ECE 6345

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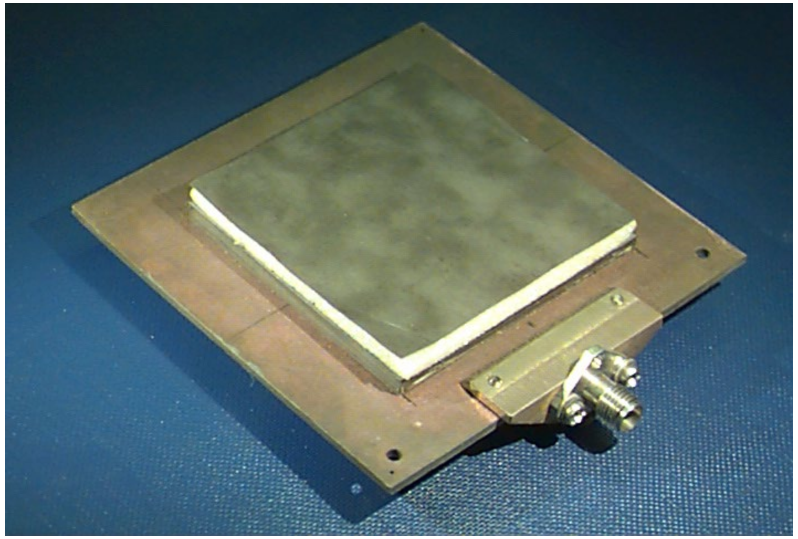
ECE Dept.



Notes 10

Overview

In this set of notes we calculate the far field of a rectangular patch using the magnetic current model.

The analysis assumes an infinite substrate, but for a truncated substrate we can use the same final result, setting the substrate permittivity to that of air (please see the discussion in Notes 7).

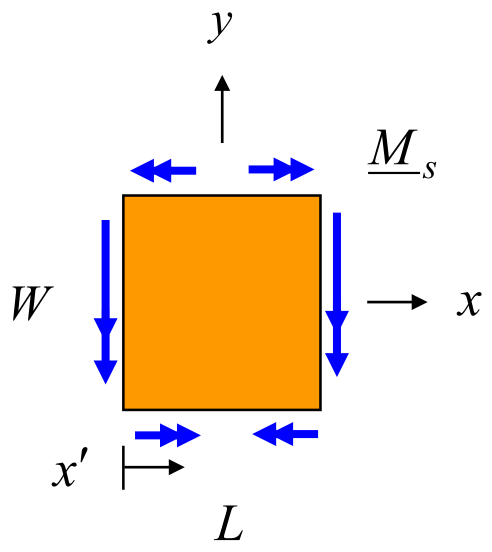
Magnetic Current Model

Assume:

$$E_z^{1,0} = -\sin\left(\frac{\pi x}{L}\right)$$

Measured from lower left corner:

$$E_z^{1,0} = \cos\left(\frac{\pi x'}{L}\right)$$



Only the radiating edges contribute to the E- and H-plane patterns, so we will ignore the non-radiating edges.

Radiating edges:

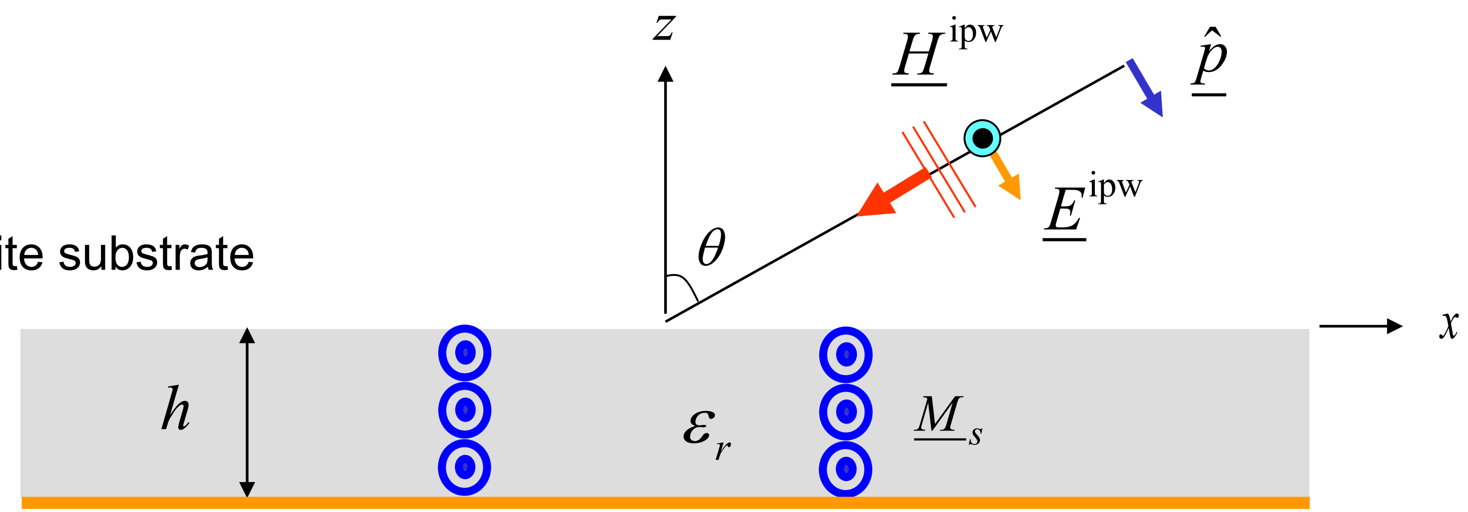
$$\underline{M}_s = -\underline{\hat{y}}$$

