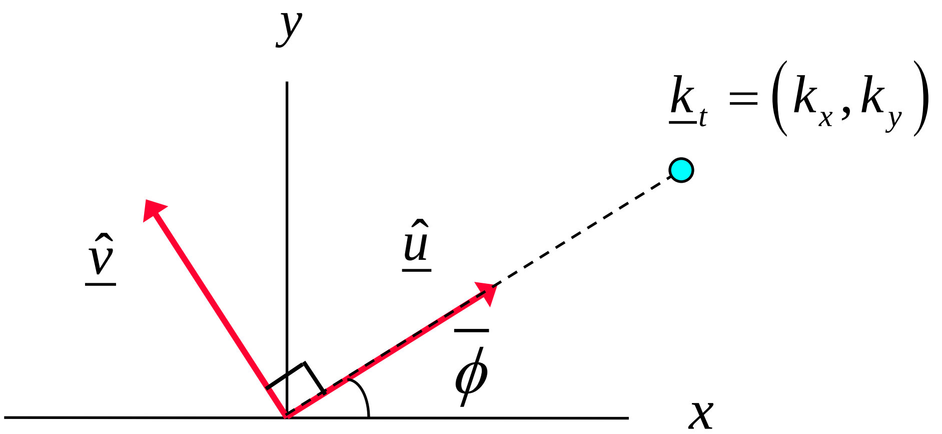
$$E_z = \frac{1}{j\omega\varepsilon_1}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_1}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

$$= \frac{1}{\omega\varepsilon_0\varepsilon_r}\left(-k_x\tilde{H}_y + k_y\tilde{H}_x\right)$$

Example (cont.)

$$\tilde{H}_x = \tilde{H}_u \left(\underline{\hat{u}} \cdot \underline{\hat{x}}\right) + \tilde{H}_v \left(\underline{\hat{v}} \cdot \underline{\hat{x}}\right)$$
$$= \tilde{H}_u \left(\cos \overline{\phi}\right) + \tilde{H}_v \left(-\sin \overline{\phi}\right)$$
$$= \tilde{H}_u \left(\frac{k_x}{k_t}\right) + \tilde{H}_v \left(-\frac{k_y}{k_t}\right)$$



$$\tilde{H}_y = \tilde{H}_u \left(\underline{\hat{u}} \cdot \underline{\hat{y}}\right) + \tilde{H}_v \left(\underline{\hat{v}} \cdot \underline{\hat{y}}\right)$$
$$\tilde{H}_u \left(\sin \overline{\phi}\right) + \tilde{H}_v \left(\cos \overline{\phi}\right)$$
$$= \tilde{H}_u \left(\frac{k_y}{k_t}\right) + \tilde{H}_v \left(\frac{k_x}{k_t}\right)$$

Example (cont.)

Hence

cancels

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0\varepsilon_r}\left(-k_x\left[\tilde{H}_u\left(\frac{k_y}{k_t}\right)+\tilde{H}_v\left(\frac{k_x}{k_t}\right)\right]+k_y\left[\tilde{H}_u\left(\frac{k_x}{k_t}\right)+\tilde{H}_v\left(-\frac{k_y}{k_t}\right)\right]\right)$$

or

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0\,\varepsilon_r}\left(\tilde{H}_v\left[-k_x^{\,2}-k_y^{\,2}\right]\frac{1}{k_t}\right)$$

or

$$\tilde{E}_z = \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\tilde{H}_v\right)$$

Example (cont.)

or

$$\tilde{E}_z\left(k_x,k_y,z\right)=\frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I^{TM}\left(z\right)$$

Hence

$$\tilde{E}_z\left(k_x,k_y,z\right)=-\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left[-\underline{\tilde{J}}_s\cdot\underline{\hat{u}}\right]$$

$$=-\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right)I_i^{TM}\left(z\right)\left[-\tilde{J}_{sx}\left(\frac{k_x}{k_t}\right)\right]$$

The result is then

$$\tilde{E}_z\left(k_x,k_y,z\right)=\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_x\right)\tilde{J}_{sx}I_i^{TM}\left(z\right)$$

Example (cont.)

Hence, in the space domain we have

$$E_z(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\omega \varepsilon_0 \varepsilon_r} (k_x)\ \tilde{J}_{sx} I_i^{TM}(z)\ e^{-j(k_x x + k_y y)} dk_x dk_y$$

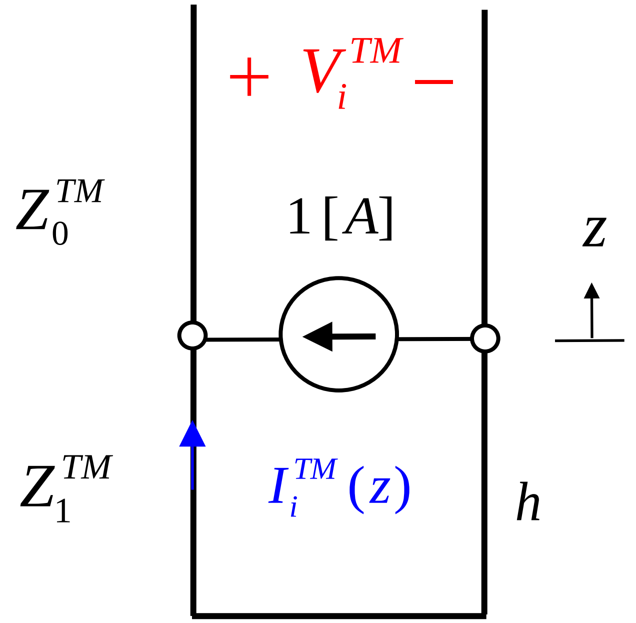
Note: Only TM_z waves contribute to the vertical electric field.

On the next page we start calculating the term $I_i^{TM}(z)$

Example (cont.)

We next calculate the function

$$I_i^{TM}(z)$$



$$I_i^{TM}\left(z=0^-\right) = -\left(\frac{V(0)}{Z_{in}^-}\right) = -\left(\frac{Z_{in}}{Z_{in}^-}\right) = -\left(\frac{1/D_m(k_t)}{j\,Z_1^{TM}\tan\left(k_{z1}h\right)}\right)$$

Example (cont.)

Because of the short circuit,

$$I_i^{TM}(z) = A\cos\left(k_{z1}(z+h)\right), \quad -h < z < 0$$

At $z = 0$: $I_i^{TM}\left(0^-\right) = A\cos\left(k_{z1}h\right)$

Therefore

$$A = \frac{I_i^{TM}\left(0^-\right)}{\cos\left(k_{z1}h\right)}$$

Hence

$$I_i^{TM}(z) = I_i^{TM}\left(0^-\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos\left(k_{z1}h\right)}\right) \qquad -h < z < 0$$

Example (cont.)

Hence

$$I_i^{TM}(z) = -\left(\frac{1/D_m(k_t)}{j\,Z_1^{TM}\tan(k_{z1}h)}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos(k_{z1}h)}\right)$$

$$-h < z < 0$$

or

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}\tan(k_{z1}h)}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\cos(k_{z1}h)}\right)$$

or

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}}\right)\left(\frac{\cos\left(k_{z1}(z+h)\right)}{\sin(k_{z1}h)}\right)$$

Example (cont.)

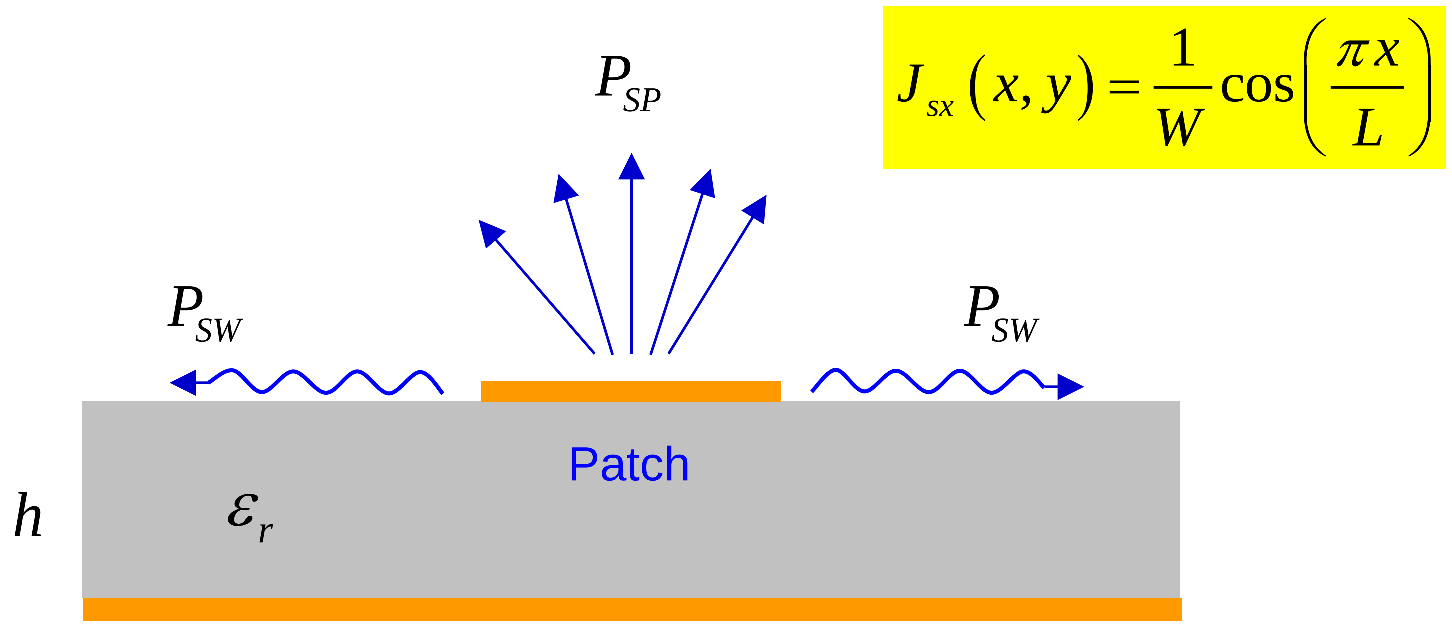
Hence

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}}\right)\left(\frac{\cos(k_{z1}(z+h))}{\sin(k_{z1}h)}\right)$$

where

$$D_m(k_x, k_y) = Y_0^{TM} - jY_1^{TM}\cot(k_{z1}h)$$

Radiated Power from Patch



Lossless substrate and metal

$$P_{TOT} = P_{SP} + P_{SW}$$

$$e_r = \frac{P_{SP}}{P_{TOT}}$$

Radiated Power (cont.)

$$P_c = -\frac{1}{2}\int_S \underline{E}\cdot\underline{J}_s^{\,*}\,dS$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x\, J_{sx}^{\,*}\, dxdy$$

Use Parseval's Theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, g^*(x,y)\, dxdy = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}\left(k_x,k_y\right)\tilde{g}^*\left(k_x,k_y\right) dk_x dk_y$$

Radiated Power (cont.)

Hence

$$P_c = -\frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}_x \tilde{J}_{sx}^{\,*} dk_x dk_y$$

From SDI analysis,

$$\tilde{E}_x = \tilde{J}_{sx} \tilde{G}_{xx}$$

where

$$\tilde{G}_{xx} = -\frac{1}{k_t^2} \left[\frac{k_x^2}{D_m(k_t)} + \frac{k_y^2}{D_e(k_t)} \right]$$

$$\tilde{J}_{sx}(k_x, k_y) = \frac{\pi L}{2} \left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2} \right] \operatorname{sinc}\left(k_y \frac{W}{2}\right)$$

$$D_m(k_t) = Y_0^{TM} - jY_1^{TM} \cot(k_{z1} h)$$

$$D_e(k_t) = Y_0^{TE} - jY_1^{TE} \cot(k_{z1} h)$$

Radiated Power (cont.)

We then have

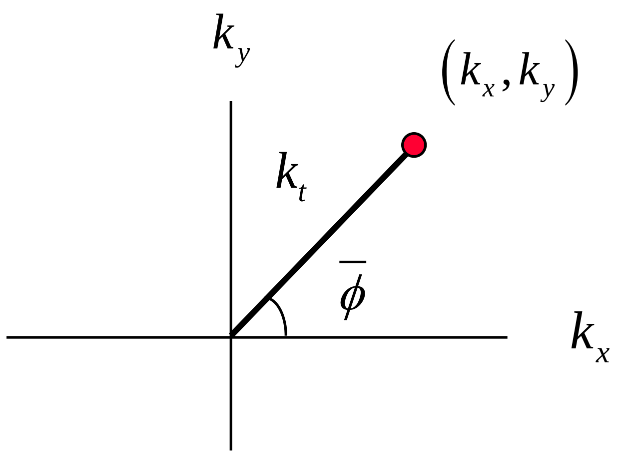
$$P_c = -\frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}_{xx} \left| \tilde{J}_{sx} \right|^2 dk_x dk_y$$

Using symmetry,

$$P_c = -\frac{1}{2\pi^2} \int_0^{\infty} \int_0^{\infty} \tilde{G}_{xx}\left(k_x, k_y\right) \left| \tilde{J}_{sx}\left(k_x, k_y\right) \right|^2 dk_x dk_y$$

Polar coordinates:

$$P_c = -\frac{1}{2\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx} \left| \tilde{J}_{sx} \right|^2 k_t dk_t d\bar{\phi}$$



Radiated Power (cont.)

$$P_c = -\frac{1}{2\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx} \left| \tilde{J}_{sx} \right|^2 k_t \, dk_t \, d\bar{\phi}$$

Note that

$\tilde{J}_{sx} =$ Real Function of $\left(k_x, k_y\right)$ for real k_x, k_y.

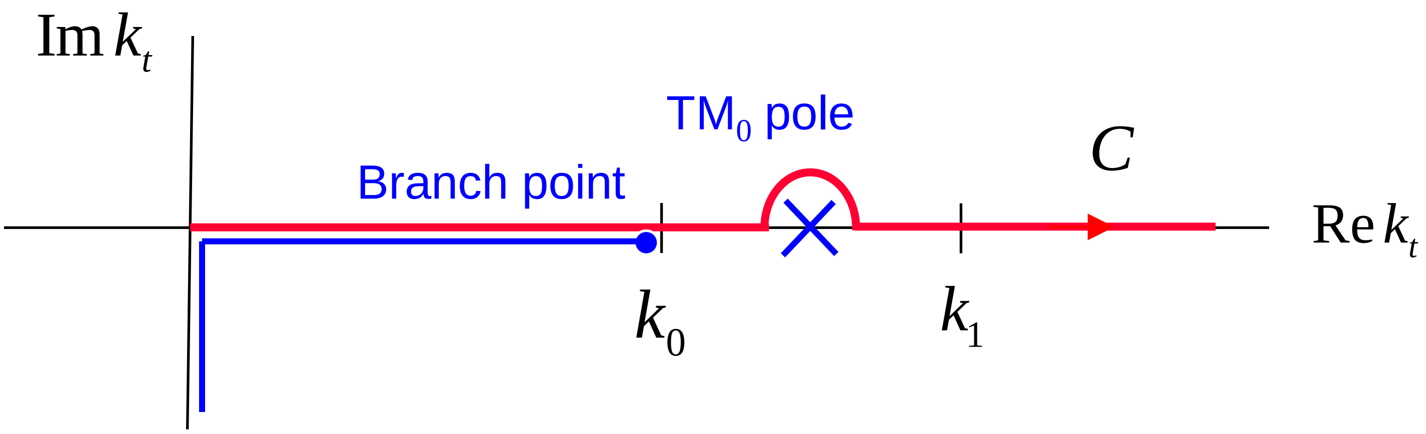
Hence

$$P_c = -\frac{1}{2\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx}\left(k_t, \bar{\phi}\right) \tilde{J}_{sx}^2\left(k_t, \bar{\phi}\right) k_t \, dk_t \, d\bar{\phi}$$

Note: $\tilde{J}_{sx}^2$ is analytic but $\left| \tilde{J}_{sx} \right|^2$ is not. This is a preferable form!

Radiated Power (cont.)

$$P_{TOT} = -\frac{1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} \int_C \tilde{G}_{xx}\left(k_t, \bar{\phi}\right) \tilde{J}_{sx}^2\left(k_t, \bar{\phi}\right) k_t \, dk_t \, d\bar{\phi}$$



For real k_t we have (proof on next page):

$$\tilde{G}_{xx} = \begin{cases} \text{complex}, & k_t < k_0 \\ \text{imag}, & k_t > k_0 \end{cases}$$

Hence, we can neglect the region $k_t > k_0$, except possibly for the pole.

Radiated Power (cont.)

