ECE 6345

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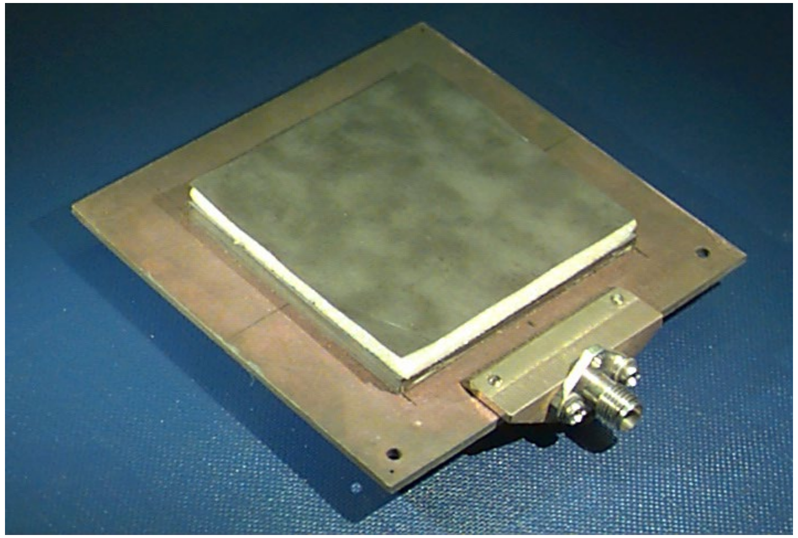

Notes 14

Overview

In this set of notes we perform the algebra necessary to evaluate the p factor in closed form (assuming a thin substrate) and to simplify the final result.

Approximation of "p"

From Notes 13 we have:

$$p = \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\left[\left|F(\theta)\right|^2 \sin^2\phi + \left|G(\theta)\right|^2 \cos^2\phi\right]\left|A(\theta,\phi)\right|^2 \sin\theta\, d\theta\, d\phi}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\left[\left|F(\theta)\right|^2 \sin^2\phi + \left|G(\theta)\right|^2 \cos^2\phi\right]\left|A(0,0)\right|^2 \sin\theta\, d\theta\, d\phi}$$

$$A(\theta,\phi) = \left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(k_x\dfrac{L}{2}\right)^2}\right]$$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$

Approximation of "p" (cont.)

Assume $h / \lambda_0 \to 0$

Then we have (from Notes 12):

$$F(\theta) \approx \left(2 j \mu_r \cos\theta\right) k_0 h$$

$$G(\theta) \approx \left(\frac{2j}{\varepsilon_r}\right)\left(n_1^2 - \sin^2\theta\right) k_0 h$$

Also assume that $n_1^2 = \varepsilon_r \mu_r >> \sin^2\theta$

This implies that the patch is fairly small (high permittivity substrate) or that the angles of significant radiation are small.

Then we have:

$$F(\theta) \approx \left(2 j \mu_r \cos\theta\right) k_0 h$$

$$G(\theta) \approx \left(2 j \mu_r\right) k_0 h$$

Approximation of "p" (cont.)

We then have:

$$\sin^2\phi\left|F(\theta)\right|^2+\cos^2\phi\left|G(\theta)\right|^2\approx 4\mu_r^2\left(k_0h\right)^2\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)$$

Therefore

$$p\approx\frac{\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\left|A(\theta,\phi)\right|^2\sin\theta\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\left|A(0,0)\right|^2\sin\theta\,d\theta\,d\phi}$$

Note: The p factor is now only a function of the patch dimensions – not the substrate.

Approximation of “p” (cont.)

The patch space (array) factor is:

$$A(\theta,\phi)=\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

or

$$A(\theta,\phi)=\left(\frac{\pi}{2}\right)^2\left(\frac{2}{\pi}WL\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

or

$$A(\theta,\phi)=\left(\frac{\pi}{2}\right)^2 A(0,0)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

Recall:

$$A(0,0)=\left(\frac{2}{\pi}WL\right)$$

$$J_{sx}^{1,0}=\cos\left(\frac{\pi x}{L}\right)$$

Approximation of “p” (cont.)

In the denominator of the p expression we have this integral:

$$\int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\sin\theta\,d\theta\,d\phi=\pi\int_0^{\pi/2}\left(\cos^2\theta+1\right)\sin\theta\,d\theta$$

$$=\pi\left(\frac{1}{3}+1\right)$$

$$=\left(\frac{4\pi}{3}\right)$$

Approximation of "p" (cont.)