$\left(\underline{\hat{p}} = \underline{\hat{\theta}} \text{ or } \underline{\hat{\phi}}\right)$

$\left(\text{shown for } \underline{\hat{p}} = \underline{\hat{\theta}}\right)$

$$\underline{M}_s = \begin{cases} -\underline{\hat{y}} & x = L \\ -\underline{\hat{y}} & x = 0 \\ -\underline{\hat{x}}\cos\left(\frac{\pi x}{L}\right) & y = W \\ \underline{\hat{x}}\cos\left(\frac{\pi x}{L}\right) & y = 0 \end{cases}$$

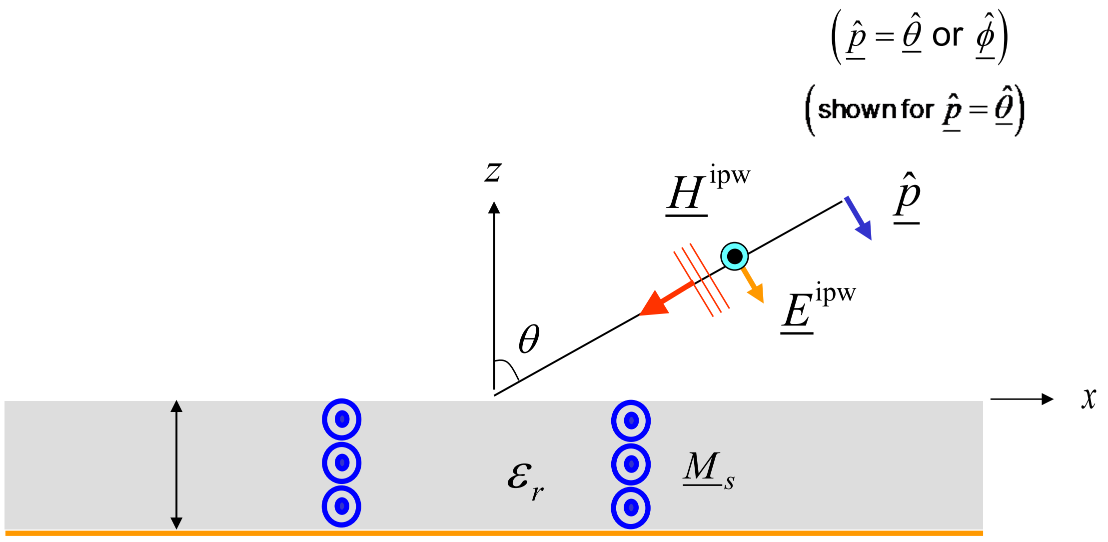
Infinite substrate



Magnetic Current Model (cont.)

From reciprocity:

$$E_p^{\text{FF}}(r,\theta,\phi) = \langle a,b \rangle$$
$$= \langle b,a \rangle$$
$$= -\int_S \left(\underline{H}^{\text{pw}} \cdot \underline{M}_s^a \right) dS$$
$$= -\int_S H_y^{\text{pw}} M_{sy}^a \, dS$$



a = radiating magnetic current
b = testing dipole ($Il = 1$)

$$S = S_L + S_R \quad \text{(left+right edges)}$$

$$M_{sy} = -1$$

Magnetic Current Model (cont.)