Proof of complex property:

$$\tilde{G}_{xx}\left(k_t,\overline{\phi}\right) = -\frac{1}{k_t^{\,2}}\left[\frac{k_x^2}{D_m\left(k_t\right)} + \frac{k_y^2}{D_e\left(k_t\right)}\right]$$

Consider the following term (D_e is similar):

always imaginary

$$D_m\left(k_t\right) = Y_0^{\,TM} - jY_1^{\,TM}\cot\left(k_{z1}h\right)$$

$$Y_0^{\,TM} = \frac{\omega\,\varepsilon_0}{k_{z0}}$$

$$k_{z0} = \left(k_0^{\,2} - k_t^{\,2}\right)^{\frac{1}{2}}$$

$$Y_1^{\,TM} = \frac{\omega\,\varepsilon_1}{k_{z1}}$$

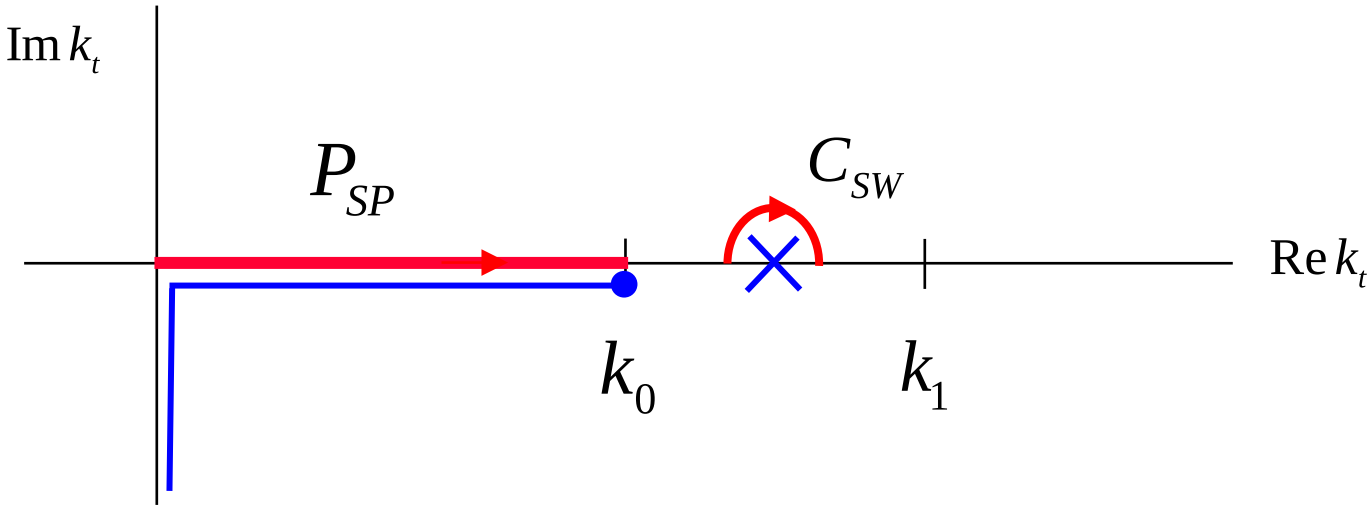
$$k_{z1} = \left(k_1^2 - k_t^2\right)^{\frac{1}{2}}$$

$$Y_0^{\,TM} = \begin{cases}\text{real}, & k_t < k_0 \\ \text{imag}, & k_t > k_0\end{cases}$$

Space-Wave Power

❖The region $k_t \in (0, k_0)$ gives the power radiated into space.

❖The residue contribution gives the power launched into the TM_0 surface wave.



$$P_{SP} = \frac{-1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} \int_0^{k_0} \tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{J}_{sx}^2\left(k_t, \overline{\phi}\right) k_t \, dk_t \, d\overline{\phi}$$

Surface-Wave Power

The pole contribution gives the surface-wave power:

$$P_{SW} = \frac{-1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} \int_{C_{SW}} \tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{J}_{sx}^2\left(k_t, \overline{\phi}\right) k_t \, dk_t \, d\overline{\phi}$$

From the residue theorem, we have

$$P_{SW} = \frac{-1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} -\pi j \operatorname{Res}\left\{\tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{J}_{sx}^2\left(k_t, \overline{\phi}\right) k_t\right\}_{k_t = k_{tp}} d\overline{\phi}$$

$$= \frac{-1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} -\pi j \, k_{tp} \, \tilde{J}_{sx}^2\left(k_{tp}, \overline{\phi}\right) \operatorname{Res}\left\{\tilde{G}_{xx}\left(k_t, \overline{\phi}\right)\right\}_{k_t = k_{tp}} d\overline{\phi}$$

$$= \frac{1}{2\pi} k_{tp} \operatorname{Re} \int_0^{\pi/2} j \; \tilde{J}_{sx}^2\left(k_{tp}, \overline{\phi}\right) \operatorname{Res}\left\{\tilde{G}_{xx}\left(k_t, \overline{\phi}\right)\right\}_{k_t = k_{tp}} d\overline{\phi}$$

Surface-Wave Power (cont.)

$$\tilde{G}_{xx}\left(k_t,\overline{\phi}\right) = -\frac{1}{k_t^{\,2}}\left[\frac{k_x^{\,2}}{D_m\left(k_t\right)} + \frac{k_y^{\,2}}{D_e\left(k_t\right)}\right]$$

$$= -\left[\frac{\cos^2\overline{\phi}}{D_m\left(k_t\right)} + \frac{\sin^2\overline{\phi}}{D_e\left(k_t\right)}\right]$$

The residue of the spectral-domain Green's function at the TM pole is

$$\operatorname{Res}\tilde{G}_{xx}\left(k_t,\overline{\phi}\right)\Big|_{k_t=k_{tp}} = -\left[\frac{\cos^2\overline{\phi}}{D_m'\left(k_{tp}\right)}\right]$$

$$= \text{ pure imaginary}$$

Note: This can be calculated in closed form, but the result is omitted here.

where

$$D_m\left(k_t\right) = Y_0^{TM} - jY_1^{TM}\cot\left(k_{z1}h\right)$$

Surface-Wave Power (cont.)

Since the transform of the current is real (assuming that k_{tp} is real), we have

$$P_{SW} = \frac{1}{2\pi} k_{tp} \int_0^{\pi/2} \tilde{J}_{sx}^2\left(k_{tp}, \overline{\phi}\right) \mathrm{Re}\left[j\,\mathrm{Res}\left\{ \tilde{G}_{xx}\left(k_t, \overline{\phi}\right)\right\}_{k_t = k_{tp}} \right] d\overline{\phi}$$

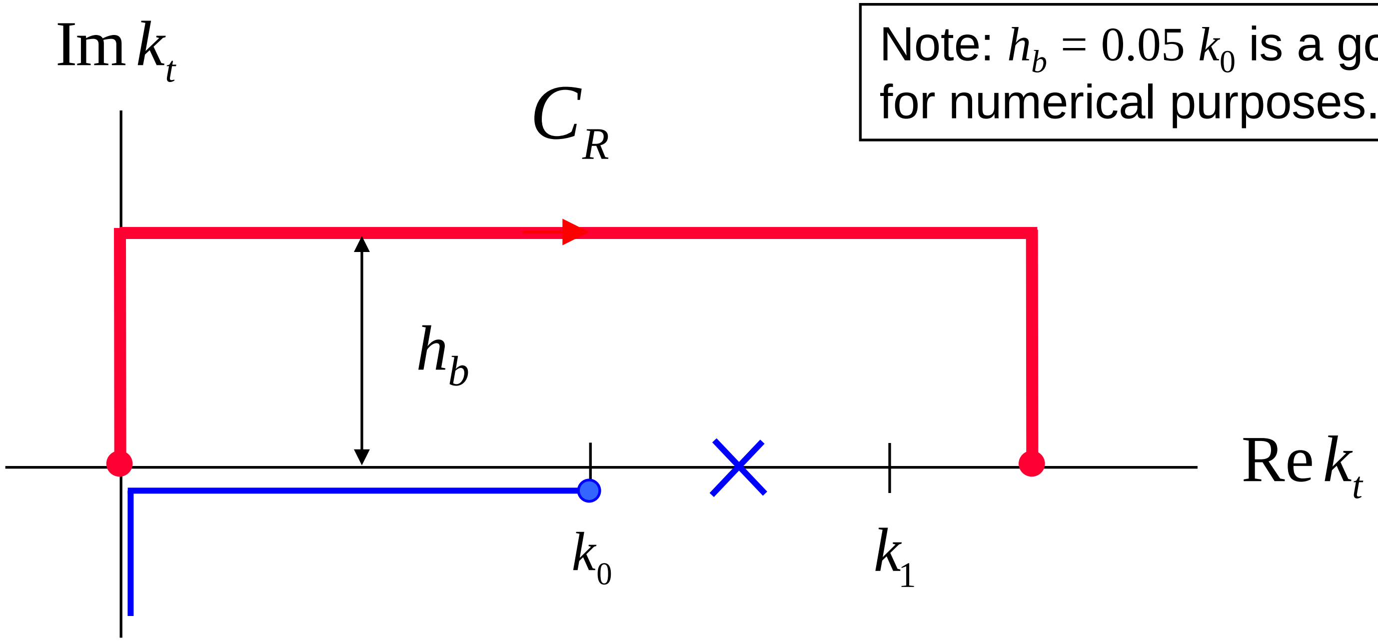
Since the residue is pure imaginary, we then have

$$P_{SW} = -\frac{1}{2\pi} k_{tp} \int_0^{\pi/2} \tilde{J}_{sx}^2\left(k_{tp}, \overline{\phi}\right) \mathrm{Im}\left[\mathrm{Res}\left\{ \tilde{G}_{xx}\left(k_t, \overline{\phi}\right)\right\}_{k_t = k_{tp}} \right] d\overline{\phi}$$

Total Power

The total power may also be calculated directly:

$$P_{TOT} = \frac{-1}{2\pi^2} \operatorname{Re} \int_0^{\pi/2} \int_{C_R} \tilde{G}_{xx}\left(k_t, \bar{\phi}\right) \tilde{J}_{sx}^{2}\left(k_t, \bar{\phi}\right) k_t \, dk_t \, d\bar{\phi}$$



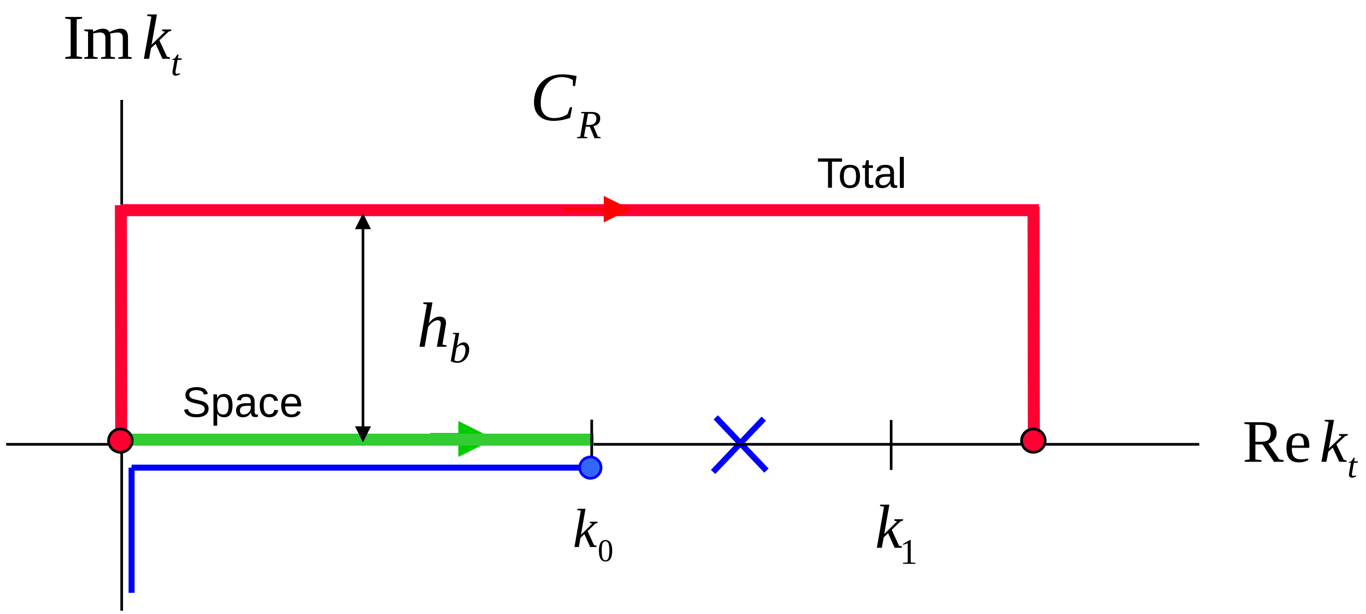
Note: $h_b = 0.05\, k_0$ is a good choice for numerical purposes.

Surface-Wave Power: Alternative Method

The surface-wave power may then be calculated from:

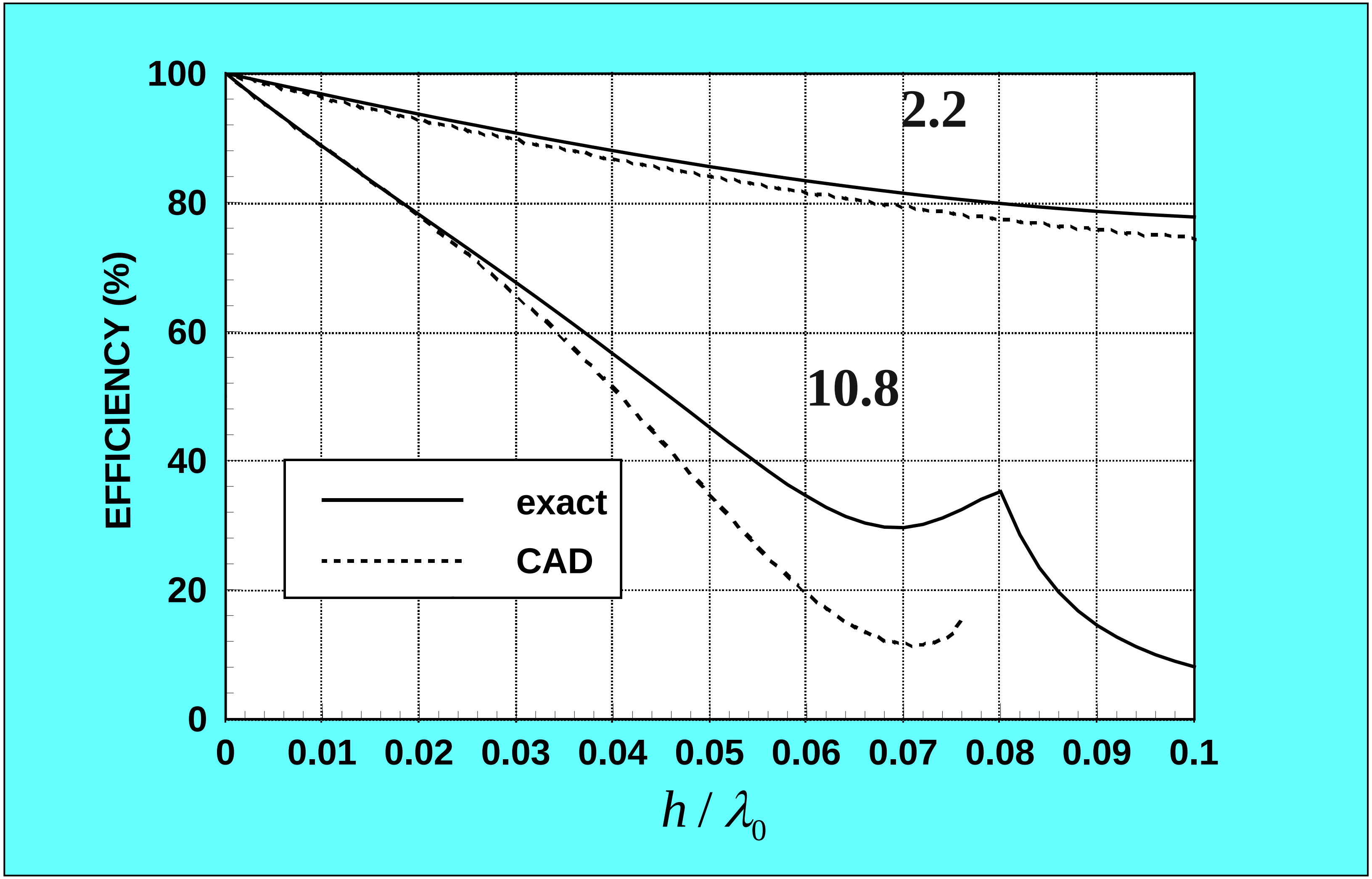
$$P_{SW} = P_{TOT} - P_{SP}$$

This avoids calculating any residues.



