ECE 6345

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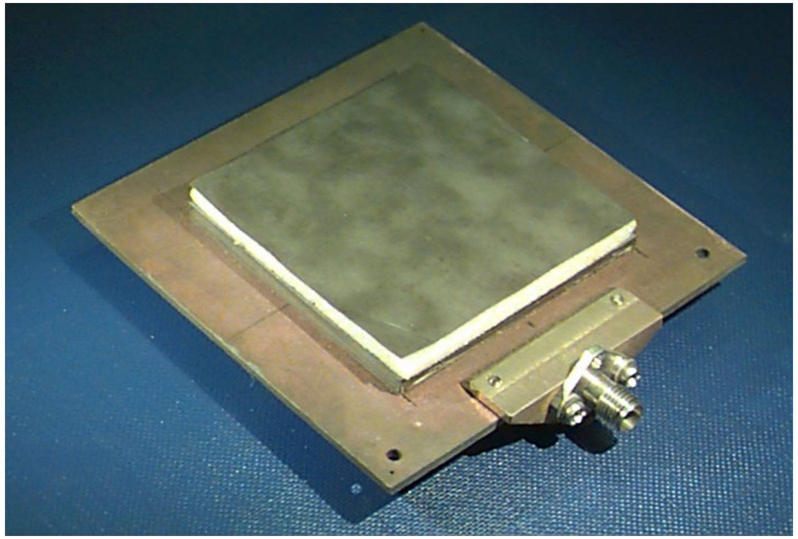


Notes 12

Overview

- In this set of notes we calculate the power radiated into space by a current source sitting on top on an infinite grounded substrate.

  - **First, we calculate the power radiated by a horizontal electric dipole.**
  - **We develop a CAD formula for the radiated power of the dipole.**
  - **Then we extend this to a finite-size patch current.**

Note:
The power radiated into space in the key ingredient for developing the CAD formula for the space-wave Q factor (Q_{sp}) of the patch. This leads to the CAD formulas for bandwidth, radiation efficiency, and input resistance.

Radiated Power of Dipole

Horizontal electric dipole in the x direction on an infinite substrate (from Notes 8):

$$E_\theta = (Il) E_0 \cos\phi \, G(\theta)$$

$$E_\phi = (Il) E_0 (-\sin\phi) F(\theta)$$

$$F(\theta) = 1 + \Gamma^{\mathrm{TE}}(\theta)$$

$$G(\theta) = \cos\theta \left(1 + \Gamma^{\mathrm{TM}}(\theta)\right)$$

$$E_0 = \frac{-j\omega\mu_0}{4\pi r} e^{-jk_0 r}$$

$$G(\theta) = \cos\theta\left(1 + \Gamma^{\mathrm{TM}}(\theta)\right) = \frac{2\cos\theta}{1 - j\left(\dfrac{\varepsilon_r \cos(\theta)}{N_1(\theta)}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

$$N_1(\theta) \equiv \sqrt{n_1^2 - \sin^2\theta}$$

$$n_1 = \sqrt{\varepsilon_r \mu_r}$$

$$F(\theta) = 1 + \Gamma^{\mathrm{TE}}(\theta) = \frac{2}{1 - j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Radiated Power of Dipole (cont.)

Poynting vector in far field:

$$S_r^{\text{dip}} = \frac{1}{2\eta_0}\left[\left|E_\theta\right|^2 + \left|E_\phi\right|^2\right]$$

Total power radiated into space:

$$P_{\text{sp}}^{\text{dip}} = \int_0^{2\pi}\int_0^{\pi/2} S_r^{\text{dip}}\left(r,\theta,\phi\right) r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2\eta_0}\left(Il\right)^2 \left|E_0\right|^2 r^2 \left[\int_0^{2\pi}\cos^2\phi \int_0^{\pi/2}\left|G(\theta)\right|^2 \sin^2\theta\, d\theta + \int_0^{2\pi}\sin^2\phi \int_0^{\pi/2}\left|F(\theta)\right|^2 \sin\theta\, d\theta\right]$$

Radiated Power of Dipole (cont.)

