ECE 6345

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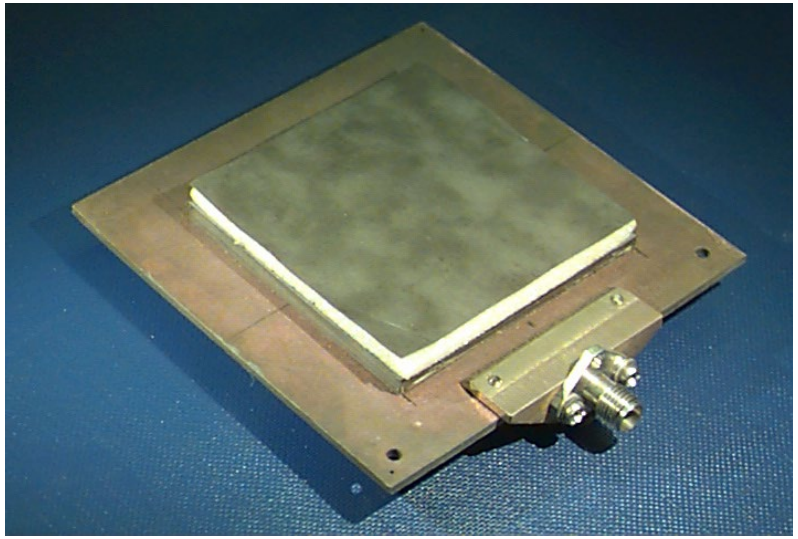
Prof. David R. Jackson
ECE Dept.



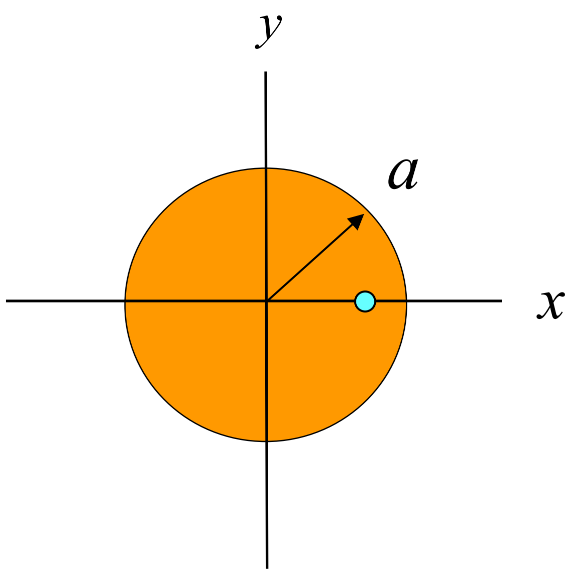
Notes 11

Overview

- In this set of notes we derive the far-field pattern of a circular patch operating in the dominant TM_{11} mode.

- We use the magnetic current model.

Circular Patch: TM_{11} Mode



$$E_z(\rho,\phi) = A\cos\phi\, J_1(k\rho)$$

The $\cos\phi$ corresponds to a probe on the x axis.

$$ka = x'_{11} = 1.841$$

Circular Patch (cont.)

Magnetic current model:

$$\underline{M}_s = -\underline{\hat{n}} \times \underline{\hat{E}}$$
$$= -\underline{\hat{\rho}} \times \underline{\hat{E}}$$