Results

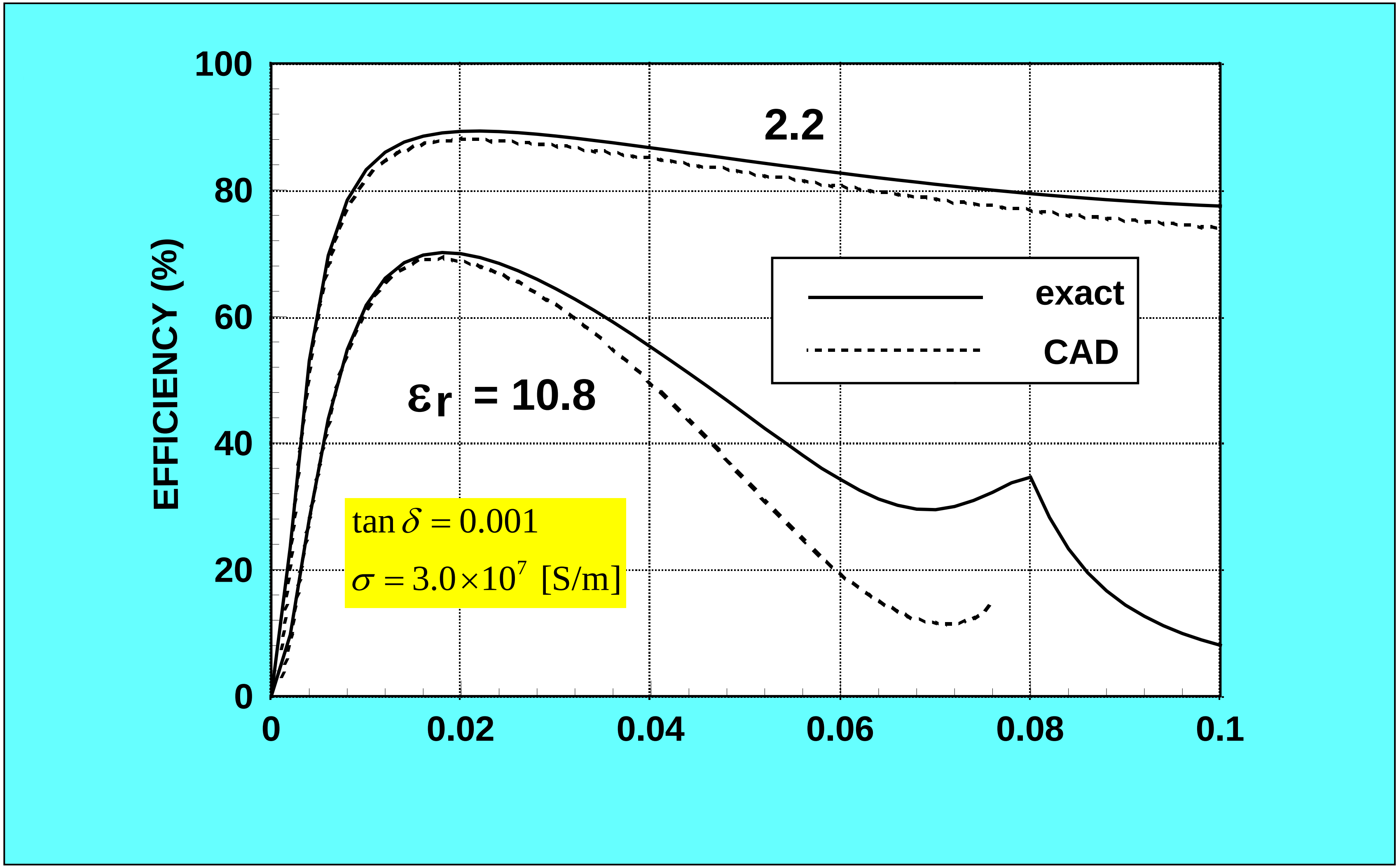
Results: Conductor and dielectric losses are neglected



$\varepsilon_r = 2.2$ or $10.8 \qquad W/L = 1.5$

Results (cont.)

Results: Accounting for all losses (including conductor and dielectric loss)



$\varepsilon_r = 2.2$ or 10.8 $\quad W/L = 1.5$

Outline

- ❖ Physical derivation of method for planar electric surface currents.
- ❖ Examples involving planar surface currents:
  - ❑ Microstrip line
  - ❑ Microstrip patch current
- ❖ General derivation (Fourier transforming Maxwell's equations) that allows for all types of sources to be included in one general derivation.
- ❖ Examples:
  - ❑ Vertical dipole over the earth (Sommerfeld problem)
  - ❑ Slot antenna covered with radome layer (magnetic current)

Overview of General SDI Derivation

In this part of the notes we derive the SDI formulation using a more mathematical, but more general, approach (we directly Fourier transform Maxwell's equations).

❖ This allows for all possible types of sources (horizontal, vertical, electric, and magnetic) to be treated in one derivation.

General SDI Method

Start with Ampere's law:

$$\nabla \times \underline{H} = \underline{J}^{i} = j\omega\varepsilon \underline{E}$$

$$\nabla = \nabla_t + \underline{\hat{z}} \frac{\partial}{\partial z}$$

where $$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

Assume a 2D spatial transform:
$$\begin{aligned} \tilde{\nabla}_t &= \underline{\hat{x}}\left(-jk_x\right) + \underline{\hat{y}}\left(-jk_y\right) \\ &= -j\left(\underline{\hat{x}}\,k_x + \underline{\hat{y}}\,k_y\right) \\ &= -j\underline{k}_t \\ &= -jk_t\,\underline{\hat{u}} \end{aligned}$$

General SDI Method (cont.)

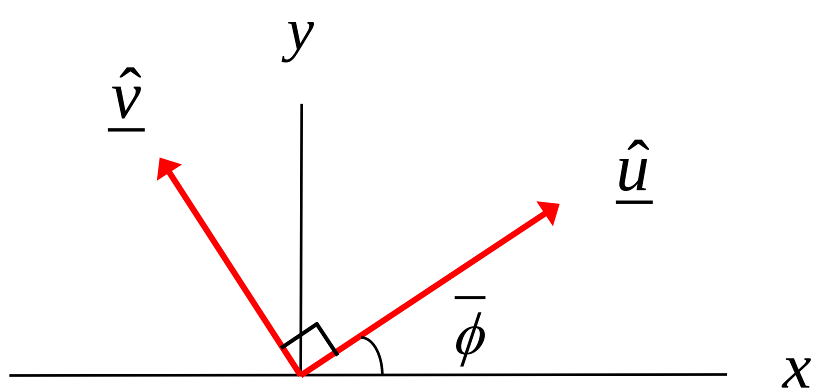
Hence we have $\left(-jk_t\,\underline{\hat{u}}+\underline{\hat{z}}\frac{\partial}{\partial z}\right)\times\underline{\tilde{H}}=\underline{\tilde{J}}^i+j\omega\varepsilon\underline{\tilde{E}}$

Next, represent the field as $\underline{\tilde{H}}_t=\underline{\hat{u}}\,\tilde{H}_u+\underline{\hat{v}}\,\tilde{H}_v$

$$=\underline{\hat{u}}\left(\underline{\tilde{H}}\cdot\underline{\hat{u}}\right)+\underline{\hat{v}}\left(\underline{\tilde{H}}\cdot\underline{\hat{v}}\right)$$

Note that

$$\underline{\hat{u}}\times\underline{\hat{v}}=\underline{\hat{z}}$$
$$\underline{\hat{z}}\times\underline{\hat{u}}=\underline{\hat{v}}$$
$$\underline{\hat{z}}\times\underline{\hat{v}}=-\underline{\hat{u}}$$



Take the $\underline{\hat{z}},\ \underline{\hat{u}},\ \underline{\hat{v}}$ components of the transformed Ampere's equation:

General SDI Method (cont.)

$$\underline{\hat{z}})\ -jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z$$

$$\underline{\hat{u}})\quad -\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u$$

$$\underline{\hat{v}})\qquad jk_t\tilde{H}_z + \frac{\partial\tilde{H}_u}{\partial z} = \tilde{J}_v^i + j\omega\varepsilon\tilde{E}_v$$

Examine the TM_z field: $\left(\tilde{E}_u, \tilde{H}_v, \tilde{E}_z\right)$ (Ignore $\underline{\hat{v}}$ equation)

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \qquad (1)$$

$$-\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u \qquad (2)$$

TM_z Fields

We wish to eliminate $\tilde{E}_z$ from Eq. (1). To do this, use Faraday's law:

$$\nabla \times \underline{E} = -\underline{M}^i - j\omega\mu\underline{H}$$

$$\Rightarrow \quad \left(-jk_t\,\underline{\hat{u}} + \underline{\hat{z}}\frac{\partial}{\partial z} \right) \times \underline{\tilde{E}} = -\underline{\tilde{M}}^i - j\omega\mu\underline{\tilde{H}}$$

Take the $\underline{\hat{v}}$ component of the transformed Faraday's Law:

$$jk_t\tilde{E}_z + \frac{\partial \underline{\tilde{E}}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v \qquad (3)$$

$$\underline{\hat{u}} \times \underline{\hat{v}} = \underline{\hat{z}}$$

$$\underline{\hat{z}} \times \underline{\hat{u}} = \underline{\hat{v}}$$

$$\underline{\hat{z}} \times \underline{\hat{v}} = -\underline{\hat{u}}$$

TM$_z$ Fields (cont.)

Substitute $\tilde{E}_z$ from (1) into (3) to eliminate $\tilde{E}_z$

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \qquad (1)$$

$$jk_t\tilde{E}_z + \frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v \qquad (3)$$

$$\Rightarrow \quad jk_t\left[\frac{1}{j\omega\varepsilon}\left(-\tilde{J}_z^i - jk_t\tilde{H}_v\right)\right] + \frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v$$

or

$$\frac{\partial \tilde{E}_u}{\partial z} + \frac{k_t^2}{j\omega\varepsilon}\tilde{H}_v + j\omega\mu\tilde{H}_v = -\tilde{M}_v^i + \frac{k_t}{\omega\varepsilon}\tilde{J}_z^i$$

TM$_z$ Fields (cont.)

Note that

$$\frac{k_t^2}{j\omega\varepsilon} + j\omega\mu = \frac{1}{j\omega\varepsilon}\left(k_t^2 - \omega^2\mu\varepsilon\right)$$

$$= \frac{1}{j\omega\varepsilon}\left(k_t^2 - k^2\right)$$

$$= \frac{-1}{j\omega\varepsilon}\left(k^2 - k_t^2\right)$$

$$= \frac{-k_z^2}{j\omega\varepsilon}$$

Hence

$$\frac{\partial \tilde{E}_u}{\partial z} - \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i \qquad (4)$$

TM$_z$ Fields (cont.)

Equations (2) and (4) are the final TM$_z$ field modeling equations:

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon} \right) \tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon} \right) \tilde{H}_v$$

TM_z Fields (cont.)

Define TEN modeling equations:

$$V^{TM}(z) \equiv \tilde{E}_u(k_x, k_y, z)$$

$$I^{TM}(z) \equiv \tilde{H}_v(k_x, k_y, z)$$

We then have

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

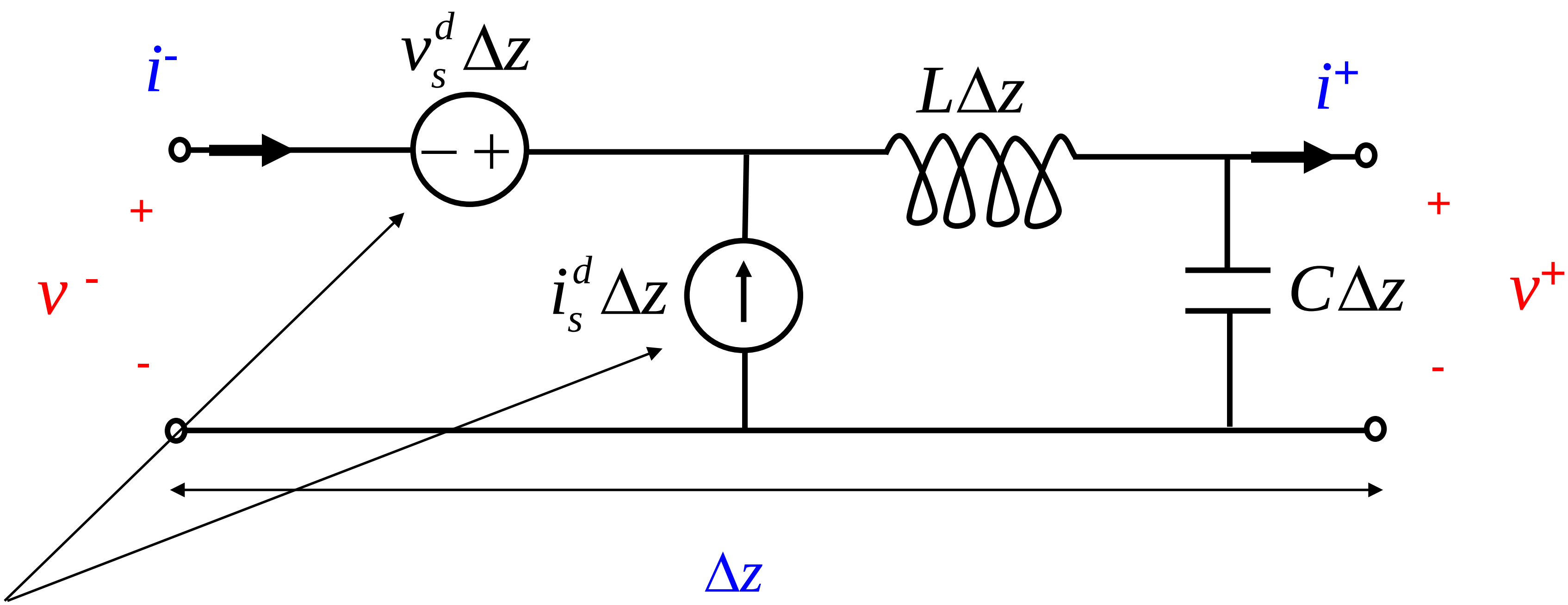
$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

⇓

$$\frac{\partial I^{TM}}{\partial z} = -j\omega\varepsilon V^{TM} - \tilde{J}_u^i$$

$$\frac{\partial V^{TM}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{TM} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

Telegrapher's Equation



Allow for distributed sources

$$v^+ - v^- = v_s^d \Delta z - (L\Delta z)\frac{\partial}{\partial t}\left(i^- + i_s^d \Delta z\right)$$

$\Delta z \to 0$ so $$\frac{\partial v}{\partial z} = v_s^d - L\frac{\partial i}{\partial t}$$

Telegrapher's Equation (cont.)

Hence, in the phasor domain,

$$\frac{\partial V}{\partial z} = -j\omega LI + V_s^d$$

Also, $$i^+ - i^- = i_s^d \Delta z - (C\Delta z)\frac{\partial v^+}{\partial t}$$

$\Delta z \to 0$ so $$\frac{\partial i}{\partial z} = i_s^d - C\frac{\partial v}{\partial t}$$

Hence, we have

$$\frac{\partial I}{\partial z} = -j\omega CV + I_s^d$$

Telegrapher's Equation (cont.)

Compare field equations for TM$_z$ fields with TL equations:

$$\frac{\partial I^{TM}}{\partial z} = -j\omega\varepsilon V^{TM} - \tilde{J}_u^i$$

$$\frac{\partial V^{TM}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{TM} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$

Telegrapher's Equation (cont.)

We then make the following identifications:

$$C = \varepsilon$$

$$\omega L = \frac{k_z^2}{\omega \varepsilon}$$

Hence

$$k_z^{TL} = \omega \sqrt{LC} = \omega \sqrt{\left(\frac{k_z^2}{\omega^2 \varepsilon} \right) \varepsilon} = k_z$$

$$Z_0^{TL} = \sqrt{\frac{L}{C}} = \sqrt{\left(\frac{k_z^2}{\omega^2 \varepsilon} \right) \frac{1}{\varepsilon}} = \frac{k_z}{\omega \varepsilon}$$

so

$$k_z^{TL} = k_z$$

$$Z_0^{TL} = \frac{k_z}{\omega \varepsilon}$$

Sources: TM$_z$

For the sources we have, for the TM$_z$ case:

$$I_s^{d^{TM}} = -\tilde{J}_u^i$$

$$V_s^{d^{TM}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

Sources: TM_z (cont.)