Performing the ϕ integrals, we have:

$$P_{\text{sp}}^{\text{dip}} = \frac{\pi}{2\eta_0}(Il)^2 \left|E_0\right|^2 r^2 \int_0^{\pi/2} \left(\left|F(\theta)\right|^2 + \left|G(\theta)\right|^2 \right) \sin\theta \, d\theta$$

Note that

$$E_0 r = \frac{-j\omega\mu_0}{4\pi} e^{-jk_0 r}$$

so

$$\left|E_0 r\right|^2 = \left(\frac{\omega\mu_0}{4\pi}\right)^2 = \left(\frac{k_0\eta_0}{4\pi}\right)^2$$

and thus

$$P_{\text{sp}}^{\text{dip}} = (Il)^2 k_0^2 \left(\frac{\eta_0}{32\pi}\right) \int_0^{\pi/2} \left[\left|F(\theta)\right|^2 + \left|G(\theta)\right|^2 \right] \sin\theta \, d\theta$$

CAD Formula for Dipole Radiated Power

Assume $$\frac{h}{\lambda_0} \to 0$$

We wish to approximate these functions:

$$F(\theta) = 1 + \Gamma^{\mathrm{TE}}(\theta)$$

$$G(\theta) = \cos\theta \left(1 + \Gamma^{\mathrm{TM}}(\theta)\right)$$

CAD Formula for Dipole Radiated Power (cont.)

$$\frac{h}{\lambda_0} \rightarrow 0$$

$$N_1(\theta) \equiv \sqrt{n_1^2 - \sin^2\theta}$$

$$n_1 = \sqrt{\varepsilon_r \mu_r}$$

Use $\cot\left(k_0 h\, N_1(\theta)\right) \approx \dfrac{1}{k_0 h\, N_1(\theta)} = \dfrac{1}{k_0 h \sqrt{n_1^2 - \sin^2\theta}}$

$$G(\theta) = \cos\theta\left(1 + \Gamma^{\mathrm{TM}}(\theta)\right) = \frac{2\cos\theta}{1 - j\left(\dfrac{\varepsilon_r \cos(\theta)}{N_1(\theta)}\right)\cot\left(k_0 h N_1(\theta)\right)} \approx \frac{2\cos\theta}{-j\left(\dfrac{\varepsilon_r \cos(\theta)}{N_1(\theta)}\right)\dfrac{1}{k_0 h\, N_1(\theta)}} = \left(\frac{2j\left(n_1^2 - \sin^2\theta\right)}{\varepsilon_r}\right)k_0 h$$

$$F(\theta) = 1 + \Gamma^{\mathrm{TE}}(\theta) = \frac{2}{1 - j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\cot\left(k_0 h N_1(\theta)\right)} \approx \frac{2}{-j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\dfrac{1}{k_0 h\, N_1(\theta)}} = \left(2j\mu_r\cos\theta\right)k_0 h$$

Note: The "1" term in the denominator of the F and G functions can be neglected for a thin substrate.

CAD Formula for Dipole Radiated Power (cont.)

The result is then:

$$G(\theta) \sim \left(\frac{2j\left(n_1^2 - \sin^2\theta\right)}{\varepsilon_r} \right) k_0 h$$

$$F(\theta) \sim \left(2j\mu_r \cos\theta\right) k_0 h$$

where $n_1 = \sqrt{\varepsilon_r \mu_r}$

CAD Formula for Dipole Radiated Power (cont.)

Recall: $$P_{\text{sp}}^{\text{dip}} = (Il)^2 \left(\frac{k_0^2 \eta_0}{32\pi} \right) \int_0^{\pi/2} \left[|F(\theta)|^2 + |G(\theta)|^2 \right] \sin\theta \, d\theta$$

We need the following integrals:

$$\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}$$

$$G(\theta) \sim \left(\frac{2j\left(n_1^2 - \sin^2\theta\right)}{\varepsilon_r} \right) k_0 h$$

$$F(\theta) \sim \left(2 j \mu_r \cos\theta\right) k_0 h$$

$$|G(\theta)|^2 \sim \left(\frac{4\left(n_1^2 - \sin^2\theta\right)^2}{\varepsilon_r^2} \right) (k_0 h)^2$$

$$|F(\theta)|^2 \sim \left(4 \mu_r^2 \cos^2\theta\right) (k_0 h)^2$$

CAD Formula for Dipole Radiated Power (cont.)