TEN modeling equation:

$$H_y^{\text{pw}}\left(x,y,z\right)=\psi_t\left(x,y\right)I\left(z\right)$$

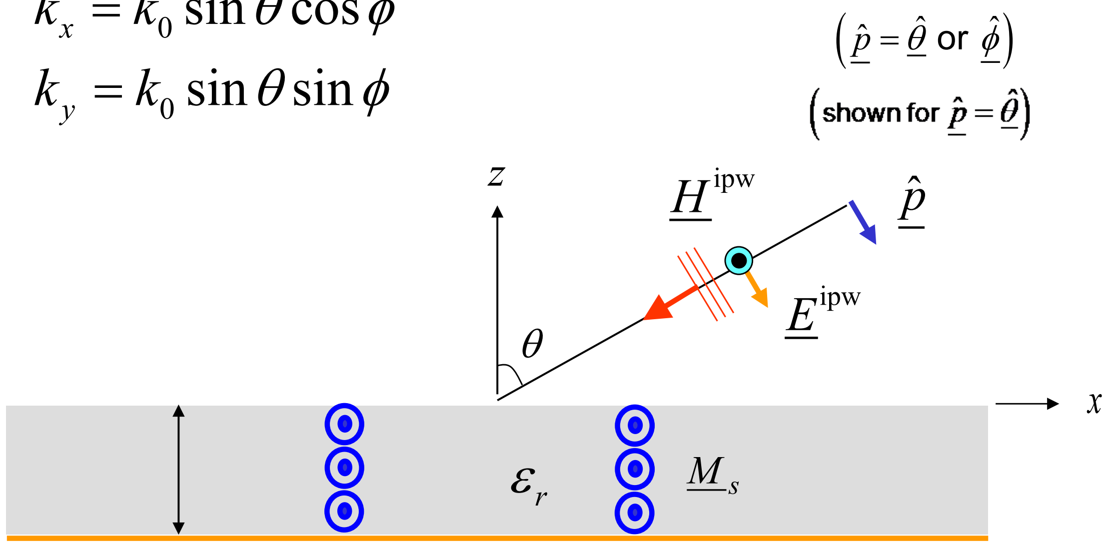
$$\psi_t\left(x,y\right)=e^{j\left(k_x x+k_y y\right)}$$

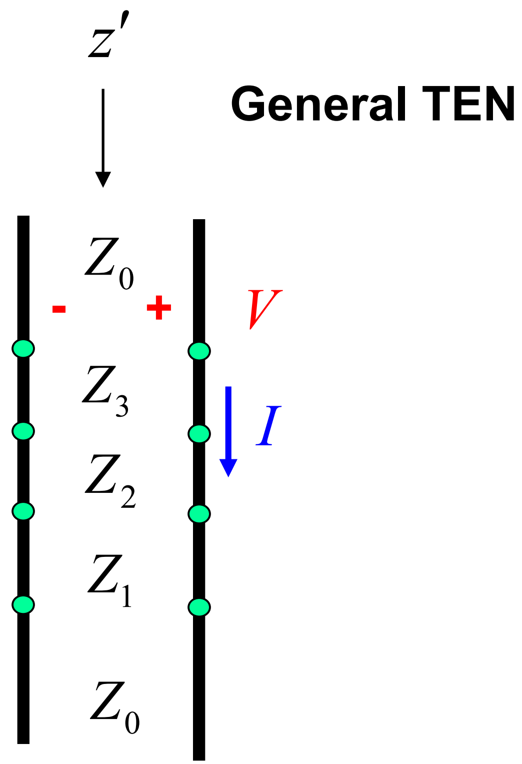
$$E_x\left(x,y,z\right)=\psi_t\left(x,y\right)\,V\left(z\right)$$

$$-H_y\left(x,y,z\right)=\psi_t\left(x,y\right)\,I\left(z\right)$$

$$k_x=k_0\sin\theta\cos\phi$$

$$k_y=k_0\sin\theta\sin\phi$$





Magnetic Current Model (cont.)

$$-H_y^{\text{pw}}(x,y,z) = \psi_t(x,y)\,I(z) = \psi_t(x,y)\,I(0)\left[\frac{\cos\left(k_{z1}(z+h)\right)}{\cos\left(k_{z1}h\right)}\right]$$

$$= -H_y^{\text{pw}}(x,y,0)\cos\left(k_{z1}(z+h)\right)\sec\left(k_{z1}h\right)$$