Hence, we have:

$$p \approx \frac{3}{4\pi}\left(\frac{\pi}{2}\right)^4 \int_0^{2\pi}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\operatorname{sinc}^2\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]^2\sin\theta\,d\theta\,d\phi$$

Using $\int_0^{2\pi}\int_0^{\pi/2}(\quad)\,d\theta\,d\phi = 4\int_0^{\pi/2}\int_0^{\pi/2}(\quad)\,d\theta\,d\phi$ and factoring out a $(\pi/2)^{-4}$, we then have

$$p \approx \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)\operatorname{sinc}^2\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{1-\left(\frac{2}{\pi}\left(k_x\frac{L}{2}\right)\right)^2}\right]^2\sin\theta\,d\theta\,d\phi$$

Approximation of "p" (cont.)

Next, use [Abromowitz & Stegun]:

$$\frac{\sin x}{x} \approx 1 + a_2 x^2 + a_4 x^4$$

$$\cos x \approx 1 + b_2 x^2 + b_4 x^4$$

$$x = k_y \frac{W}{2}$$

$$x = k_x \frac{L}{2}$$

for $0 \leq x \leq \frac{\pi}{2}$

where

$$a_2 = -0.16605$$
$$a_4 = 0.00761$$
$$b_2 = -0.49670$$
$$b_4 = 0.03705$$

Note: These are <u>not</u> Taylor series, but are approximations that are more uniformly accurate over the entire range.

Approximation of "p" (cont.)

$$\left[\frac{\cos\left(k_x \frac{L}{2}\right)}{1-\left(\frac{2}{\pi}\left(k_x\frac{L}{2}\right)\right)^2}\right]^2$$

The cosine term may thus be approximated as:

$$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx \left[1+b_2x^2+b_4x^4\right]\left[1+\left(\frac{2}{\pi}x\right)^2+\left(\frac{2}{\pi}x\right)^4\right]$$

where we have used use a Taylor series for $\left[1-\left(\frac{2}{\pi}x\right)^2\right]^{-1}$ $\left(\frac{1}{1-z}=1+z+z^2+\ldots\right)$

We then have (keeping terms up to x^4):

$$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx 1+x^2\left[b_2+\frac{4}{\pi^2}\right]+x^4\left(b_4+b_2\frac{4}{\pi^2}+\frac{16}{\pi^4}\right)$$

Approximation of "*p*" (cont.)

Define: $$c_2 = b_2 + \frac{4}{\pi^2}$$

$$c_4 = b_4 + b_2\frac{4}{\pi^2} + \frac{16}{\pi^4}$$

The numerical values are:

$$c_2 = -0.0914153$$

$$c_4 = 7.884 \times 10^{-7} \approx 0$$

Then $$\frac{\cos x}{1-\left(\frac{2}{\pi}x\right)^2} \approx 1 + c_2 x^2 + \cancel{c_4} x^4$$

Approximation of "p" (cont.)

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left(\cos^2\theta \sin^2\phi + \cos^2\phi \right) \operatorname{sinc}^2\left(k_y \frac{W}{2} \right) \left[\frac{\cos\left(k_x \frac{L}{2} \right)}{1 - \left(\frac{2}{\pi} \left(k_x \frac{L}{2} \right) \right)^2} \right]^2 \sin\theta \, d\theta \, d\phi$$

We then have:

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left(\cos^2\theta \sin^2\phi + \cos^2\phi \right)$$

$$\cdot \left[1 + a_2 \left(\frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 + a_4 \left(\frac{k_0 W}{2} \sin\theta \sin\phi \right)^4 \right]^2$$

$$\cdot \left[1 + c_2 \left(\frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 \right]^2 \sin\theta \; d\theta \, d\phi$$

Approximation of "p" (cont.)

Take the squares, and the neglect the following terms:

$$a_4^2,\ a_4 a_2,\ c_2^2$$

We then have:

$$p \approx \frac{3}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\left(\cos^2\theta\sin^2\phi+\cos^2\phi\right)$$

$$\cdot\left[1+2a_2\left(\frac{k_0 W}{2}\sin\theta\sin\phi\right)^2+\left(a_2^2+2a_4\right)\left(\frac{k_0 W}{2}\sin\theta\sin\phi\right)^4\right]$$

$$\cdot\left[1+2c_2\left(\frac{k_0 L}{2}\sin\theta\cos\phi\right)^2\right]\sin\theta d\theta\, d\phi$$

$$a_2 = -0.16605$$
$$a_4 = 0.00761$$
$$b_2 = -0.49670$$
$$b_4 = 0.03705$$
$$c_2 = -0.0914153$$
$$c_4 = 7.884\times10^{-7} \approx 0$$

Approximation of "p" (cont.)