$$M_{s\phi} = E_z(a,\phi) = A\cos\phi\, J_1(ka)$$

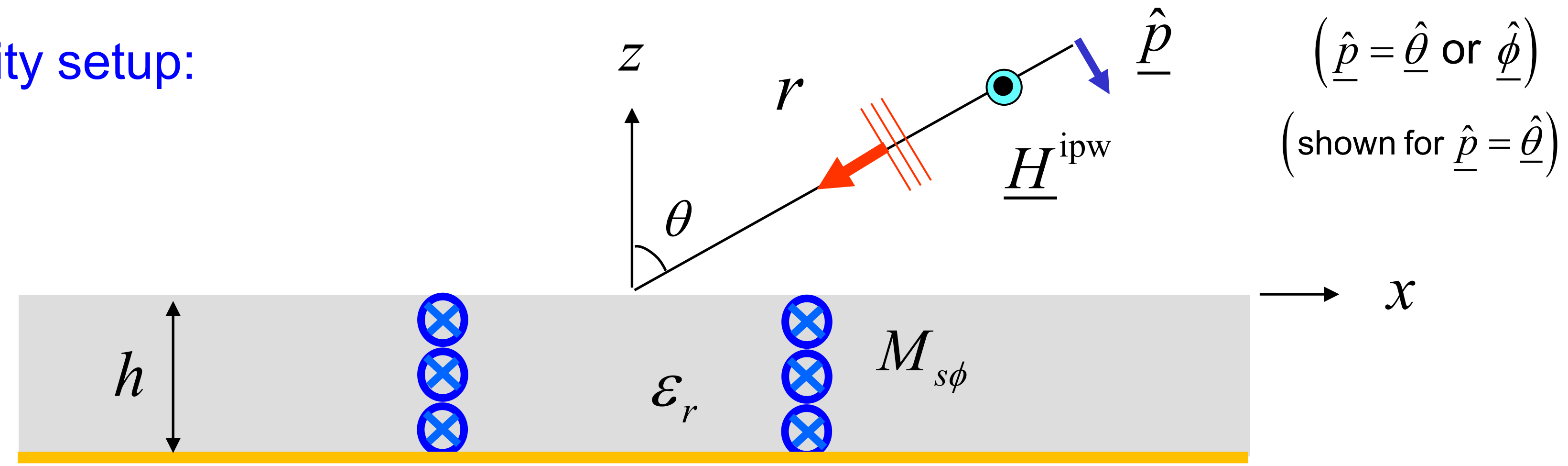
Choose $A = \dfrac{1}{J_1(ka)}$

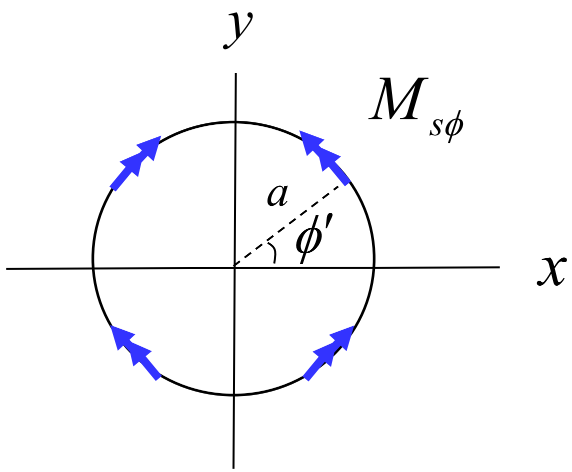
$$M_{s\phi} = \cos\phi$$

$\left(V_0 = -h \text{ [V] at patch edge on } x \text{ axis}\right)$

Far Field of Circular Patch

Reciprocity setup:





Far Field of Circular Patch (cont.)

Far-field:

$$E_p^{\mathrm{FF}}(r,\theta,\phi) = \langle a,b \rangle$$
$$= \langle b,a \rangle$$
$$= -\int_S \underline{H}^{\mathrm{pw}} \cdot \underline{M}_s^a \, dS$$
$$= -\int_S H_{\phi'}^{\mathrm{pw}}(\rho',\phi',z') \cos\phi' \, dS'$$
$$= -\int_0^{2\pi} \int_{-h}^{0} H_{\phi'}^{\mathrm{pw}}(a,\phi',z') \cos\phi' \, a \, dz' \, d\phi'$$

$$\underline{M}_s^a = \underline{\hat{\phi}} \cos\phi$$

The primes here denotes source coordinates.

$$E_p^{\mathrm{FF}}(r,\theta,\phi) = -\int_0^{2\pi} \int_{-h}^{0} H_{\phi'}^{\mathrm{pw}}(a,\phi',z') \cos\phi' \, a \, dz' \, d\phi'$$

$$H_{\phi'}^{\mathrm{pw}} = H_x^{\mathrm{pw}}(-\sin\phi') + H_y^{\mathrm{pw}}(\cos\phi')$$

Far Field of Circular Patch (cont.)