$$k_{z1} = k_0 N_1(\theta)$$

$$N_1(\theta) \equiv \sqrt{n_1^2 - \sin^2(\theta)}$$

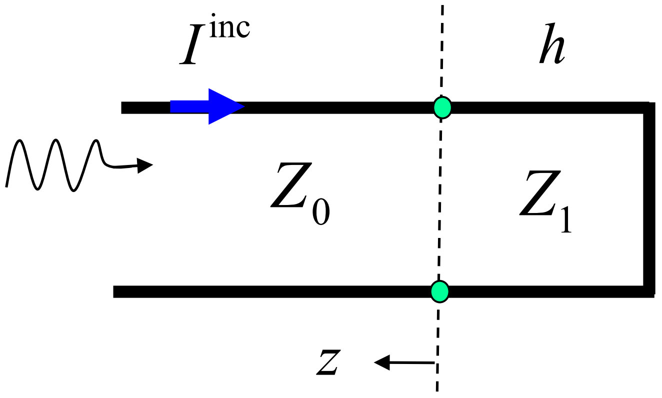
$$n_1 = \sqrt{\varepsilon_r \mu_r}$$

$$H_y^{\text{pw}}(x,y,z) = H_y^{\text{pw}}(x,y,0)\cos\left(k_{z1}(z+h)\right)\sec\left(k_{z1}h\right)$$

$$I(0) = I^{\text{inc}}(0)\left(1-\Gamma^{\text{TM/TE}}\right)$$

Also,

$$H_y^{\text{pw}}(x,y,0) = H_y^{\text{ipw}}(x,y,0)\left(1-\Gamma^{\text{TM/TE}}\right)$$



$$\text{TM}_z:\ \underline{\hat{p}} = \underline{\hat{\theta}}$$

$$\text{TE}_z:\ \underline{\hat{p}} = \underline{\hat{\phi}}$$

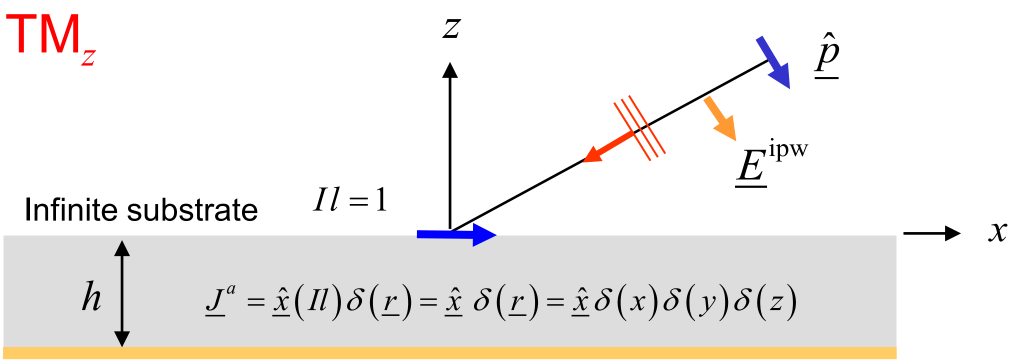
Magnetic Current Model (cont.)

$\underline{\hat{p}} = \underline{\hat{\theta}}$:

$$\underline{H}^{\text{ipw}} = -\underline{\hat{\phi}} \frac{E_\theta^{\text{ipw}}}{\eta_0} = -\underline{\hat{\phi}} \left(\frac{E_0}{\eta_0} \right) \psi(x, y, z) = -\underline{\hat{\phi}} \left(\frac{E_0}{\eta_0} \right) \psi_t(x, y)\, e^{jk_z z}$$

$$H_y^{\text{ipw}}(x, y, 0) = \left(\frac{E_0}{\eta_0} \right) (-\cos\phi) \psi_t(x, y)$$

$$\left(\underline{\hat{\phi}} \cdot \underline{\hat{y}} = \cos\phi \right)$$



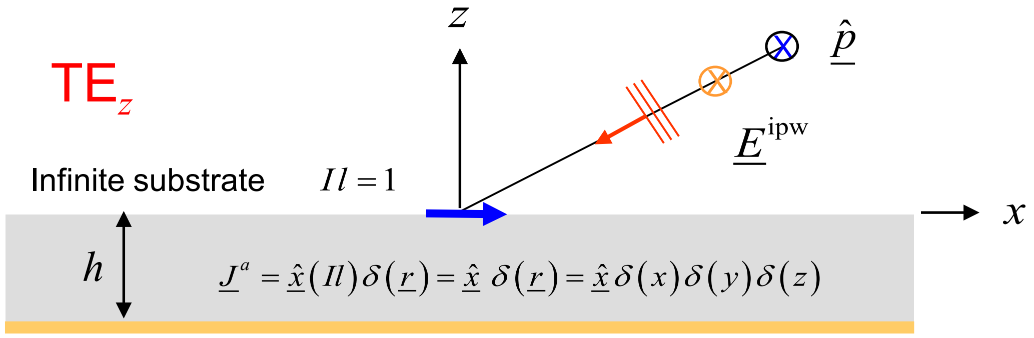
$$E_0 = -\left(\frac{j\omega\mu_0}{4\pi r} \right) e^{-jk_0 r}$$