Special case: planar horizontal surface current sources:

Assume

$$\underline{J}^{i}(x,y,z)=\underline{J}_{s}^{i}(x,y)\,\delta(z)$$

$$\underline{M}^{i}(x,y,z)=\underline{M}_{s}^{i}(x,y)\,\delta(z)$$

Then we have

$$I_s^{d^{TM}}=-\tilde{J}_u^{i}=-\tilde{J}_{su}^{i}\,\delta(z)$$

$$V_s^{d^{TM}}=-\tilde{M}_v^{i}=-\tilde{M}_{sv}^{i}\,\delta(z)$$

These correspond to lumped current and voltage sources:

$$I_s^{TM}=-\tilde{J}_{su}^{i}$$ *lumped* parallel current source

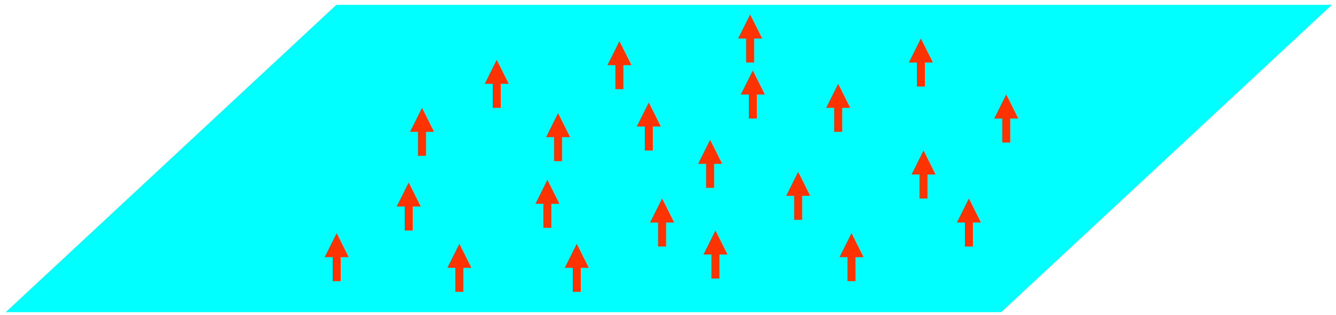
$$V_s^{TM}=-\tilde{M}_{sv}^{i}$$ *lumped* series voltage source

Sources: TM$_z$ (cont.)

For a vertical electric current:

Assume $J_z^i(x,y,z) = J_{sz}^i(x,y)\,\delta(z)$

"planar vertical current distribution"



Sources: TM$_z$ (cont.)

$$J_z^i\left(x,y,z\right)=J_{sz}^i(x,y)\,\delta(z)$$

$$V_s^{d^{TM}}=\left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i(k_x,k_y)=\left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^i(k_x,k_y)\,\delta\left(z\right)$$

This corresponds to a *lumped* series voltage source:

$$V_s^{TM}=\left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^i(k_x,k_y)$$

Sources: TM$_z$ (cont.)

Special case: vertical electric dipole

$$J_z^i(x,y,z) = \delta(x)\delta(y)\delta(z)$$

$$J_{sz}^i(x,y) = \delta(x)\delta(y)$$

$$\Rightarrow \quad \tilde{J}_{sz}^i(k_x,k_y) = 1$$

Hence $$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)$$

TE_z Fields

Use duality:

$$\underline{E} \rightarrow \underline{H}$$

$$\underline{H} \rightarrow -\underline{E}$$

$$\underline{J}^{\,i} \rightarrow \underline{M}^{\,i}$$

$$\underline{M}^{\,i} \rightarrow -\underline{J}^{\,i}$$

$$\varepsilon \rightleftarrows \mu$$

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^{\,i} - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^{\,i} + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^{\,i} + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM_z

$$-\frac{\partial \tilde{E}_v}{\partial z} = -\tilde{M}_u^{\,i} - j\omega\mu \tilde{H}_u$$

$$\frac{\partial \tilde{H}_u}{\partial z} = +\tilde{J}_v^{\,i} + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^{\,i} - \left(\frac{k_z^2}{j\omega\mu}\right)\tilde{E}_v$$

TE_z

TE$_z$ (cont.)

Define

$$V^{TE}(z) \equiv -\tilde{E}_v(k_x, k_y, z)$$
$$I^{TE}(z) \equiv \tilde{H}_u(k_x, k_y, z)$$

$$\frac{\partial V^{TE}}{\partial z} = -j\omega\mu I^{TE} - \tilde{M}_u^i$$
$$\frac{\partial I^{TE}}{\partial z} = \left(\frac{k_z^2}{j\omega\mu}\right) V^{TE} + \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$
$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

We then identify

$$L = \mu$$
$$\omega C = \frac{k_z^2}{\omega\mu}$$

$$\Rightarrow$$

$$k_z^{TL} = k_z$$
$$Z_0^{TE} = \frac{\omega\mu}{k_z}$$

TE_z (cont.)

For the sources, we have

$$V_s^{d^{TE}} = -\tilde{M}_u^i$$

$$I_s^{d^{TE}} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of planar horizontal surface currents:

$$V_s^{TE} = -\tilde{M}_{su}^i$$

lumped series voltage source

$$I_s^{TE} = +\tilde{J}_{sv}^i$$

lumped parallel current source

Sources: TE$_z$ (cont.)

For a vertical magnetic current:

Assume $$M_z^i(x,y,z) = M_{sz}^i(x,y)\,\delta(z)$$

This corresponds to a *lumped* parallel current source:

Then we have $$I_s^{TM} = \left(\frac{k_t}{\omega u}\right)\tilde{M}_{sz}^i\left(k_x,k_y\right)$$

Sources: TE$_z$ (cont.)

Special case: vertical magnetic dipole

$$M_z^i(x,y,z) = \delta(x)\delta(y)\delta(z)$$

$$M_{sz}^i(x,y) = \delta(x)\delta(y)$$

$$\Rightarrow \quad \tilde{M}_{sz}^i(k_x,k_y) = 1$$

Hence $\quad I_s^{TM} = \left(\dfrac{k_t}{\omega u}\right)$

Summary of Modeling Formulas

Results for 3D (volumetric) sources

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

Horizontal

$$I_s^{d\,TM} = -\tilde{J}_u^{\,i}$$
$$V_s^{d\,TM} = -\tilde{M}_v^{\,i}$$

$$I_s^{d\,TE} = +\tilde{J}_v^{\,i}$$
$$V_s^{d\,TE} = -\tilde{M}_u^{\,i}$$

Vertical

$$V_s^{d\,TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^{\,i}$$

$$I_s^{d\,TE} = \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^{\,i}$$

Distributed sources:
either parallel current sources
or series voltage sources

Summary of Modeling Formulas (cont.)

Results for 2D (planar) sources

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

Horizontal

$$I_s^{TM} = -\tilde{J}_{su}^{\,i}$$
$$V_s^{TM} = -\tilde{M}_{sv}^{\,i}$$

$$I_s^{TE} = +\tilde{J}_{sv}^{\,i}$$
$$V_s^{TE} = -\tilde{M}_{su}^{\,i}$$

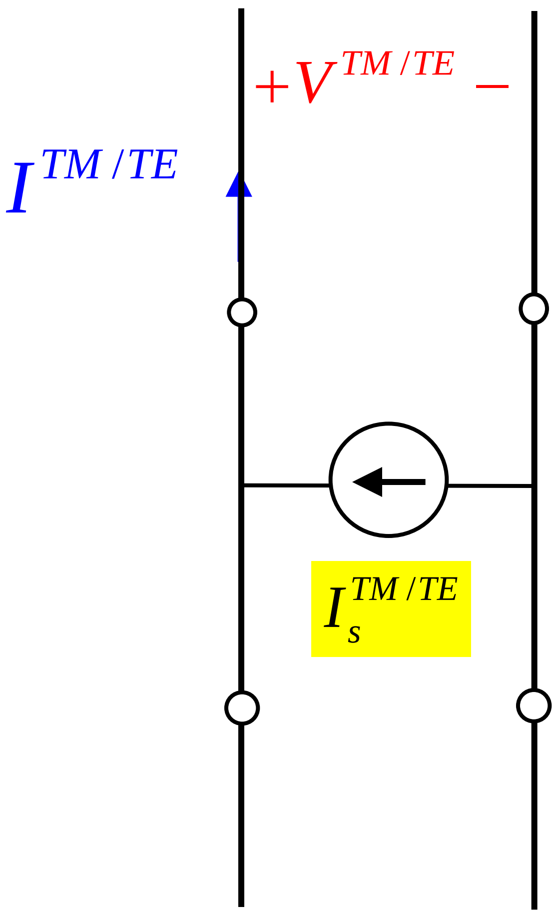
Vertical

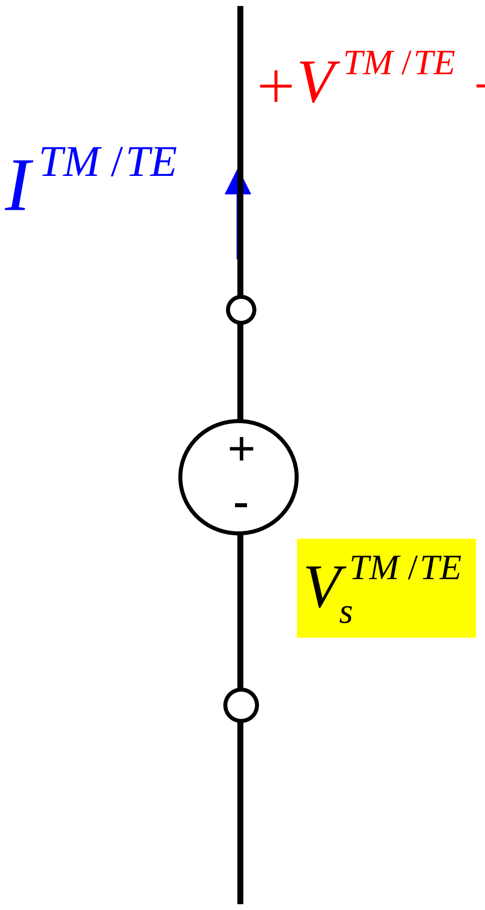
$$V_s^{TM} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_{sz}^{\,i}$$

$$I_s^{TE} = \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_{sz}^{\,i}$$

Lumped sources:
either parallel current sources
or series voltage sources

Summary of Modeling Formulas (cont.)



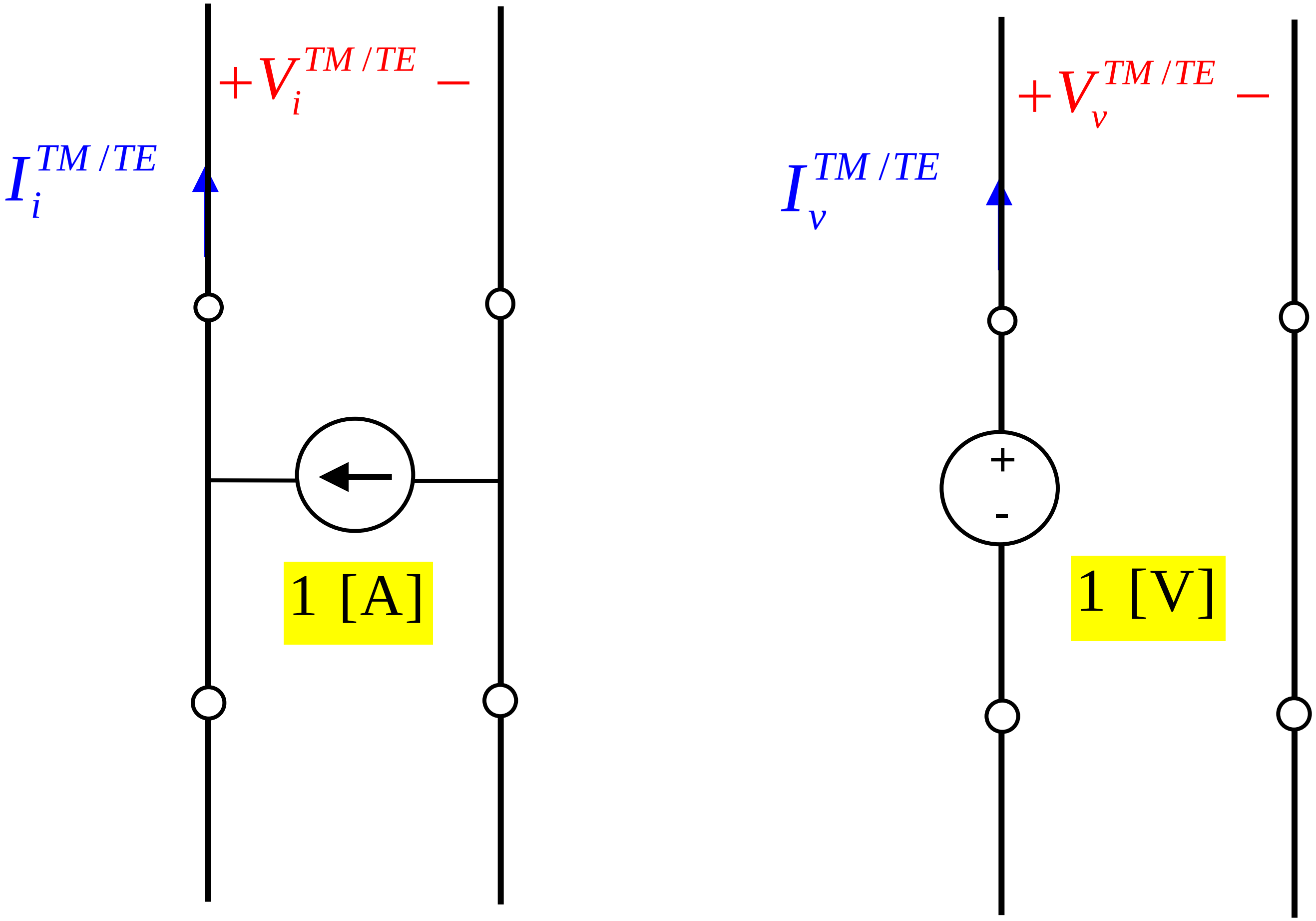


$$V^{TM/TE}(z) = V_i^{TM/TE}(z)\, I_s^{TM/TE}$$
$$I^{TM/TE}(z) = I_i^{TM/TE}(z)\, I_s^{TM/TE}$$

$$V^{TM/TE}(z) = V_v^{TM/TE}(z)\, V_s^{TM/TE}$$
$$I^{TM/TE}(z) = I_v^{TM/TE}(z)\, V_s^{TM/TE}$$

Summary of Modeling Formulas (cont.)

Michalski functions



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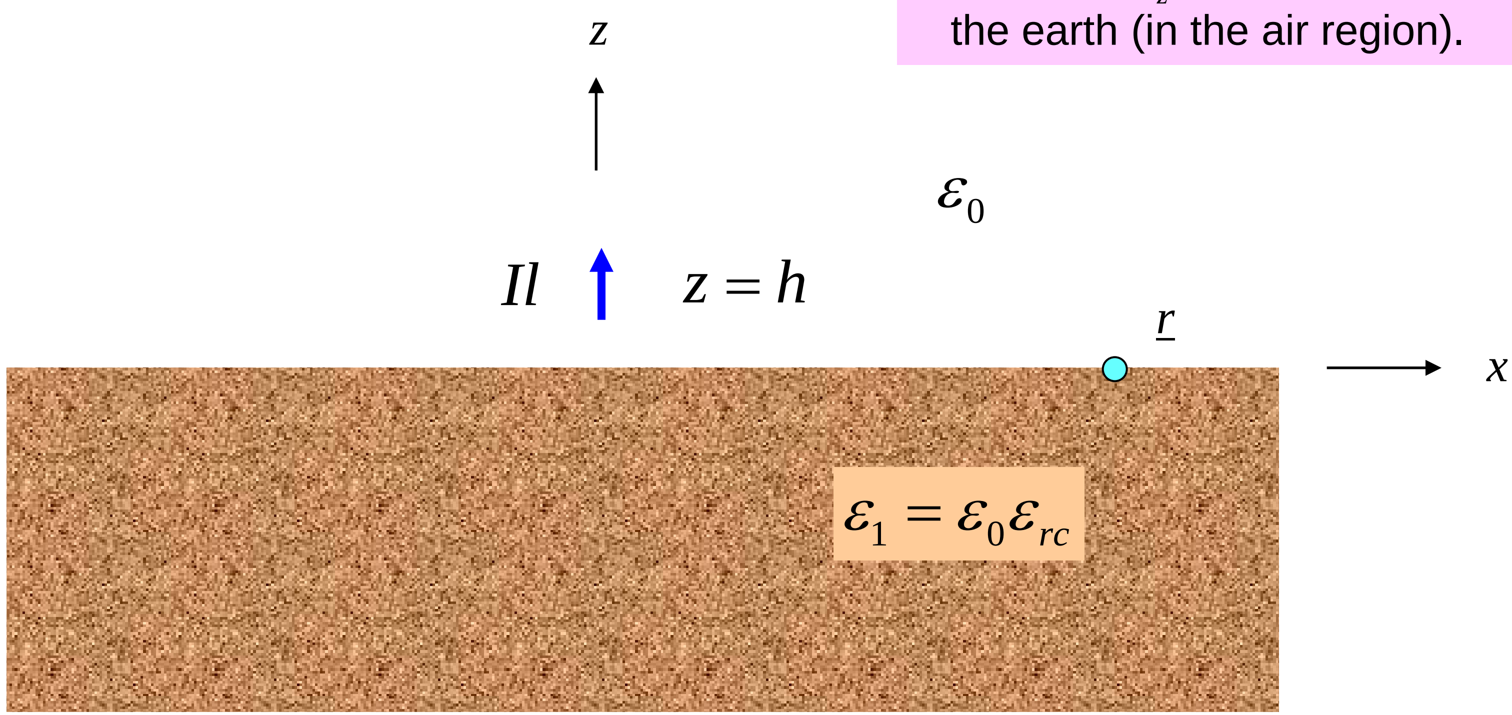
  - ❑ Slot antenna covered with radome layer (magnetic current)

Sommerfeld Problem

In this part of the notes we use SDI theory to solve the classical "Sommerfeld problem" of a vertical dipole over an semi-infinite earth.