Inside the substrate we have (see Notes 10):

$$H_{x,y}^{\text{pw}}\left(x',y',z'\right)=H_{x,y}^{\text{pw}}\left(0,0,0\right)\,e^{j\left(k_x x'+k_y y'\right)}\left[\sec\left(k_{z1}h\right)\cos\left(k_{z1}(z'+h)\right)\right]$$

$$\left(k_{z1}=k_0 N_1(\theta)\right)$$

$$N_1\left(\theta\right)\equiv\sqrt{n_1^2-\sin^2\left(\theta\right)}$$

$$n_1=\sqrt{\varepsilon_r\mu_r}$$

The exponent term may be put in cylindrical coordinates as follows ($\rho' = a$):

$$\begin{aligned}k_x x'+k_y y'&=\left(k_0\sin\theta\cos\phi\right)\left(a\cos\phi'\right)+\left(k_0\sin\theta\sin\phi\right)\left(a\sin\phi'\right)\\&=k_0\,a\sin\theta\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)\\&=k_0\sin\theta\cos\left(\phi'-\phi\right)\end{aligned}$$

Far Field of Circular Patch (cont.)

Hence, we have:

$$H_{\phi'}^{\mathrm{pw}} = H_x^{\mathrm{pw}}\left(-\sin\phi'\right) + H_y^{\mathrm{pw}}\left(\cos\phi'\right)$$

$$\Rightarrow \quad H_{\phi'}^{\mathrm{pw}} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'+h\right) e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)} \cdot\left[-\sin\phi' H_x^{\mathrm{pw}}(0,0,0) + \cos\phi' H_y^{\mathrm{pw}}(0,0,0)\right]$$

Since the horizontal magnetic field components are modeled as current in the TEN, we have

$$H_{x,y}^{\mathrm{pw}}\left(0,0,0\right) = H_{x,y}^{\mathrm{ipw}}\left(0,0,0\right)\left(1-\Gamma(\theta)\right)$$

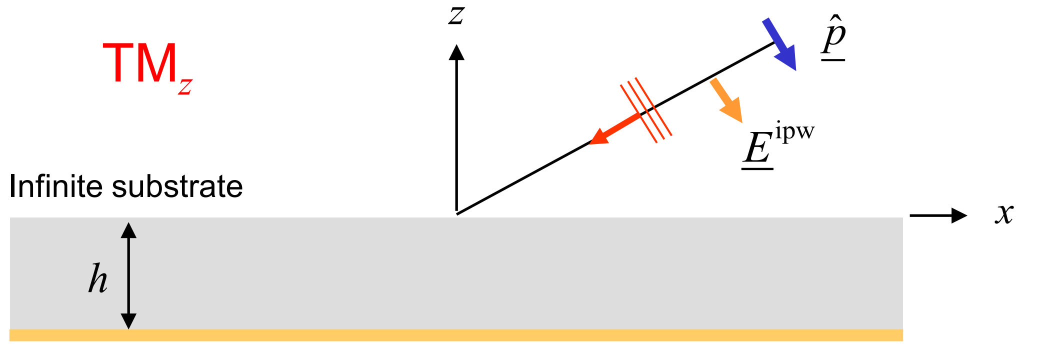
$$p = \theta : \mathrm{TM}, \quad p = \phi : \mathrm{TE}$$

Far Field of Circular Patch (cont.)

$\text{TM}_z \quad \left(\underline{\hat{p}} = \underline{\hat{\theta}}\right)$

$$H_x^{\text{ipw}}(0,0,0) = \left(-\frac{E_0}{\eta_0}\underline{\hat{\phi}}\right)\cdot\underline{\hat{x}} = \frac{E_0}{\eta_0}(\sin\phi)$$

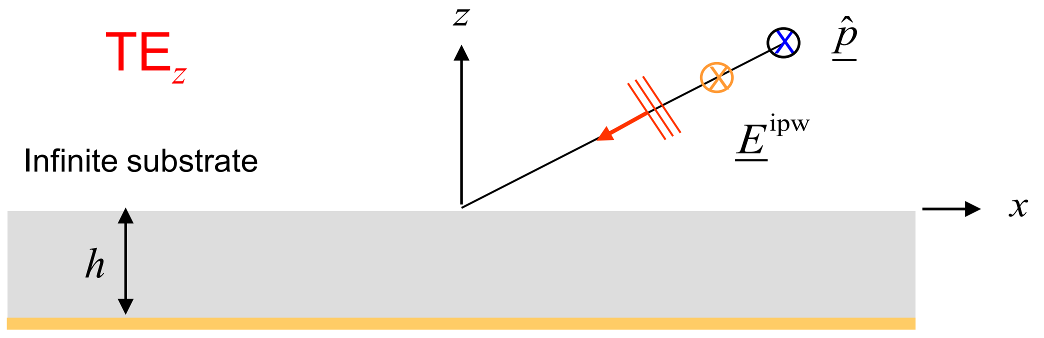
$$H_y^{\text{ipw}}(0,0,0) = \left(-\frac{E_0}{\eta_0}\underline{\hat{\phi}}\right)\cdot\underline{\hat{y}} = \frac{E_0}{\eta_0}(-\cos\phi)$$