$\underline{\hat{p}} = \underline{\hat{\phi}}$:

$$\underline{H}^{\text{ipw}} = \underline{\hat{\theta}} \frac{E_\phi^{\text{ipw}}}{\eta_0} = \underline{\hat{\theta}} \left(\frac{E_0}{\eta_0} \right) \psi(x, y, z) = \underline{\hat{\theta}} \left(\frac{E_0}{\eta_0} \right) \psi_t(x, y)\, e^{jk_z z}$$

$$H_y^{\text{ipw}}(x, y, 0) = \left(\frac{E_0}{\eta_0} \right) (\cos\theta \sin\phi) \psi_t(x, y)$$

$$\left(\underline{\hat{\theta}} \cdot \underline{\hat{y}} = \cos\theta \sin\phi \right)$$

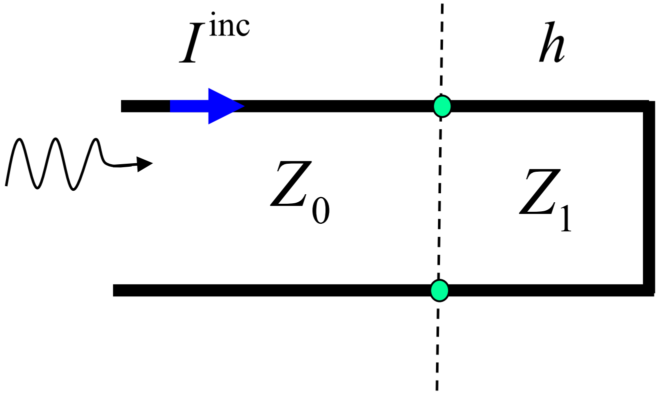


Magnetic Current Model (cont.)

$\underline{\hat{p}} = \underline{\hat{\theta}}$:

$$H_y^{\text{ipw}}(x,y,0) = \left(\frac{E_0}{\eta_0}\right)(-\cos\phi)\psi_t(x,y)$$

$$I^{\text{inc}} = \left(\frac{E_0}{\eta_0}\right)(\cos\phi),\ \ E_\theta\ (\text{TM}_z)$$



$\underline{\hat{p}} = \underline{\hat{\phi}}$:

$$H_y^{\text{ipw}}(x,y,0) = \left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi)\psi_t(x,y)$$

$$I^{\text{inc}} = -\left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi),\ \ E_\phi\ (\text{TE}_z)$$

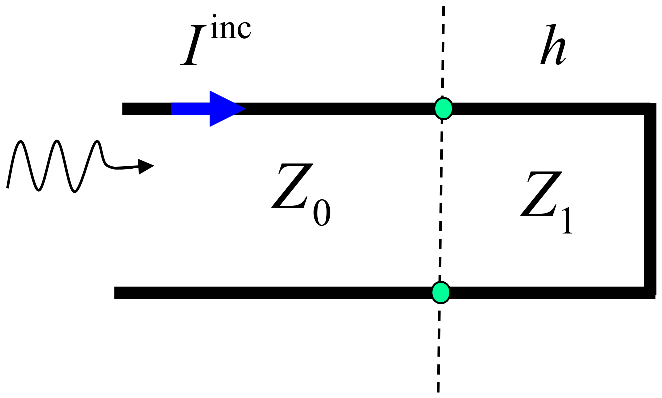
Magnetic Current Model (cont.)

Therefore, we have:

$$-H_y^{\text{pw}}(x,y,z)=e^{j(k_x x+k_y y)}I^{\text{inc}}\left(1-\Gamma^{\text{TM/TE}}\right)\sec(k_{z1}h)\cos k_{z1}(z+h)$$

$$I^{\text{inc}}=\left(\frac{E_0}{\eta_0}\right)(\cos\phi),\ \ E_\theta\ \ (\text{TM}_z)$$

$$I^{\text{inc}}=-\left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi),\ \ E_\phi\ \ (\text{TE}_z)$$