We then have, accounting for all of the constants in front of the integrals:

$$P_{\text{sp}}^{\text{dip}} = \left(Il\right)^2 \left(\frac{k_0^2 \eta_0}{32\pi}\right) \left(k_0 h\right)^2 4 \left[\mu_r^2 \left(\frac{1}{3}\right) + \frac{1}{\varepsilon_r^2} \left(n_1^4 (1) - 2 n_1^2 \left(\frac{2}{3}\right) + \frac{8}{15} \right) \right]$$

Substituting for the index of refraction, we have:

$$P_{\text{sp}}^{\text{dip}} = \left(Il\right)^2 \left(k_0 h\right)^2 k_0^2 \left(\frac{\eta_0}{8\pi}\right) \left[\mu_r^2 \left(\frac{1}{3}\right) + \mu_r^2 \left(1\right) - 2 \frac{\mu_r}{\varepsilon_r} \left(\frac{2}{3}\right) + \frac{1}{\varepsilon_r^2} \left(\frac{8}{15}\right) \right]$$

Recall : $n_1 = \sqrt{\varepsilon_r \mu_r}$

CAD Formula for Dipole Radiated Power (cont.)

$$P_s^{\text{dip}} = (Il)^2 (k_0 h)^2 k_0^2 \left(\frac{\eta_0}{8\pi}\right)\left[\mu_r^2\left(\frac{1}{3}\right) + \mu_r^2(1) - 2\frac{\mu_r}{\varepsilon_r}\left(\frac{2}{3}\right) + \frac{1}{\varepsilon_r^2}\left(\frac{8}{15}\right)\right]$$

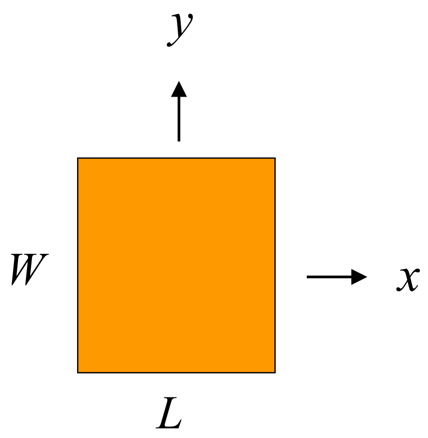
Note that $[\;] = \mu_r^2\left(\frac{4}{3} - \frac{4}{3}\frac{1}{\varepsilon_r\mu_r} + \frac{8}{15}\frac{1}{\mu_r^2\varepsilon_r^2}\right)$

$$= \mu_r^2\left(\frac{4}{3}\right)\left(1 - \frac{1}{n_1^2} + \frac{2/5}{n_1^4}\right)$$

Define $c_1 \equiv 1 - \frac{1}{n_1^2} + \frac{2/5}{n_1^4}$

Then $$P_s^{\text{dip}} \approx (Il)^2 (k_0 h)^2 k_0^2 \left(\frac{\eta_0}{6\pi}\right)\mu_r^2 c_1$$

Radiated Power of Patch



Assume that

$$J_{sx}^{1,0} = \cos\left(\frac{\pi x}{L}\right)$$

We find the equivalent dipole moment of the patch:

$$(Il)_{\text{patch}} = \int_{-\frac{L}{2}}^{+\frac{L}{2}} \int_{-\frac{W}{2}}^{+\frac{W}{2}} J_{sx}^{1,0}(x,y)\, dy\, dx$$

$$= W \int_{-\frac{L}{2}}^{+\frac{L}{2}} \cos\left(\frac{\pi x}{L}\right) dx$$

$$= W\left(\frac{2L}{\pi}\right)$$

Radiated Power of Patch (cont.)

$$\left(Il\right)_{\text{patch}} = \left(\frac{2}{\pi}WL\right)$$

Neglecting the space factor, $P_{\text{sp}}^{\text{patch}} \approx P_{\text{sp}}^{\text{dip}}$

where

$$P_{\text{sp}}^{\text{dip}} = \left(\frac{2}{\pi}WL\right)^2 \left(k_0 h\right)^2 k_0^2 \left(\frac{\eta_0}{6\pi}\right)\mu_r^2 c_1$$

This formula may be improved by accounting for the patch space factor, which leads to the introduction of the "p factor" that is discussed in the next set of notes.