Goal: Find E_z on the surface of the earth (in the air region).



Sommerfeld Problem (cont.)

Planar vertical electric current :

$$V_s^{TM} = \left(\frac{k_t}{\omega \varepsilon_c} \right) \tilde{J}_{sz}$$

$\left(\text{where } J_z(x, y, z) = J_{sz}(x, y)\delta(z)\right)$

(the impressed source current)

For a vertical electric dipole of amplitude Il, we have

$$J_z(x, y) = \left(Il\,\delta(x)\delta(y) \right)\delta(z)$$

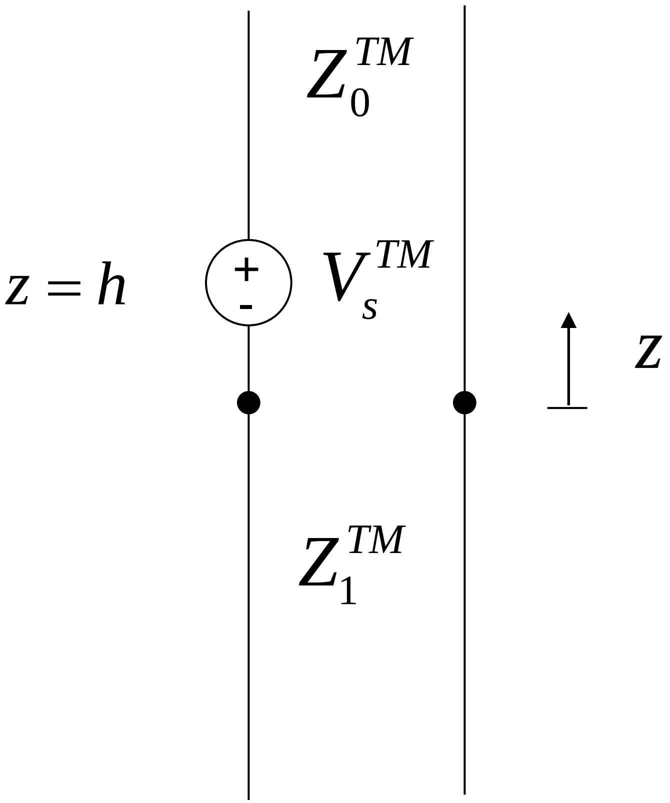
Hence $J_{sz}(x, y) = Il\,\delta(x)\delta(y)$

Therefore, we have

$$V_s^{TM} = \left(\frac{k_t}{\omega \varepsilon_c} \right)(Il)$$

Sommerfeld Problem (cont.)

TEN:



$$V_s^{TM} = Il\left(\frac{k_t}{\omega\varepsilon_0}\right)$$

The vertical electric dipole excites TM_z waves only.

Sommerfeld Problem (cont.)

Find E_z (x,y,z) inside the air region ($z > 0$):

$$\nabla \times \underline{H} = j\omega\varepsilon_0 \underline{E}$$

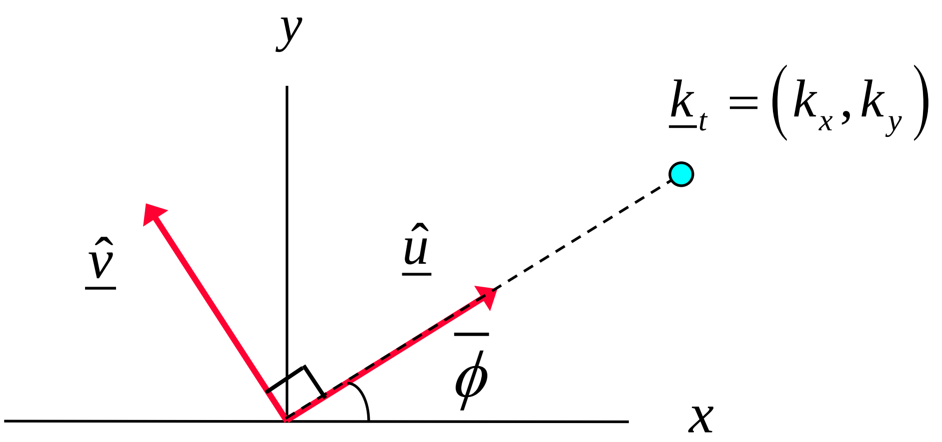
$$E_z = \frac{1}{j\omega\varepsilon_0}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_0}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

$$= \frac{1}{\omega\varepsilon_0}\left(-k_x\tilde{H}_y + k_y\tilde{H}_x\right)$$

Sommerfeld Problem (cont.)

$$\tilde{H}_x = \tilde{H}_u\left(\underline{\hat{u}}\cdot\underline{\hat{x}}\right)+\tilde{H}_v\left(\underline{\hat{v}}\cdot\underline{\hat{x}}\right)$$
$$= \tilde{H}_u\left(\cos\overline{\phi}\right)+\tilde{H}_v\left(-\sin\overline{\phi}\right)$$
$$= \tilde{H}_u\left(\frac{k_x}{k_t}\right)+\tilde{H}_v\left(-\frac{k_y}{k_t}\right)$$



$$\tilde{H}_y = \tilde{H}_u\left(\underline{\hat{u}}\cdot\underline{\hat{y}}\right)+\tilde{H}_v\left(\underline{\hat{v}}\cdot\underline{\hat{y}}\right)$$
$$\tilde{H}_u\left(\sin\overline{\phi}\right)+\tilde{H}_v\left(\cos\overline{\phi}\right)$$
$$= \tilde{H}_u\left(\frac{k_y}{k_t}\right)+\tilde{H}_v\left(\frac{k_x}{k_t}\right)$$

Sommerfeld Problem (cont.)

Hence

cancels

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0}\left(-k_x\left[\tilde{H}_u\left(\frac{k_y}{k_t}\right)+\tilde{H}_v\left(\frac{k_x}{k_t}\right)\right]+k_y\left[\tilde{H}_u\left(\frac{k_x}{k_t}\right)+\tilde{H}_v\left(-\frac{k_y}{k_t}\right)\right]\right)$$

or

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0}\left(\tilde{H}_v\left[-k_x^{\,2}-k_y^{\,2}\right]\frac{1}{k_t}\right)$$

or

$$\tilde{E}_z = \frac{-1}{\omega\varepsilon_0}\left(k_t\tilde{H}_v\right)$$

Sommerfeld Problem (cont.)

Hence

$$\tilde{E}_z\left(k_x,k_y,z\right)=\frac{-1}{\omega\varepsilon_0}\left(k_t\right)I^{TM}\left(z\right)$$

We use the Michalski normalized current function:

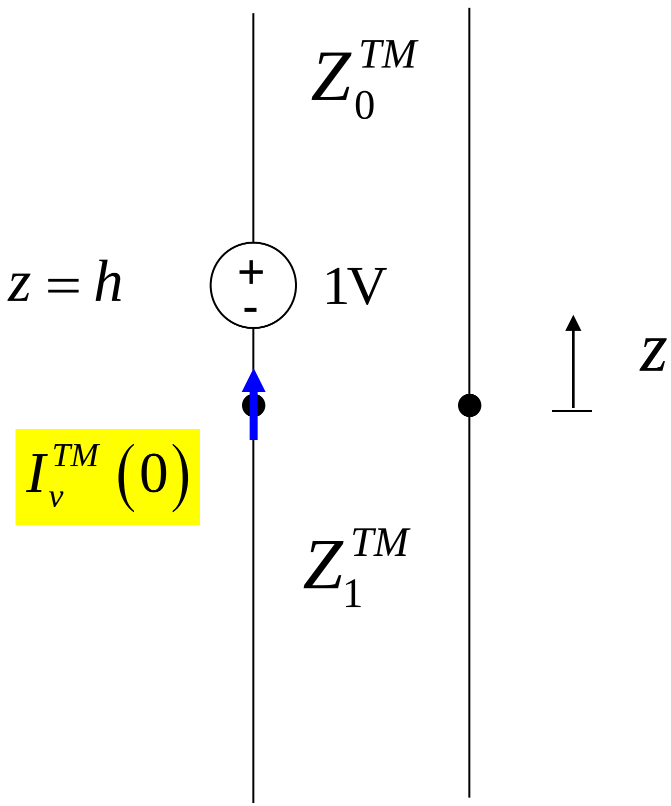
$$I^{TM}\left(z\right)=I_v^{TM}\left(z\right)V_s^{TM}=I_v^{TM}\left(z\right)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right]$$

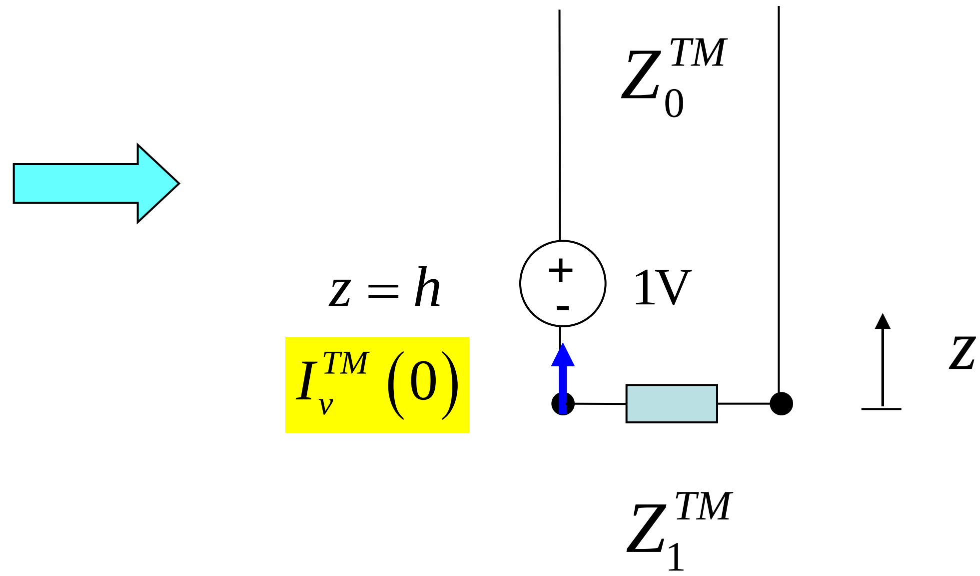
(The v subscript indicates a 1V series source.)

We need to calculate the Michalski normalized current function at $z = 0$ (since we want the field on the surface of the earth).

Sommerfeld Problem (cont.)

Calculation of the Michalski normalized current function

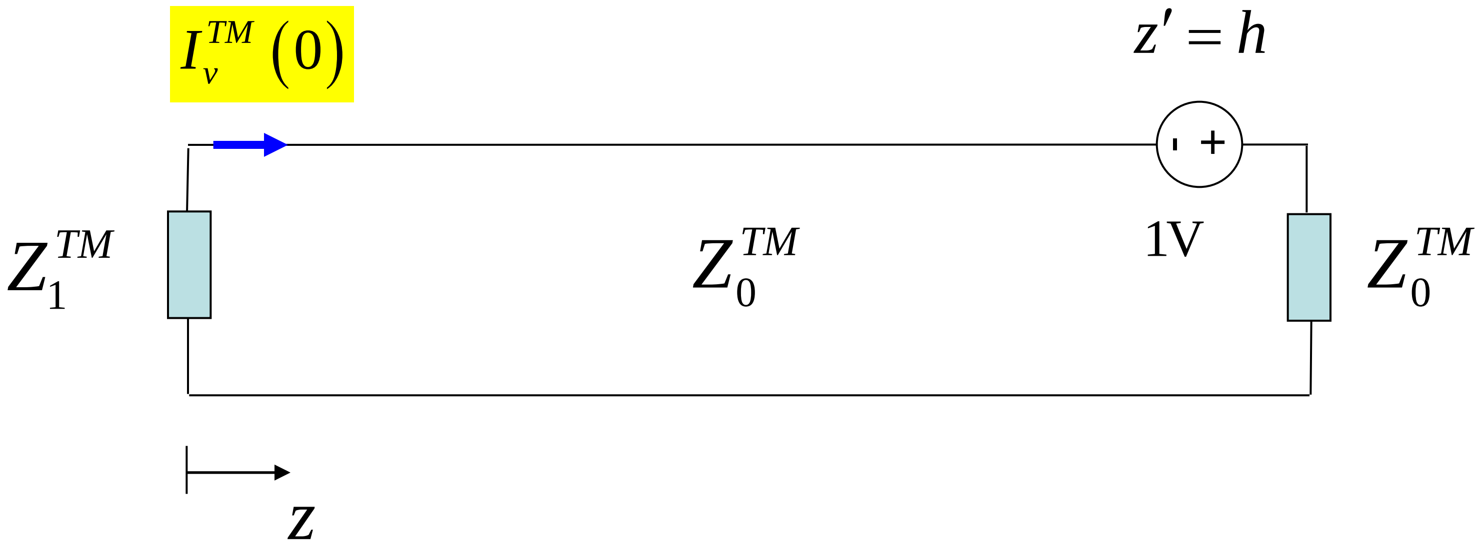




$$Z_0^{TM} = \frac{k_{z0}}{\omega \varepsilon_0} \qquad k_{z0} = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

$$Z_1^{TM} = \frac{k_{z1}}{\omega \varepsilon_1} \qquad k_{z1} = \sqrt{k_1^2 - k_x^2 - k_y^2}$$

Sommerfeld Problem (cont.)



This figure shows how to calculate the Michalski normalized current function: it will be calculated later.

Sommerfeld Problem (cont.)

Return to the calculation of the field:

$$\tilde{E}_z\left(k_x,k_y,0\right)=\frac{-1}{\omega\varepsilon_0}\left(k_t\right)I_v^{TM}\left(0\right)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right]$$

Hence we have

$$E_z\left(x,y,0\right)=\frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-1}{\omega\varepsilon_0}\left(k_t\right)I_v^{TM}\left(0\right)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right]e^{-j\left(k_x x+k_y y\right)}\,dk_x dk_y$$

or

$$E_z\left(x,y,0\right)=-\frac{Il}{\left(2\pi\right)^2}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I_v^{TM}\left(0\right)\,e^{-j\left(k_x x+k_y y\right)}\,k_t^2\,dk_x dk_y$$

Sommerfeld Problem (cont.)

$$E_z(x,y,0) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_v^{TM}(0)\, e^{-j(k_x x + k_y y)}\, k_t^2\, dk_x dk_y$$

Note: $I_v^{TM}(0)$ is only a function of k_t

Change to polar coordinates:

$$x = \rho\cos\phi \qquad k_x = k_t\cos\bar{\phi}$$

$$y = \rho\sin\phi \qquad k_y = k_t\sin\bar{\phi}$$

$$dk_x\, dk_y \to k_t\, dk_t\, d\bar{\phi}$$

Sommerfeld Problem (cont.)

$$E_z(x,y,0) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_v^{TM}(0)\, e^{-j\left(k_x x + k_y y\right)}\, k_t^2\, dk_x dk_y$$

Switch to polar coordinates

$$dk_x\, dk_y \rightarrow k_t\, dk_t\, d\bar{\phi}$$

$$E_z(x,y,0) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{0}^{\infty} I_v^{TM}(0)\, k_t^3 \int_{0}^{2\pi} e^{-j\left(k_t \cos\bar{\phi}\,\rho\cos\phi + k_t \sin\bar{\phi}\,\rho\sin\phi\right)}\, d\bar{\phi}\, dk_t$$

$$E_z(x,y,0) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{0}^{\infty} I_v^{TM}(0)\, k_t^3 \int_{0}^{2\pi} e^{-j\left(k_t\rho\right)\cos\left(\bar{\phi}-\phi\right)}\, d\bar{\phi}\, dk_t$$

Sommerfeld Problem (cont.)

$$E_z(x,y,0) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^{\infty} I_v^{TM}(0)\, k_t^3 \int_0^{2\pi} e^{-j(k_t\rho)\cos(\bar{\phi}-\phi)}\, d\bar{\phi}$$

Use $\alpha = \bar{\phi} - \phi$

$$\int_0^{2\pi} e^{-j(k_t\rho)\cos(\bar{\phi}-\phi)}\, d\bar{\phi} = \int_{-\phi}^{2\pi-\phi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha = \int_0^{2\pi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha$$

We see from this result that the vertical field of the vertical electric dipole should not vary with angle ϕ.

Integral identity: $I = \int_0^{2\pi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha = 2\pi J_0(k_t\rho)$

Sommerfeld Problem (cont.)