We can thus now consider

$$M_{sy}=-1$$

$$E_p^{\text{FF}}(r,\theta,\phi)=-\int_{S_L+S_R}H_y^{\text{pw}}(x,y,z)(-1)\,dS$$

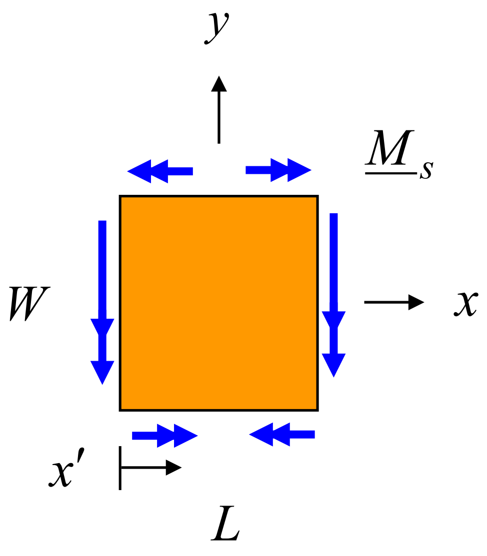
Magnetic Current Model (cont.)

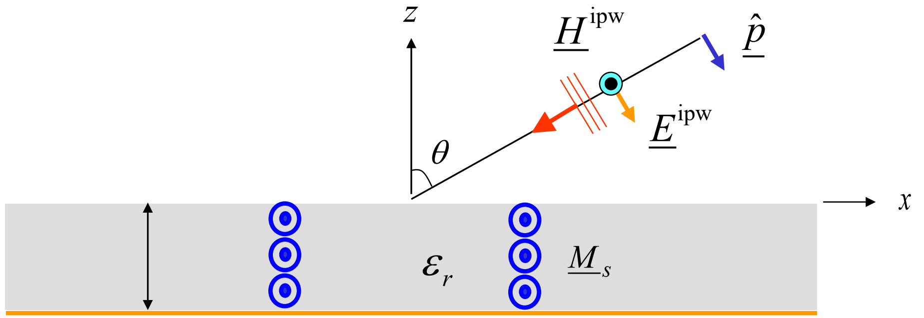
Consider the right edge S_R :

$$x = \frac{L}{2}$$

$$y \in \left(-\frac{W}{2}, \frac{W}{2}\right)$$

$$z \in (-h, 0)$$





Therefore, we have:

$$E_p^R(r,\theta,\phi) = -\int_{-h}^{0}\int_{-\frac{W}{2}}^{+\frac{W}{2}} e^{j\left(k_x \frac{L}{2}\right)} e^{jk_y y} I^{\text{inc}}\left(1-\Gamma^{\text{TM/TE}}\right)\sec\left(k_{z1}h\right)\cos k_{z1}\left(z+h\right)\,dy\,dz$$

Magnetic Current Model (cont.)

$$E_p^R(r,\theta,\phi) = -I^{\text{inc}}\left(1-\Gamma^{\text{TM/TE}}\right)\sec\left(k_{z1}h\right)e^{j\left(k_x\frac{L}{2}\right)}\underbrace{\int_{-\frac{W}{2}}^{+\frac{W}{2}} e^{jk_y y}\,dy}_{\text{integral1}}\;\underbrace{\int_{-h}^{0}\cos k_{z1}(z+h)\,dz}_{\text{integral2}}$$

Integral 1: $W\,\text{sinc}\left(\dfrac{k_y W}{2}\right)$

Integral 2: $h\,\text{sinc}\left(k_{z1}h\right)$

Hence, we have:

$$E_p^R(r,\theta,\phi) = -I^{\text{inc}}\left(1-\Gamma^{\text{TM/TE}}\right)e^{jk_x\frac{L}{2}}(Wh)\,\text{sinc}\left(\frac{k_y W}{2}\right)\text{tanc}(k_{z1}h)$$

where

$$\text{tanc}(x) \equiv \frac{\tan x}{x} \qquad \left(\sec(x)\,\text{sinc}(x) = \text{tanc}(x)\right)$$

Magnetic Current Model (cont.)

For S_L : Replace $\frac{L}{2} \to -\frac{L}{2}$ ⇨ $e^{jk_x \frac{L}{2}} \to e^{-jk_x \frac{L}{2}}$

For $S_L + S_R$: $e^{jk_x \frac{L}{2}} + e^{-jk_x \frac{L}{2}}$ ⇨ $2\cos\left(k_x \frac{L}{2}\right)$

Hence, we have:

$$E_p^{\mathrm{FF}}(r,\theta,\phi) = -I^{\mathrm{inc}}\left(1-\Gamma^{\mathrm{TM/TE}}\right)2\cos\left(k_x \frac{L}{2}\right)(Wh)\,\mathrm{sinc}\left(k_y \frac{W}{2}\right)\mathrm{tanc}\left(k_{z1}h\right)$$

$$I^{\mathrm{inc}} = \left(\frac{E_0}{\eta_0}\right)(\cos\phi),\; E_\theta \;(\mathrm{TM}_z)$$

$$I^{\mathrm{inc}} = -\left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi),\; E_\phi \;(\mathrm{TE}_z)$$

Magnetic Current Model (cont.)