$\text{TE}_z \quad \left(\underline{\hat{p}} = \underline{\hat{\phi}}\right)$

$$H_x^{\text{ipw}}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{x}} = \frac{E_0}{\eta_0}(\cos\theta\cos\phi)$$

$$H_y^{\text{ipw}}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{y}} = \frac{E_0}{\eta_0}(\cos\theta\sin\phi)$$



Far Field E_θ

$\text{TM}_z \quad \left(\underline{\hat{p}} = \underline{\hat{\theta}}\right)$

$$H_{\phi'}^{\text{pw}} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'+h\right)\, e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)} \cdot\left[-\sin\phi'\, H_x^{\text{pw}}(0,0,0)+\cos\phi' H_y^{\text{pw}}(0,0,0)\right]$$

$$H_x^{\text{ipw}}\left(0,0,0\right)=\frac{E_0}{\eta_0}\left(\sin\phi\right) \qquad H_y^{\text{ipw}}\left(0,0,0\right)=\frac{E_0}{\eta_0}\left(-\cos\phi\right)$$

Substituting for H_x and H_y, we have:

$$H_{\phi'}^{\text{pw}} = \sec\left(k_{z1}h\right)\cos k_{z1}\left(z'+h\right)e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)}\left(\frac{E_0}{\eta_0}\right)\left[-\sin\phi'\sin\phi-\cos\phi'\cos\phi\right]\left(1-\Gamma^{\text{TM}}(\theta)\right)$$

$$E_p^{\text{FF}}\left(r,\theta,\phi\right) = -\int_0^{2\pi}\int_{-h}^{0} H_{\phi'}^{\text{pw}}\left(a,\phi',z'\right)\cos\phi'\, a\, dz'\, d\phi'$$

Note: $\left[-\sin\phi'\sin\phi-\cos\phi'\cos\phi\right] = -\cos\left(\phi'-\phi\right)$

Hence, we have:

$$E_\theta^{\text{FF}}\left(r,\theta,\phi\right) = \int_0^{2\pi}\int_{-h}^{0}\frac{E_0}{\eta_0}\sec\left(k_{z1}h\right)\cos k_{z1}(z'+h)\, e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)} \cdot\left(1-\Gamma^{\text{TM}}(\theta)\right)\left(\cos(\phi'-\phi)\right)\cos\phi'\, a\, dz'\, d\phi'$$

Far Field E_θ (cont.)

For the z' integral we have:

$$\sec(k_{z1}h)\int_{-h}^{0}\cos k_{z1}(z'+h)\,dz' = h\,\mathrm{tanc}(k_{z1}h)$$

so that

$$E_\theta^{\mathrm{FF}}(r,\theta,\phi) = a\left(\frac{E_0}{\eta_0}\right)h\,\mathrm{tanc}(k_{z1}h)\left(1-\Gamma^{\mathrm{TM}}(\theta)\right)I_q^{\mathrm{TM}}$$

where

$$I_q^{\mathrm{TM}} \equiv \int_0^{2\pi} e^{jq\cos(\phi'-\phi)}\cos(\phi'-\phi)\cos\phi'\,d\phi'$$

$$q \equiv (k_0 a)\sin\theta$$

Let $\phi'' = \phi' - \phi$ then $I_q^{\mathrm{TM}} = \int_{-\phi}^{2\pi-\phi} e^{jq\cos(\phi'')}\cos(\phi'')\cos(\phi''+\phi)\,d\phi''$

Far Field E_θ (cont.)