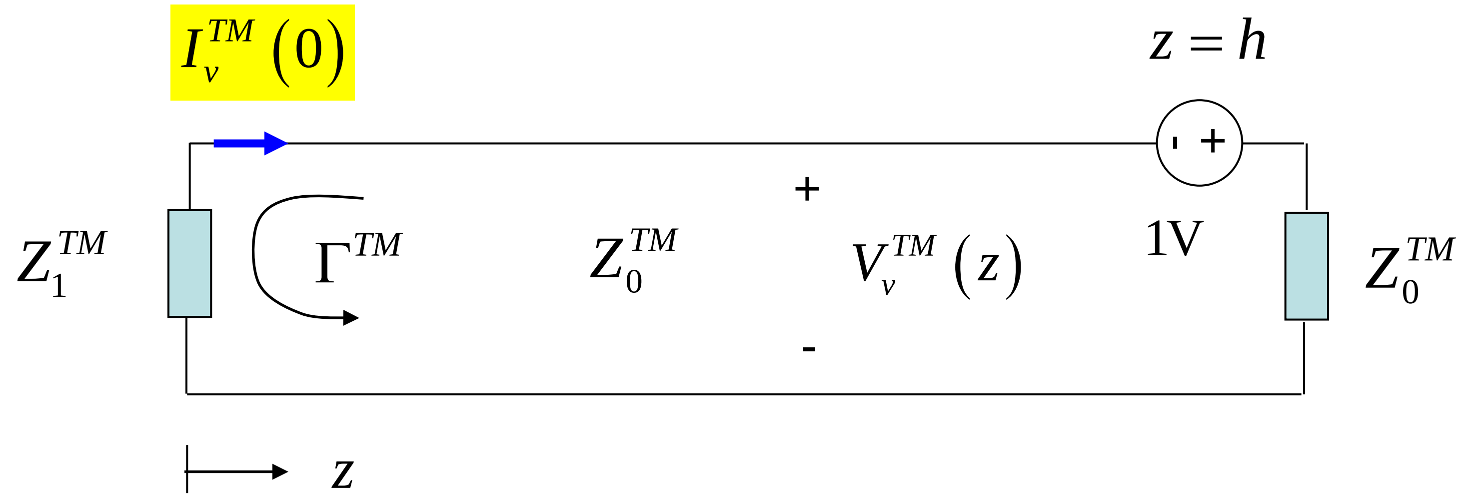
Hence we have

$$E_z\left(\rho,0\right)=-\frac{Il}{\left(2\pi\right)}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_0^\infty J_0\left(k_t\rho\right)\,I_v^{TM}\left(0\right)\,k_t^3\,dk_t$$

This is the “Sommerfeld form” of the field.

Sommerfeld Problem (cont.)

We now return to the calculation of the Michalski normalized current function.



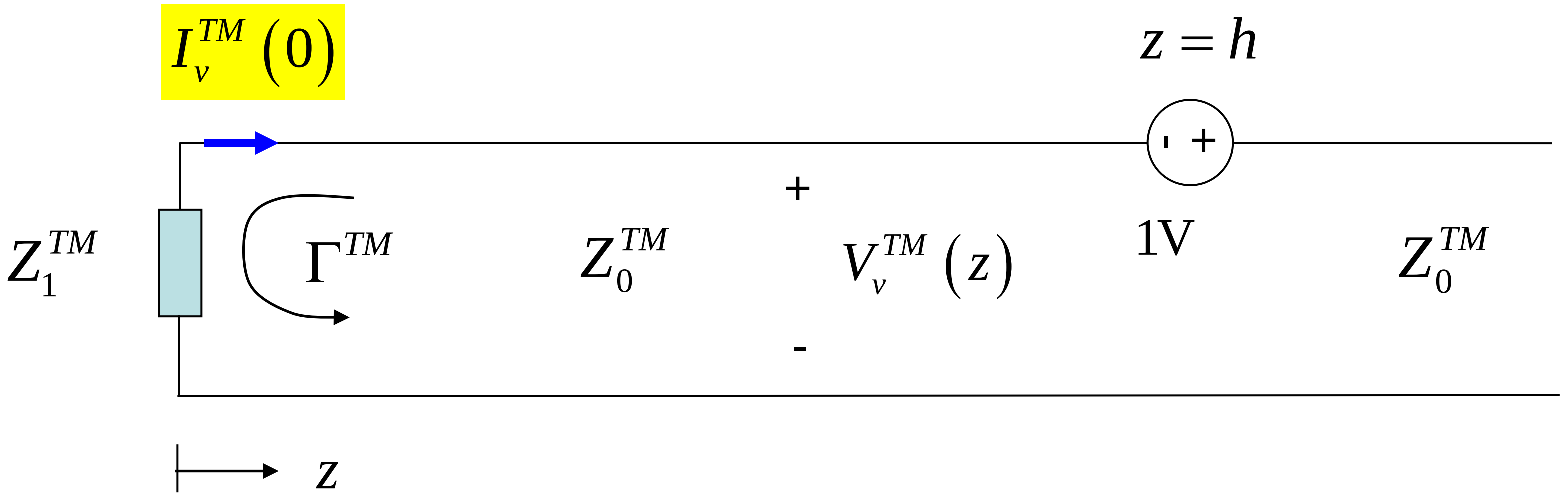
$0 < z < h$

$$V_v^{TM}(z) = Ae^{-jk_{z0}z} + Be^{+jk_{z0}z}$$
$$= B\left(e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

$$\Gamma^{TM} = \frac{A}{B} = \frac{Z_1^{TM} - Z_0^{TM}}{Z_1^{TM} + Z_0^{TM}}$$

$$I_v^{TM}(z) = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

Sommerfeld Problem (cont.)



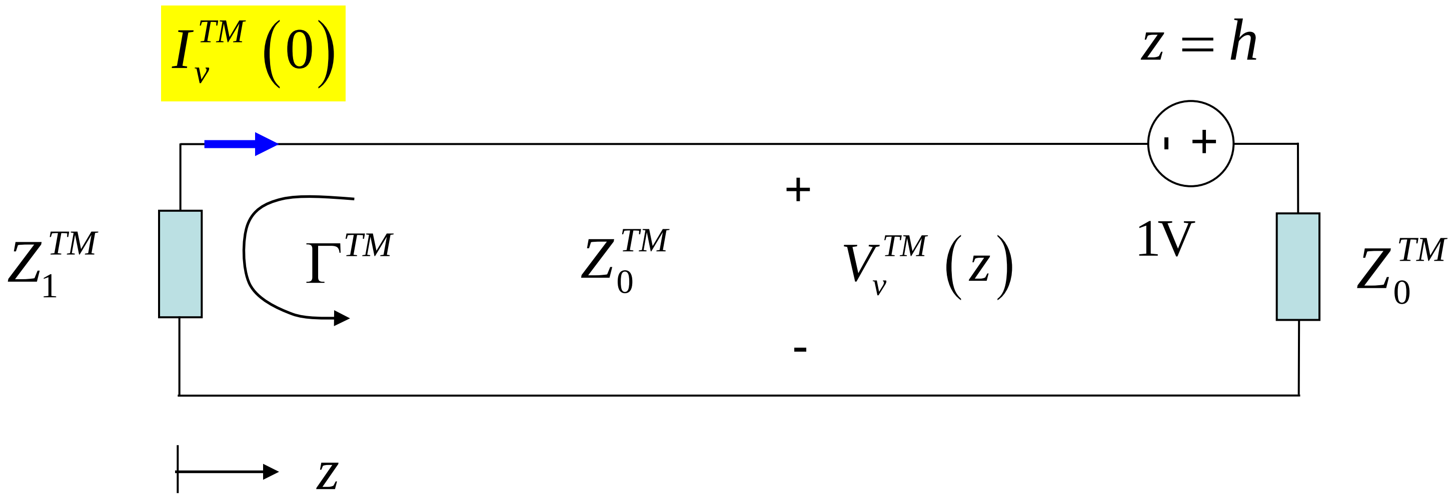
$$V_v^{TM}(z) = Ce^{-jk_{z0}z}$$

$z > h$

$$I_v^{TM}(z) = \frac{C}{Z_0^{TM}} e^{-jk_{z0}z}$$

Here we visualize the transmission line as infinite beyond the voltage source.

Sommerfeld Problem (cont.)



Boundary conditions:

$$V_v^{TM}(h^+) - V_v^{TM}(h^-) = 1$$

$$I_v^{TM}(h^+) = I_v^{TM}(h^-)$$

Sommerfeld Problem (cont.)

Hence we have

$$V_v^{TM}\left(h^+\right)-V_v^{TM}\left(h^-\right)=1$$

$$\Rightarrow \quad Ce^{-jk_{z0}h}-B\left(e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)=1$$

$$I_v^{TM}\left(h^+\right)=I_v^{TM}\left(h^-\right)$$

$$\Rightarrow \quad \frac{C}{Z_0^{TM}}e^{-jk_{z0}h}=\frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)$$

Sommerfeld Problem (cont.)

Substitute the first of these into the second one:

$$Ce^{-jk_{z0}h} = 1 + B\left(e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

$$\frac{C}{Z_0^{TM}} e^{-jk_{z0}h} = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

This gives us

$$\frac{1}{Z_0^{TM}}\left[1 + B\left(e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)\right] = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

Sommerfeld Problem (cont.)

$$\frac{1}{Z_0^{TM}}\left[1+B\left(e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)\right]=\frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)$$

⇓

$$\left[1+B\left(e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)\right]=B\left(-e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)$$

cancels

⇓

$$B\left(2e^{+jk_{z0}h}\right)=-1$$

Hence we have

$$B=-\frac{1}{2}e^{-jk_{z0}h}$$

Sommerfeld Problem (cont.)

For the current we then have

$$I_v^{TM}(z) = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

with $B = -\frac{1}{2} e^{-jk_{z0}h}$

Hence

$$I_v^{TM}(z) = -\frac{1}{2Z_0^{TM}} e^{-jk_{z0}h}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

For $z = 0$:

$$I_v^{TM}(0) = \frac{1}{2Z_0^{TM}}\left(1 - \Gamma^{TM}\right) e^{-jk_{z0}h}$$

Sommerfeld Problem (cont.)

We thus have

$$E_z(\rho,0) = -\frac{Il}{(2\pi)}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^\infty J_0(k_t\rho)\, I_v^{TM}(0)\, k_t^3\, dk_t$$

with $I_v^{TM}(0,h) = \frac{1}{2Z_0^{TM}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}$

Hence

$$E_z(\rho,0) = -\frac{Il}{(2\pi)}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^\infty J_0(k_t\rho)\left[\frac{1}{2Z_0^{TM}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\, dk_t$$

or

$$E_z(\rho,0) = -\frac{Il}{(2\pi)}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^\infty J_0(k_t\rho)\left[\frac{1}{2\left(\frac{k_{z0}}{\omega\varepsilon_0}\right)}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\, dk_t$$

Sommerfeld Problem (cont.)

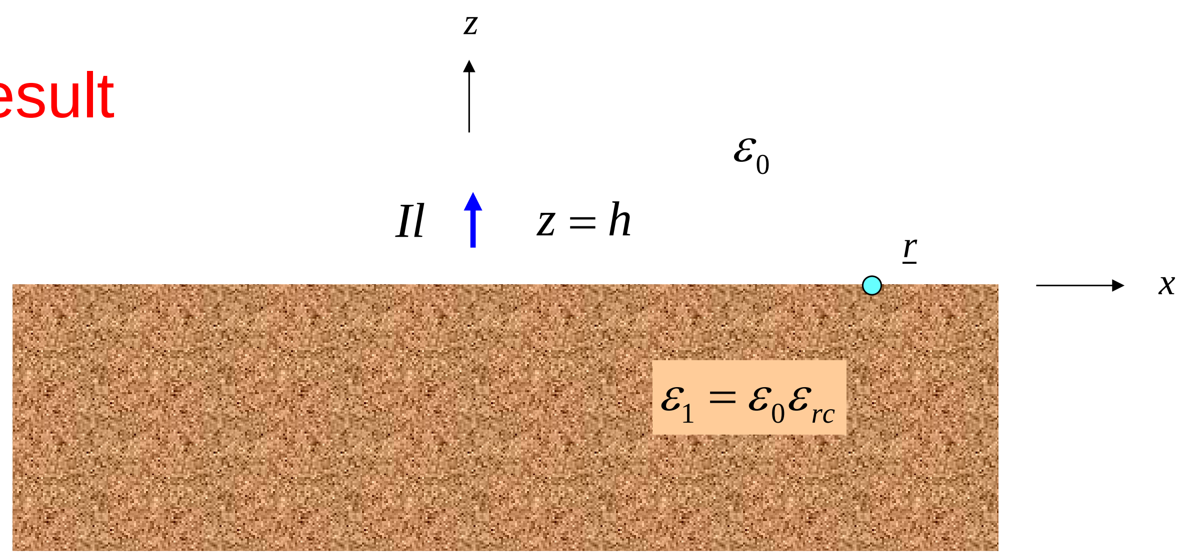
$$E_z(\rho,0) = -\frac{Il}{(2\pi)}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^{\infty} J_0(k_t\rho)\left[\frac{1}{2\left(\dfrac{k_{z0}}{\omega\varepsilon_0}\right)}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

Simplify

$$E_z(\rho,0) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\int_0^{\infty} J_0(k_t\rho)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

Sommerfeld Problem (cont.)

Final Result



$$E_z\left(\rho,0^+\right) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\int_0^\infty J_0\left(k_t\rho\right)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

$$\Gamma^{TM} = \Gamma^{TM}\left(k_t\right) = \frac{Z_1^{TM} - Z_0^{TM}}{Z_1^{TM} + Z_0^{TM}}$$

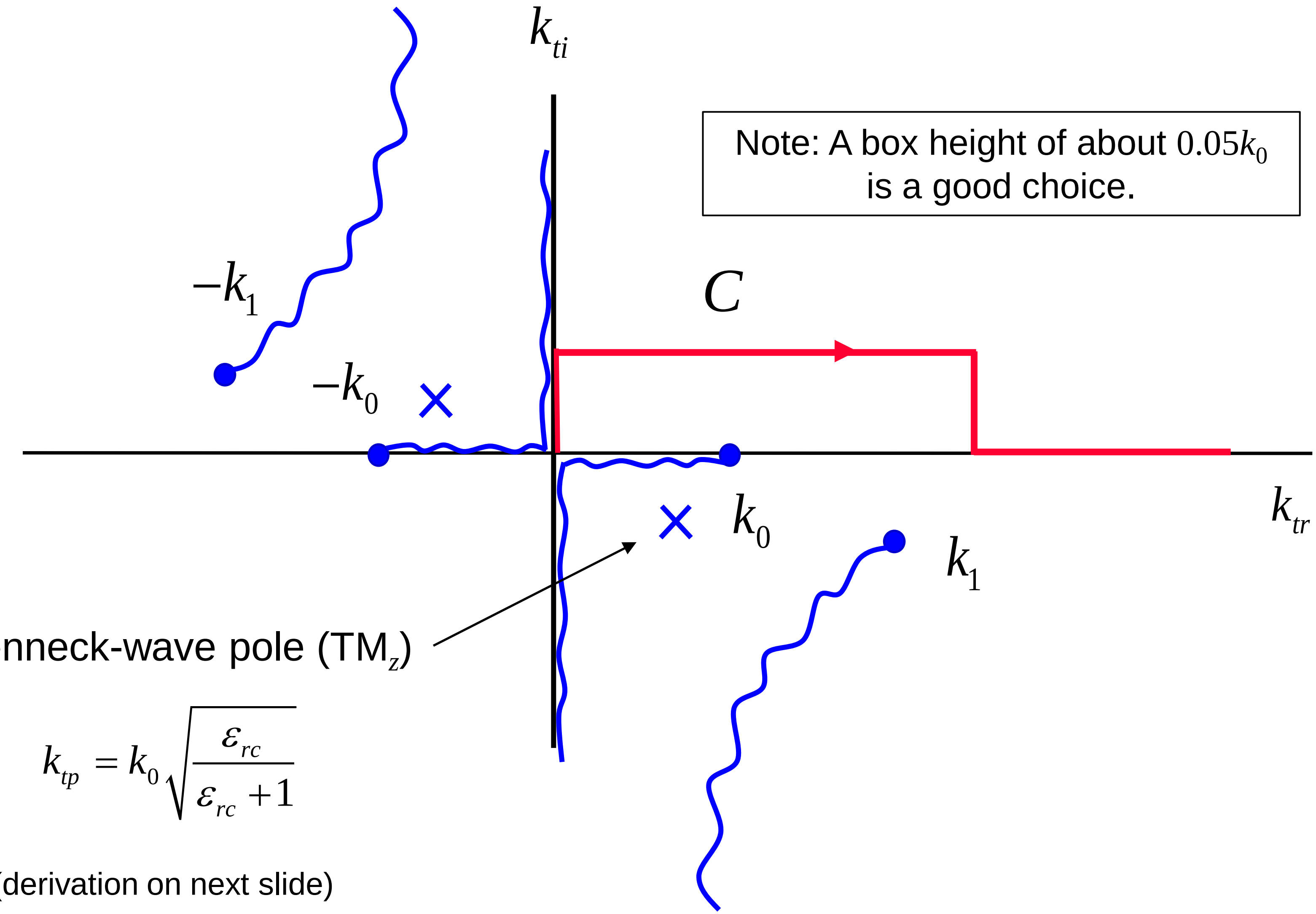
$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0}$$

$$Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1}$$

$$k_{z0} = \sqrt{k_0^2 - k_t^2}$$

$$k_{z1} = \sqrt{k_1^2 - k_t^2}$$

Sommerfeld Problem (cont.)