Substituting for I^{inc} in the TM_z and TE_z cases, we have:

$$E_\theta^{\text{FF}}(r,\theta,\phi) = -2Wh\left(\frac{E_0}{\eta_0}\right)\cos\phi\left(1-\Gamma^{\text{TM}}(\theta)\right)\cos\left(k_x\frac{L}{2}\right)\text{sinc}\left(k_y\frac{W}{2}\right)\text{tanc}\left(k_{z1}h\right)$$

$$E_\phi^{\text{FF}}(r,\theta,\phi) = 2Wh\left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi)\left(1-\Gamma^{\text{TE}}(\theta)\right)\cos\left(k_x\frac{L}{2}\right)\text{sinc}\left(k_y\frac{W}{2}\right)\text{tanc}\left(k_{z1}h\right)$$

$$E_0 = \left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

$$k_x = k_0\sin\theta\cos\phi$$
$$k_y = k_0\sin\theta\sin\phi$$

Magnetic Current Model (cont.)

Examine the reflection coefficient terms:

$$1-\Gamma = 1-\left(\frac{Z_{\text{in}}-Z_0}{Z_{\text{in}}+Z_0}\right)$$
$$= \frac{2Z_0}{Z_{\text{in}}+Z_0}$$

$\Gamma = \Gamma^{\text{TM}}$ for $\underline{\hat{p}} = \underline{\hat{\theta}}$

$\Gamma = \Gamma^{\text{TE}}$ for $\underline{\hat{p}} = \underline{\hat{\phi}}$

$$Z_{\text{in}}^{\text{TM/TE}} = jZ_1^{\text{TM/TE}} \tan\left(k_{z1}h\right)$$
$$= jZ_1^{\text{TM/TE}} \tan\left(k_0 N_1\left(\theta\right)h\right)$$

$$N_1(\theta) \equiv \sqrt{n_1^2 - \sin^2\theta} = \sqrt{\varepsilon_r\mu_r - \sin^2\theta}$$

$$Z_1^{\text{TM}} = \frac{\eta_0}{\varepsilon_r} N_1\left(\theta\right)$$

$$Z_1^{\text{TE}} = \frac{\eta_0\mu_r}{N_1\left(\theta\right)}$$

$$Z_0^{\text{TM}} = \eta_0 \cos\theta$$

$$Z_0^{\text{TE}} = \frac{\eta_0}{\cos\theta}$$

Magnetic Current Model (cont.)

Substituting the impedance formulas into the formulas for the reflection coefficients, and then simplifying the results, we have:

$$1-\Gamma^{\mathrm{TM}}(\theta)=\frac{2}{1+j\left(\dfrac{N_1(\theta)\sec\theta}{\varepsilon_r}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

$$1-\Gamma^{\mathrm{TE}}(\theta)=\frac{2}{1+j\left(\dfrac{\mu_r\cos\theta}{N_1(\theta)}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

Summary

$$E_\theta^{\mathrm{FF}}(r,\theta,\phi) = -2Wh\left(\frac{E_0}{\eta_0}\right)\cos\phi\left(1-\Gamma^{\mathrm{TM}}(\theta)\right)\cos\left(k_x\frac{L}{2}\right)\mathrm{sinc}\left(k_y\frac{W}{2}\right)\mathrm{tanc}\left(k_{z1}h\right)$$

$$E_\phi^{\mathrm{FF}}(r,\theta,\phi) = 2Wh\left(\frac{E_0}{\eta_0}\right)(\cos\theta\sin\phi)\left(1-\Gamma^{\mathrm{TE}}(\theta)\right)\cos\left(k_x\frac{L}{2}\right)\mathrm{sinc}\left(k_y\frac{W}{2}\right)\mathrm{tanc}\left(k_{z1}h\right)$$

$$1-\Gamma^{\mathrm{TM}}(\theta) = \frac{2}{1+j\left(\dfrac{N_1(\theta)\sec\theta}{\varepsilon_r}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