Next, after multiplying the two terms in square brackets together, we also neglect the following terms:

$$a_4 c_2 , a_2^2 c_2$$

$a_2 = -0.16605$

$a_4 = 0.00761$

$b_2 = -0.49670$

$b_4 = 0.03705$

$c_2 = -0.0914153$

$c_4 = 7.884 \times 10^{-7} \approx 0$

We then have:

$$p \approx \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left(\cos^2\theta \sin^2\phi + \cos^2\phi \right)$$

$$\cdot \left[1 + 2a_2 \left(\frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 + \left(a_2^2 + 2a_4 \right) \left(\frac{k_0 W}{2} \sin\theta \sin\phi \right)^4 \right.$$

$$\left. + 2c_2 \left(\frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 + 4a_2 c_2 \left(\frac{k_0 W}{2} \sin\theta \sin\phi \right)^2 \left(\frac{k_0 L}{2} \sin\theta \cos\phi \right)^2 \right]$$

$$\cdot \sin\theta \, d\theta \, d\phi$$

Approximation of "p" (cont.)

Expanding, we have:

$$p = \frac{3}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \{ \; \sin^2\phi \cos^2\theta \sin\theta + \cos^2\phi \sin\theta$$

$$+ 2a_2 \left(\frac{k_0 W}{2}\right)^2 \sin^4\phi \sin^3\theta \cos^2\theta + 2a_2 \left(\frac{k_0 W}{2}\right)^2 \sin^2\phi \cos^2\phi \sin^3\theta$$

$$+ \left(a_2^2 + 2a_4\right) \left(\frac{k_0 W}{2}\right)^4 \sin^6\phi \sin^5\theta \cos^2\theta + \left(a_2^2 + 2a_4\right) \left(\frac{k_0 W}{2}\right)^4 \sin^4\phi \cos^2\phi \sin^5\theta$$

$$+ 2c_2 \left(\frac{k_0 L}{2}\right)^2 \cos^2\phi \sin^2\phi \cos^2\theta \sin^3\theta + 2c_2 \left(\frac{k_0 L}{2}\right)^2 \cos^4\phi \sin^3\theta$$

$$+ 4a_2 c_2 \left(\frac{k_0 W}{2}\right)^2 \left(\frac{k_0 L}{2}\right)^2 \sin^4\phi \cos^2\phi \sin^5\theta \cos^2\theta + 4a_2 c_2 \left(\frac{k_0 W}{2}\right)^2 \left(\frac{k_0 L}{2}\right)^2 \cos^4\phi \sin^2\phi \sin^5\theta \; \} d\theta \, d\phi$$

Approximation of “*p*” (cont.)

All of the ϕ integrals may now be done in closed form:

$$\int_0^{\frac{\pi}{2}} \sin^2\phi \, d\phi = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2\phi \cos^2\phi \, d\phi = \frac{\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\phi \, d\phi = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^6\phi \, d\phi = \frac{5\pi}{32}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\phi \, d\phi = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin^2\phi \cos^4\phi \, d\phi = \frac{\pi}{32}$$

$$\int_0^{\frac{\pi}{4}} \cos^4\phi \, d\phi = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\phi \cos^2\phi \, d\phi = \frac{\pi}{32}$$

Approximation of "p" (cont.)

All of the θ integrals may also be done in closed form:

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}$$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta = \frac{2}{15}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^2\theta \, d\theta = \frac{8}{105}$$

Approximation of "p" (cont.)

This yields:

$$
\begin{aligned}
p = &\frac{3}{\pi} \\
&\cdot\Big\{ \frac{\pi}{4}\frac{1}{3} + \frac{\pi}{4}1 + 2a_2\left(\frac{k_0 W}{2}\right)^2 \frac{3\pi}{16}\frac{2}{15} + 2a_2\left(\frac{k_0 W}{2}\right)^2 \frac{\pi}{16}\frac{2}{3} \\
&+\left(a_2^2 + 2a_4\right)\left(\frac{k_0 W}{2}\right)^4 \frac{5\pi}{32}\frac{8}{105} + \left(a_2^2 + 2a_4\right)\left(\frac{k_0 W}{2}\right)^4 \frac{\pi}{32}\frac{8}{15} \\
&+2c_2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{16}\frac{2}{15} + 2c_2\left(\frac{k_0 L}{2}\right)^2 \frac{3\pi}{16}\frac{2}{3} \\
&+4a_2 c_2\left(\frac{k_0 W}{2}\right)^2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{32}\frac{8}{105} + 4a_2 c_2\left(\frac{k_0 W}{2}\right)^2\left(\frac{k_0 L}{2}\right)^2 \frac{\pi}{32}\frac{8}{15} \Big\}
\end{aligned}
$$

Approximation of "p" (cont.)

Simplifying, we obtain

$$p = 1 + \frac{a_2}{10}(k_0 W)^2 + \left(a_2^2 + 2a_4\right)\left(\frac{3}{560}\right)(k_0 W)^4 + c_2\left(\frac{1}{5}\right)(k_0 L)^2 + a_2 c_2\left(\frac{1}{70}\right)(k_0 W)^2 (k_0 L)^2$$

$$a_2 = -0.16605$$
$$a_4 = 0.00761$$
$$c_2 = -0.0914153$$