$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc} + 1}}$$

(derivation on next slide)

ε_{rc} = complex relative permittivity of the earth (accounting for the conductivity.)

Sommerfeld Problem (cont.)

Zenneck-wave pole (TM_z)

The TRE is:

$$\overleftarrow{Z}^{TM} = -\overrightarrow{Z}^{TM}$$

$$\Downarrow$$

$$Z_0^{TM} = -Z_1^{TM}$$

$$\Downarrow$$

$$\frac{k_{z0}}{\omega\varepsilon_0} = -\frac{k_{z1}}{\omega\varepsilon_1}$$

$$\varepsilon_{rc} k_{z0} = -k_{z1}$$

$$\Downarrow$$

$$\varepsilon_{rc}^2 \left(k_0^2 - k_{tp}^2\right) = \left(k_1^2 - k_{tp}^2\right)$$

$$\Downarrow$$

$$k_{tp}^2 \left(\varepsilon_{rc}^2 - 1\right) = \varepsilon_{rc}^2 k_0^2 - k_1^2$$

$$k_{tp}^2 \left(\varepsilon_{rc}^2 - 1\right) = \varepsilon_{rc}^2 k_0^2 - k_0^2 \varepsilon_{rc}$$

$$\Downarrow$$

$$k_{tp}^2 = \frac{\varepsilon_{rc}^2 k_0^2 - k_0^2 \varepsilon_{rc}}{\varepsilon_{rc}^2 - 1}$$

$$\Downarrow$$

$$k_{tp}^2 = k_0^2 \varepsilon_{rc} \left(\frac{\varepsilon_{rc} - 1}{\varepsilon_{rc}^2 - 1}\right)$$

$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc} + 1}}$$

Note: Both vertical wavenumbers (k_{z0} and k_{z1}) are proper.
(Power flows downward in both regions.)

Sommerfeld Problem (cont.)

Alternative form: The path is extended to the entire real axis.

$$E_z\left(\rho,0^+\right)=\int_0^\infty J_0\left(k_t\rho\right)\mathrm{Odd}\left(k_t\right)dk_t$$

We use

$$J_0\left(x\right)=\frac{1}{2}\left(H_0^{(1)}\left(x\right)+H_0^{(2)}\left(x\right)\right)$$

$$H_0^{(1)}\left(x\right)=-H_0^{(2)}\left(-x\right)$$

Transform the $H_0^{(1)}$ term:

$$\int_0^\infty \mathrm{Odd}\left(k_t\right)H_0^{(1)}\left(k_t\rho\right)dk_t=-\int_0^\infty \mathrm{Odd}\left(k_t\right)H_0^{(2)}\left(-k_t\rho\right)dk_t$$

$$=\int_0^{-\infty} \mathrm{Odd}\left(-k_t'\right)H_0^{(2)}\left(k_t'\rho\right)dk_t' \qquad \text{Use } k_t'=-k_t$$

$$=\int_{-\infty}^{0} \mathrm{Odd}\left(k_t'\right)H_0^{(2)}\left(k_t'\rho\right)dk_t'$$

$$=\int_{-\infty}^{0} \mathrm{Odd}\left(k_t\right)H_0^{(2)}\left(k_t\rho\right)dk_t$$

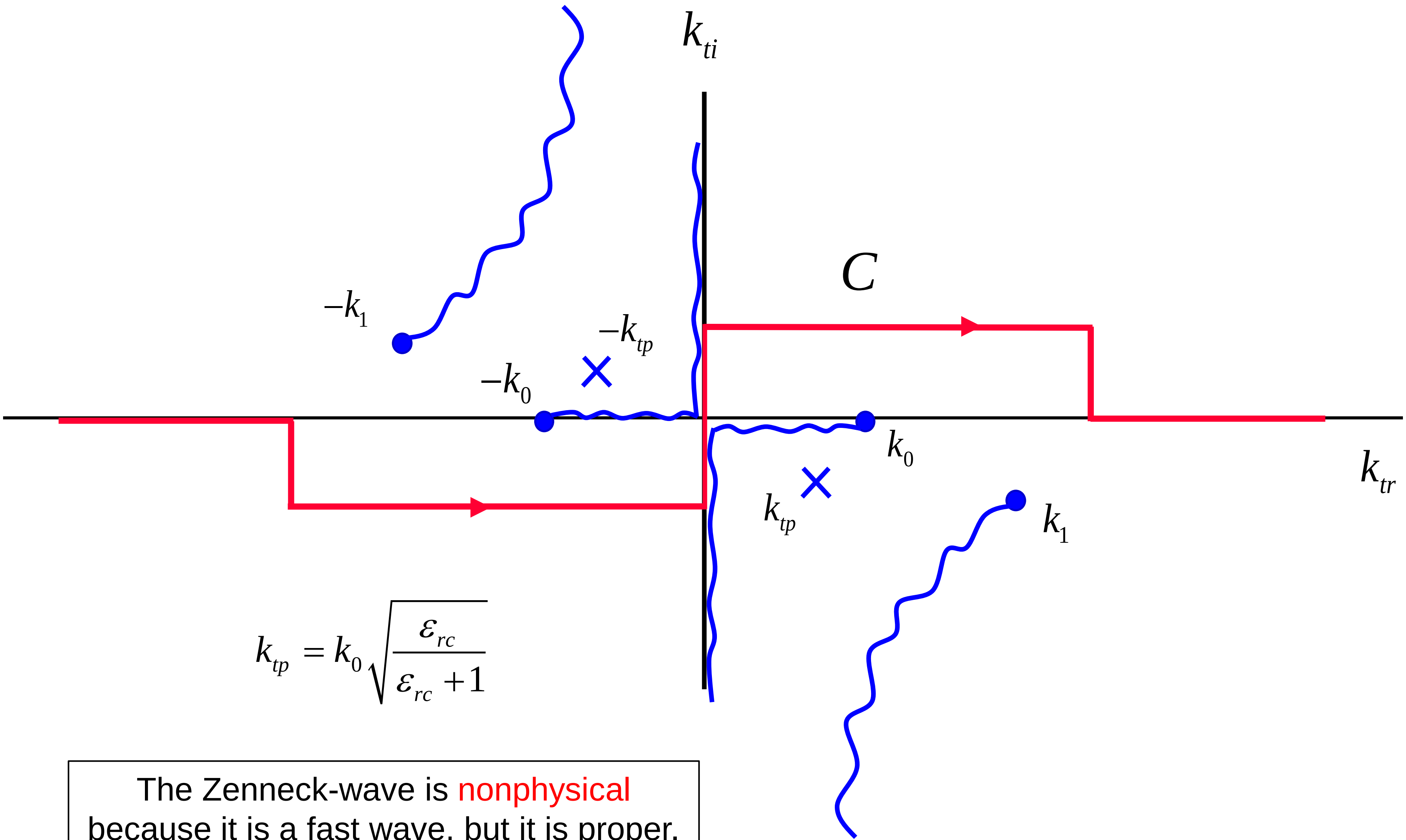
Sommerfeld Problem (cont.)

Hence we have

$$E_z(\rho,0) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\frac{1}{2}\int_{-\infty}^{\infty} H_0^{(2)}(k_t\rho)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

This is a convenient form for deforming the path.

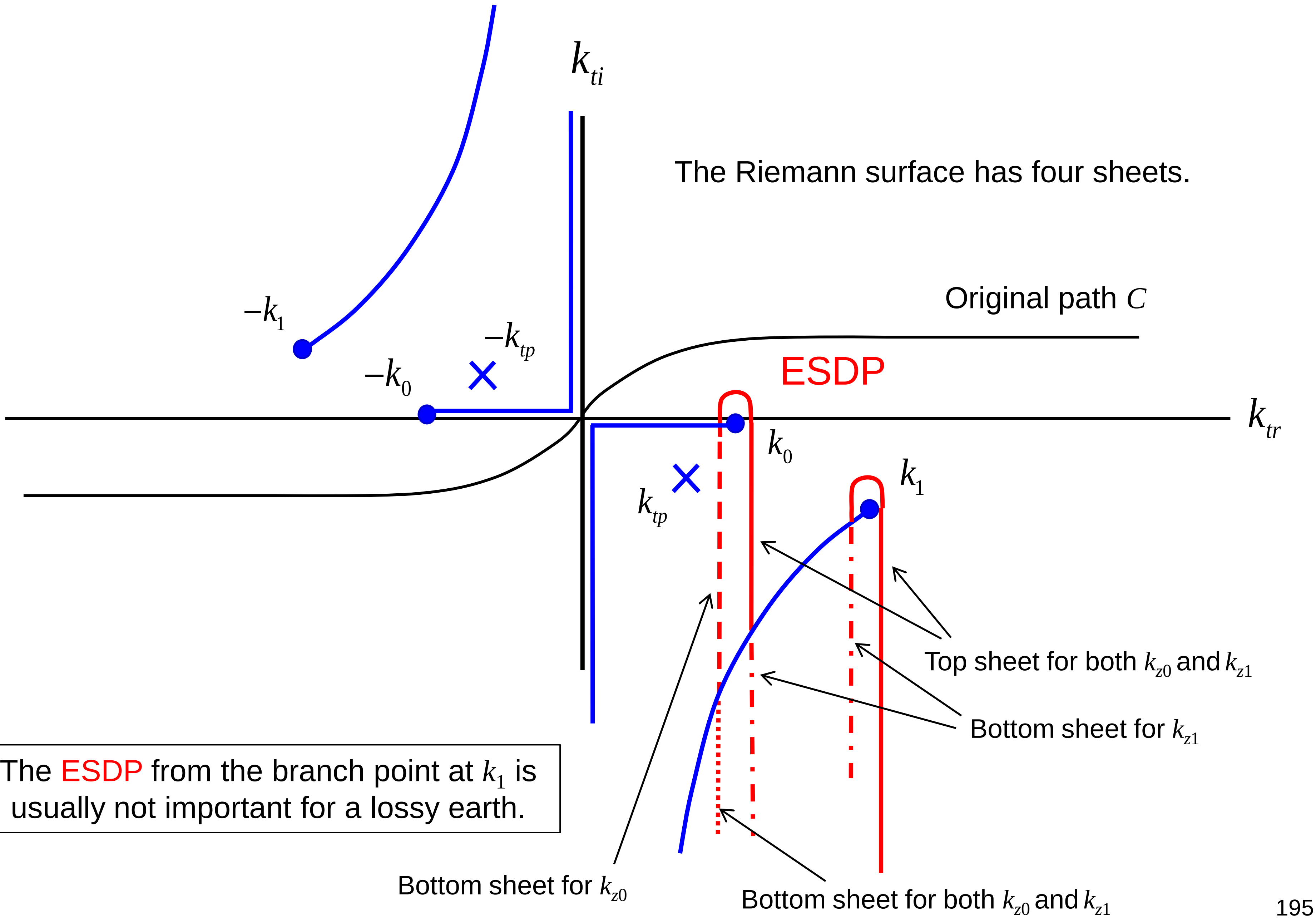
Sommerfeld Problem (cont.)



$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc} + 1}}$$

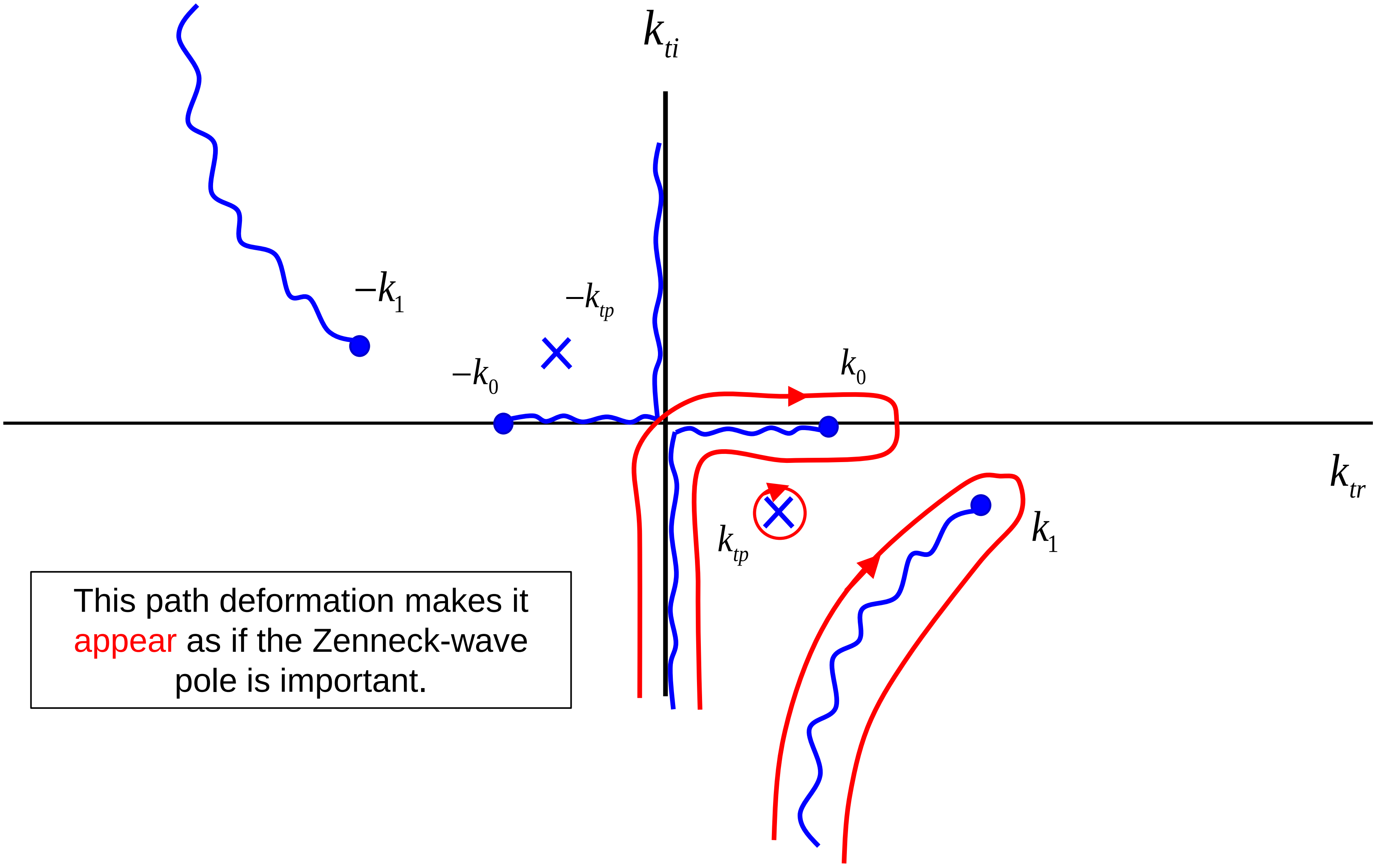
The Zenneck-wave is nonphysical because it is a fast wave, but it is proper. It is not captured when deforming to the ESDP (vertical path from k_0).

Sommerfeld Problem (cont.)



The ESDP from the branch point at k_1 is usually not important for a lossy earth.

Sommerfeld Problem (cont.)



This path deformation makes it appear as if the Zenneck-wave pole is important.

Sommerfeld Problem (cont.)

Throughout much of the 20th century, a controversy raged about the "reality of the Zenneck wave."

- Arnold Sommerfeld predicted a surface-wave like field coming from the residue of the Zenneck-wave pole (1909).
- People took measurements and could not find such a wave.
- Hermann Weyl solved the problem in a different way and did not get the Zenneck wave (1919).
- Some people (Norton, Niessen) blamed it on a sign error that Sommerfeld had made, though Sommerfeld never admitted to a sign error.
- Eventually it was realized that there was no sign error (Collin, 2004).
- The limitation in Sommerfeld's original asymptotic analysis (which shows a Zenneck-wave term) is that the pole must be well separated from the branch point – the asymptotic expansion that he used neglects the effects of the pole on the branch point (the saddle point in the steepest-descent plane).
- When the asymptotic evaluation of the branch-cut integral around k_0 includes the effects of the pole, it turns out that there is no Zenneck-wave term in the total solution (branch-cut integrals + pole-residue term).
- The easiest way to explain the fact that the Zenneck wave is not important far away is that the pole is not captured in deforming to the ESDP paths.

Sommerfeld Problem (cont.)

R. E. Collin, “Hertzian Dipole Radiating Over a Lossy Earth or Sea: Some Early and Late 20th-Century Controversies,” AP-S Magazine, pp. 64-79, April 2004.