$$1-\Gamma^{\mathrm{TE}}(\theta) = \frac{2}{1+j\left(\dfrac{\mu_r\cos\theta}{N_1(\theta)}\right)\tan\left(k_0 h\, N_1(\theta)\right)}$$

$$E_0 = \left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

$$k_x = k_0\sin\theta\cos\phi$$
$$k_y = k_0\sin\theta\sin\phi$$

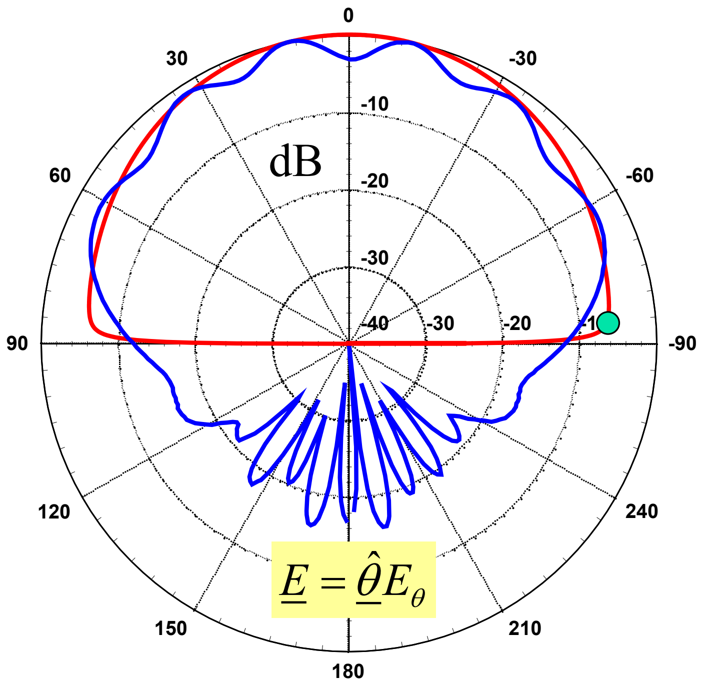
$$k_{z1} = k_0 N_1(\theta)$$
$$N_1(\theta) \equiv \sqrt{n_1^2 - \sin^2\theta}$$

$$n_1 = \sqrt{\mu_r\varepsilon_r}$$

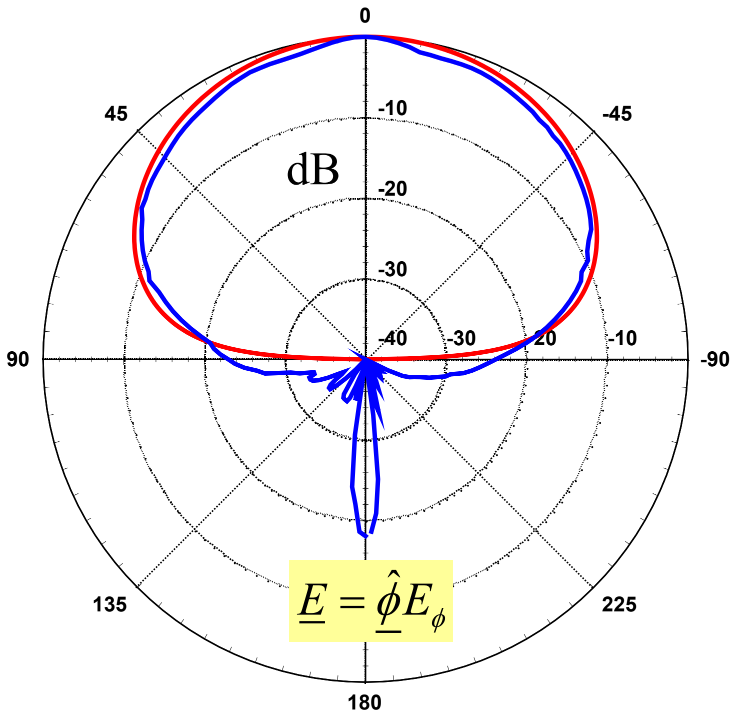
Assumption: $E_z^{1,0} = -\sin\left(\dfrac{\pi x}{L}\right)$

Summary

E-plane pattern ($\phi = 0^o$)



H-plane pattern ($\phi = 90^o$)



Red: infinite substrate and ground plane

Blue: 1 meter diameter ground plane

Comments:

- The E plane is broader than the H plane.
- The E-plane pattern "tucks in" and tends to zero at the horizon due to the presence of the infinite substrate (green dot). (As the substrate gets thinner, the tuck-in point approaches 90°.)