We have that

$$\cos\left(\phi''+\phi\right)=\cos\phi''\cos\phi-\sin\phi''\sin\phi$$

and $$\int_{-\phi}^{2\pi-\phi}(\ \ )\,d\phi''=\int_0^{2\pi}(\ \ )\,d\phi''$$

so that

$$I_q^{\mathrm{TM}}\equiv\int_0^{2\pi}e^{jq\cos(\phi'')}\cos\left(\phi''\right)\cos\left(\phi''+\phi\right)d\phi''$$

$$=\cos\phi\int_0^{2\pi}e^{jq\cos\phi''}\cos^2\phi''\,d\phi''$$

$$-\sin\phi\underline{\int_0^{2\pi}e^{jq\cos\phi''}\sin\phi''\cos\phi''\,d\phi''}$$

Integrates to zero (odd function)

Now use $\cos^2\phi''=1-\sin^2\phi''$

Far Field E_θ (cont.)

$$I_q^{\mathrm{TM}} = \cos\phi \int_0^{2\pi} e^{jq\cos\phi''}\,d\phi'' - \cos\phi \int_0^{2\pi} e^{jq\cos\phi''} \sin^2\phi''\,d\phi''$$

Now we use the following identity:

$$\int_0^{2\pi} e^{jq\cos\phi''} \sin^{2n}\phi''\,d\phi'' = J_n(q)\left[\frac{2^{n+1}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{q^n}\right]$$

$$n = 0, 1, 2 \ldots$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Note : $\Gamma(x) = (x-1)!$

Far Field E_θ (cont.)

Hence

$$\int_0^{2\pi} e^{jq\cos\phi''} d\phi'' = 2\pi\, J_0(q) \qquad (n=0)$$

and

$$\int_0^{2\pi} e^{jq\cos\phi''} \sin^2\phi'' d\phi'' = 2\pi \left(\frac{J_1(q)}{q}\right) \qquad (n=1)$$

and thus

$$I_q^{\mathrm{TM}} = \cos\phi\ (2\pi)\left[J_0(q) - \frac{J_1(q)}{q}\right]$$

Next, use

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

so that

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

Far Field E_θ (cont.)

Hence, we have:

$$I_q^{\mathrm{TM}} = 2\pi \cos\phi \, J_1'(q)$$

$$E_\theta^{\mathrm{FF}}\left(r,\theta,\phi\right) = a\left(\frac{E_0}{\eta_0}\right) h \,\mathrm{tanc}(k_{z1}h)\left(1-\Gamma^{\mathrm{TM}}(\theta)\right) I_q^{\mathrm{TM}}$$

The far field is then:

$$E_\theta^{\mathrm{FF}}\left(r,\theta,\phi\right) = \frac{E_0}{\eta_0}(ah)\,\mathrm{tanc}(k_{z1}h)\,Q(\theta)\left[2\pi \cos\phi \, J_1'(k_0 a \sin\theta)\right]$$

where

$$Q(\theta) = 1 - \Gamma^{\mathrm{TM}}(\theta)$$

Far Field E_ϕ

TE$_z$ $\left(\underline{\hat{p}} = \underline{\hat{\phi}}\right)$

$$E_\phi^{\text{FF}}(r,\theta,\phi) = -\int_0^{2\pi}\int_{-h}^{0} H_{\phi'}^{\text{pw}}(a,\phi',z')\cos\phi'\, a\, dz'\, d\phi'$$

$$H_{\phi'}^{\text{pw}}(a,\phi',z') = \sec(k_{z1}h)\cos k_{z1}(z'+h)\, e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)}$$
$$\cdot\left[-\sin\phi' H_x^{\text{pw}}(0,0,0) + \cos\phi' H_y^{\text{pw}}(0,0,0)\right]$$

Performing similar steps for the TE$_z$ case, we have:

$$H_{\phi'}^{\text{pw}}(a,\phi',z') = \sec(k_{z1}h)\cos k_{z1}(z'-h)\left(1-\Gamma^{\text{TE}}(\theta)\right) e^{j(k_0 a)\sin\theta\cos(\phi'-\phi)}$$
$$\cdot\left(\frac{E_0}{\eta_0}\right)\left[-\sin\phi'\cos\theta\cos\phi + \cos\phi'\cos\theta\sin\phi\right]$$

$$H_x^{\text{ipw}}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{x}} = \frac{E_0}{\eta_0}(\cos\theta\cos\phi)$$

$$H_y^{\text{ipw}}(0,0,0) = \left(\frac{E_0}{\eta_0}\underline{\hat{\theta}}\right)\cdot\underline{\hat{y}} = \frac{E_0}{\eta_0}(\cos\theta\sin\phi)$$