Hertzian Dipole Radiating Over a Lossy Earth or Sea: Some Early and Late 20th-Century Controversies

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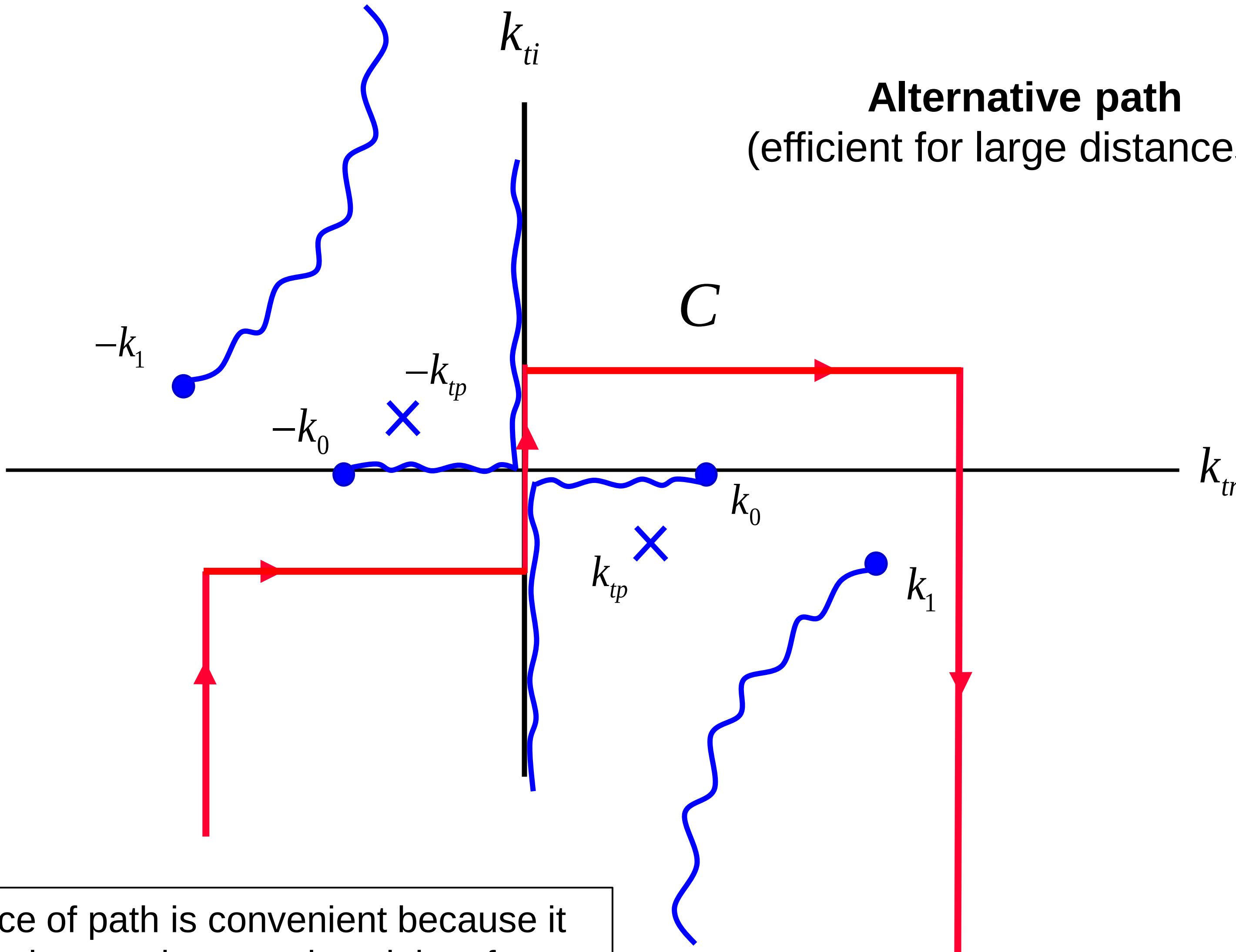
Abstract

This paper presents a contemporary solution to the problem of radiation from a vertical Hertzian dipole over a lossy Earth. Sommerfeld's 1909 solution to the problem is re-examined. It is demonstrated that a change in sign in the square root of the numerical distance is mathematically not allowed. Thus, the sign error that has been claimed in the technical literature for more than 65 years is a myth. Recent work by King and Sandler is also examined. It is found that due to an incorrect asymptotic expansion of the complementary error function for the problem of a lossy earth or sea covered with a thin dielectric layer, a trapped surface wave was missed in their solution.

Keywords: Dipole antennas; electromagnetic radiation; Zenneck surface wave; asymptotic solution; electromagnetic surface waves

Sommerfeld Problem (cont.)



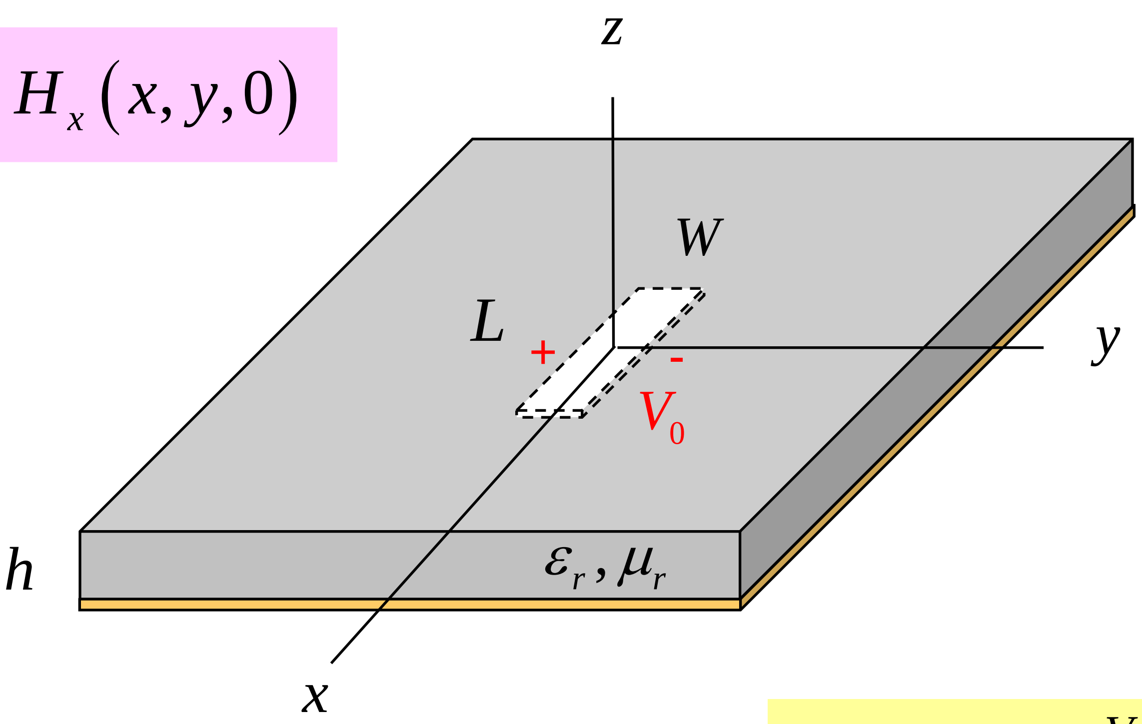
This choice of path is convenient because it stays on the top sheet, and yet it has fast convergence as the distance ρ increases, due to the Hankel function.

Outline

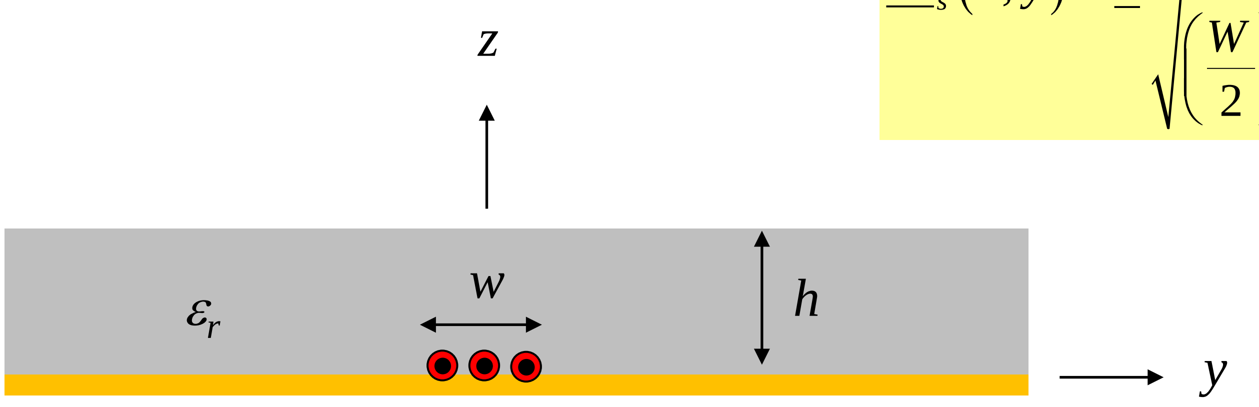
- ❖ Physical derivation of method for planar electric surface currents.
- ❖ Examples involving planar surface currents:
  - ❑ Microstrip line
  - ❑ Microstrip patch current
- ❖ General derivation (Fourier transforming Maxwell's equations) that allows for all types of sources to be included in one general derivation.
- ❖ Examples:
  - ❑ Vertical dipole over the earth (Sommerfeld problem)
  - ❑ Slot antenna covered with radome layer (magnetic current)

Slot Antenna with Radome

Find $H_x(x, y, 0)$



$$\underline{M}_s(x, y) = \underline{\hat{x}} \frac{V_0 / \pi}{\sqrt{\left(\frac{W}{2}\right)^2 - y^2}} \cos\left(\frac{\pi x}{L}\right)$$



Slot Antenna with Radome (cont.)

$$\tilde{H}_x(k_x,k_y) = \tilde{H}_u(\underline{\hat{u}}\cdot\underline{\hat{x}}) + \tilde{H}_v(\underline{\hat{v}}\cdot\underline{\hat{x}})$$
$$= \tilde{H}_u(\cos\overline{\phi}) + \tilde{H}_v(-\sin\overline{\phi})$$
$$= I^{TE}\left(\frac{k_x}{k_t}\right) + I^{TM}\left(-\frac{k_y}{k_t}\right)$$
$$= I_v^{TE}\left(-\tilde{M}_{su}\right)\left(\frac{k_x}{k_t}\right) + I_v^{TM}\left(-\tilde{M}_{sv}\right)\left(-\frac{k_y}{k_t}\right)$$
$$= I_v^{TE}\left(-\tilde{M}_{sx}\right)(\underline{\hat{u}}\cdot\underline{\hat{x}})\left(\frac{k_x}{k_t}\right) + I_v^{TM}\left(-\tilde{M}_{sx}\right)(\underline{\hat{v}}\cdot\underline{\hat{x}})\left(-\frac{k_y}{k_t}\right)$$
$$= I_v^{TE}\left(-\tilde{M}_{sx}\right)\left(\frac{k_x}{k_t}\right)\left(\frac{k_x}{k_t}\right) + I_v^{TM}\left(-\tilde{M}_{sx}\right)\left(-\frac{k_y}{k_t}\right)\left(-\frac{k_y}{k_t}\right)$$
$$= I_v^{TE}\left(-\tilde{M}_{sx}\right)\left(\frac{k_x}{k_t}\right)^2 + I_v^{TM}\left(-\tilde{M}_{sx}\right)\left(\frac{k_y}{k_t}\right)^2$$

$$V^{TM} = \tilde{E}_u$$
$$I^{TM} = \tilde{H}_v$$
$$V^{TE} = -\tilde{E}_v$$
$$I^{TE} = \tilde{H}_u$$

Horizontal

$$I_s^{TM} = -\tilde{J}_{su}^{\,i}$$
$$V_s^{TM} = -\tilde{M}_{sv}^{\,i}$$

$$I_s^{TE} = +\tilde{J}_{sv}^{\,i}$$
$$V_s^{TE} = -\tilde{M}_{su}^{\,i}$$

Slot Antenna with Radome (cont.)

Hence

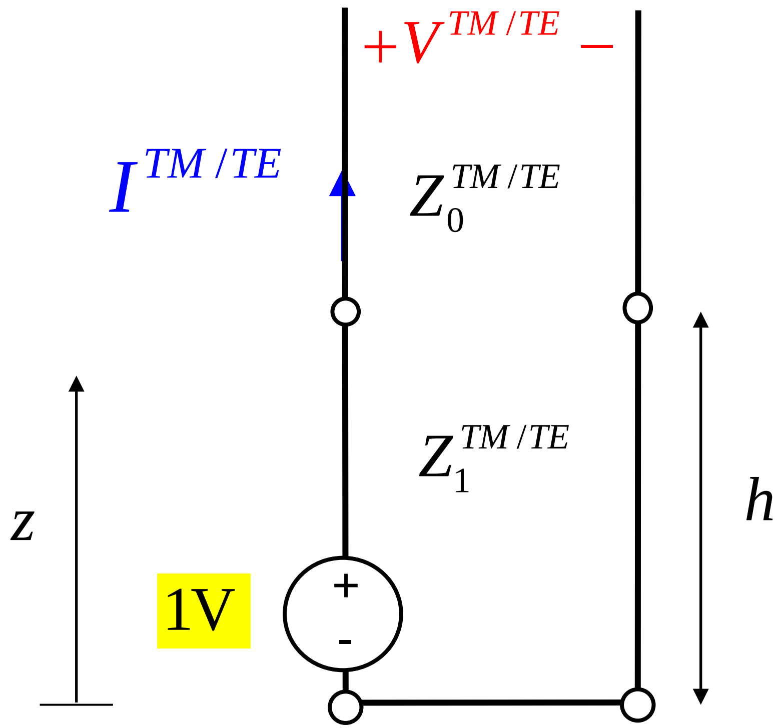
$$\tilde{H}_x\left(k_x,k_y\right) = -\tilde{M}_{sx}\left[I_v^{TE}\left(\frac{k_x}{k_t}\right)^2 + I_v^{TM}\left(\frac{k_y}{k_t}\right)^2 \right]$$

For the slot current we have

$$\underline{\tilde{M}}_s\left(k_x,k_y\right) = \underline{\hat{x}}V_0 J_0\left(\frac{k_y W}{2}\right)\left[\left(\frac{\pi L}{2}\right)\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2}\right]$$

We next calculate the Michalski functions.

Slot Antenna with Radome (cont.)



$$I_v^{TM}(0) = \frac{1}{Z_{in}^{TM}}$$

$$I_v^{TE}(0) = \frac{1}{Z_{in}^{TE}}$$

$$Z_0^{TM} = \frac{k_{z0}}{\omega \varepsilon_0}$$

$$Z_1^{TM} = \frac{k_{z0}}{\omega \varepsilon_1}$$

$$Z_0^{TE} = \frac{\omega \mu_0}{k_{z0}}$$

$$Z_1^{TE} = \frac{\omega \mu_0}{k_{z1}}$$

$$Z_{in}^{TM/TE} = Z_1^{TM/TE}\left[\frac{Z_0^{TM/TE} + jZ_1^{TM/TE}\tan(k_{z1}h)}{Z_1^{TM/TE} + jZ_0^{TM/TE}\tan(k_{z1}h)}\right]$$

Slot Antenna with Radome (cont.)

Input impedance of slot:

$$P_c = \frac{1}{2}\frac{\left|V_0\right|^2}{\left(Z_{in}^{slot}\right)^*} \quad \Rightarrow \quad Z_{in}^{slot} = \frac{\left|V_0\right|^2}{2P_c^*}$$

$$P_c = -\frac{1}{2}\int\limits_{slot} M_{sx} H_x^* dS$$

Therefore

$$P_c^* = -\frac{1}{2}\int\limits_{slot} M_{sx}^* H_x dS$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M_{sx}^* H_x \, dx \, dy$$

$$= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{M}_{sx}^* \tilde{H}_x \, dk_x \, dk_y$$

(Parseval's theorem)

Slot Antenna with Radome (cont.)

Hence, we have

$$P_c^* = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tilde{M}_{sx}\right|^2 \left[I_v^{TE}(0)\left(\frac{k_x}{k_t}\right)^2 + I_v^{TM}(0)\left(\frac{k_y}{k_t}\right)^2 \right] dk_x\, dk_y$$

so that

$$Z_{in}^{slot} = \frac{\left|V_0\right|^2}{\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tilde{M}_{sx}\right|^2 \left[I_v^{TE}(0)\left(\frac{k_x}{k_t}\right)^2 + I_v^{TM}(0)\left(\frac{k_y}{k_t}\right)^2 \right] dk_x\, dk_y}$$

Slot Antenna with Radome (cont.)

Integrating in polar coordinates, and using symmetry to reduce the integration to the first quadrant, we have

$$Z_{in}^{slot} = \frac{\left|V_0\right|^2}{4\left(\frac{1}{2\pi}\right)^2 \int_0^{\pi/2}\int_0^{\infty} \left|\tilde{M}_{sx}\right|^2 \left[I_v^{TE}(0)\cos^2\bar{\phi} + I_v^{TM}(0)\sin^2\bar{\phi} \right] k_t \, dk_t \, d\bar{\phi}}$$

with

$$\underline{\tilde{M}}_s\left(k_x, k_y\right) = \underline{\hat{x}} V_0 J_0\left(\frac{k_y W}{2}\right) \left[\left(\frac{\pi L}{2}\right) \frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2} \right]$$

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