Using reciprocity and performing the integration in z', we have:

$$E_\phi^{\text{FF}}(r,\theta,\phi) = a\left(\frac{E_0}{\eta_0}\right)(h)\,\text{tanc}(k_{z1}h)\left(1-\Gamma^{\text{TE}}(\theta)\right)$$
$$\cdot\cos\theta\int_0^{2\pi} e^{jq\cos(\phi'-\phi)}\sin(\phi'-\phi)\cos\phi'\, d\phi'$$

Far Field E_ϕ (cont.)

Evaluating the integral, we have:

$$I_q^{\mathrm{TE}} \equiv \int_0^{2\pi} e^{jq\cos(\phi'-\phi)} \sin(\phi'-\phi)\cos\phi' d\phi'$$

$$= \int_0^{2\pi} e^{jq\cos\phi''} \sin\phi'' \cos(\phi''+\phi)\, d\phi''$$

$$= \int_0^{2\pi} e^{jq\cos\phi''} \sin\phi'' \left[\cos\phi''\cos\phi - \sin\phi''\sin\phi\right] d\phi''$$

$$= \cos\phi \underline{\int_0^{2\pi} e^{jq\cos\phi''} \sin\phi''\cos\phi'' d\phi''} - \sin\phi \int_0^{2\pi} e^{jq\cos\phi''} \sin^2\phi''\, d\phi''$$

integrates to zero
(odd function)

$$= -\sin\phi\, 2\pi \left(\frac{J_1(q)}{q}\right)$$

Far Field E_ϕ (cont.)

Hence, we have:

$$E_\phi^{\text{FF}}\left(r,\theta,\phi\right)=\frac{E_0}{\eta_0}(ah)\,\text{tanc}\left(k_{z1}h\right)\left[-\sin\phi\,2\pi\left(\frac{J_1\left(k_0 a\sin\theta\right)}{k_0 a\sin\theta}\right)\right]P(\theta)$$

where

$$P(\theta)\equiv\cos\theta\left(1-\Gamma^{\text{TE}}(\theta)\right)$$

Far Field (Summary)

$$E_\theta^{\mathrm{FF}}\left(r,\theta,\phi\right)=\frac{E_0}{\eta_0}(ah)\,\mathrm{tanc}(k_{z1}h)\,Q(\theta)\,2\pi\cos\phi\,J_1'(k_0 a\sin\theta)$$

$$E_\phi^{\mathrm{FF}}\left(r,\theta,\phi\right)=-\frac{E_0}{\eta_0}(ah)\,\mathrm{tanc}\left(k_{z1}h\right)\sin\phi\,2\pi\left(\frac{J_1\left(k_0 a\sin\theta\right)}{k_0 a\sin\theta}\right)P(\theta)$$

where

$$P(\theta)\equiv\cos\theta\left(1-\Gamma^{\mathrm{TE}}(\theta)\right)$$

$$Q(\theta)=1-\Gamma^{\mathrm{TM}}(\theta)$$

$$1-\Gamma^{\mathrm{TM}}\left(\theta\right)=\frac{2}{1+j\left(\dfrac{N_1(\theta)\sec\theta}{\varepsilon_r}\right)\tan\left(k_0 h\,N_1(\theta)\right)}$$

$$1-\Gamma^{\mathrm{TE}}\left(\theta\right)=\frac{2}{1+j\left(\dfrac{\mu_r\cos\theta}{N_1(\theta)}\right)\tan\left(k_0 h\,N_1(\theta)\right)}$$

$$E_0=\left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

$$k_x=k_0\sin\theta\cos\phi$$
$$k_y=k_0\sin\theta\sin\phi$$

$$k_{z1}=k_0 N_1\left(\theta\right)$$
$$N_1\left(\theta\right)\equiv\sqrt{n_1^2-\sin^2\theta}$$

$$n_1=\sqrt{\mu_r\varepsilon_r}$$

Assumption: $E_z^{1,1}=\cos\phi'$