ECE 6345

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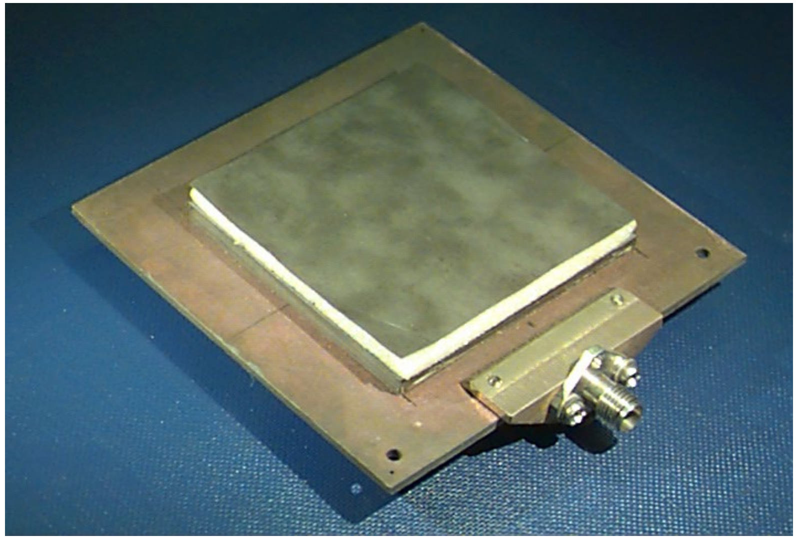

Notes 19

Approximation of “p_c” (cont.)

In this set of notes we approximate the p_c factor for the circular patch to obtain an approximate closed-form CAD formula for it.

Approximation of "p_c" (cont.)

From Notes 18 we have:

$$p_c \equiv \frac{I_c}{I_0} = \frac{\int_0^{\pi/2} C(\theta)\,d\theta}{\int_0^{\pi/2} C_0(\theta)\,d\theta}$$

The p_c term gives the ratio of the power radiated by the actual patch to the power radiated if we ignore the array factor, and collapse the magnetic current down to a single dipole.

$$C(\theta) = \operatorname{tanc}^2(k_{z1} h)\left[|Q(\theta)|^2 J_1'^2(k_0 a \sin\theta) + |P(\theta)|^2 J_{\text{inc}}^2(k_0 a \sin\theta)\right]\sin\theta$$

$$C_0(\theta) \equiv C(\theta)\Big|_{a\to 0}$$

$$p_c \approx 3\int_0^{\pi/2} \sin\theta\left[J_1'^2(k_0 a \sin\theta) + \cos^2\theta\, J_{\text{inc}}^2(k_0 a \sin\theta)\right]d\theta$$

The goal is to approximate this integral for the p_c factor in closed form.

Approximation of "p_c" (cont.)

$$p_c \approx 3\int_0^{\pi/2} \sin\theta \left[J_1'^2\left(k_0 a \sin\theta\right) + \cos^2\theta \;\, J_{\mathrm{inc}}^2\left(k_0 a \sin\theta\right)\right] d\theta$$

Using

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

we have

$$p_c \approx 3\int_0^{\pi/2} \sin\theta \left[\left(J_0(x) - \frac{J_1(x)}{x}\right)^2 + \cos^2\theta \left(\frac{J_1(x)}{x}\right)^2\right] d\theta$$

where

$$x = k_0 a \sin\theta$$

We next approximate the Bessel function terms appearing in this expression.

Approximation of "p_c" (cont.)

From Abramowitz and Stegun, we have the following approximations:

$$J_0(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + a_{10} x^{10} + a_{12} x^{12}$$

$$a_0 = 1.0$$

$$a_2 = -0.24999997$$

$$a_4 = 0.015624948$$

$$a_6 = -4.340008 \times 10^{-4}$$

$$a_8 = 6.77456 \times 10^{-6}$$

$$a_{10} = -6.6799 \times 10^{-8}$$

$$a_{12} = 3.951 \times 10^{-10}$$

error $< 5 \times 10^{-8}, \quad 0 < x \leq +3$

We keep (a_0, a_2, a_4, a_6) error < 0.001 for $x \leq x'_{11} = 1.8412$

Approximation of "p_c" (cont.)

Also, from Abramowitz and Stegun, we have:

$$\frac{J_1(x)}{x} = b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + b_8 x^8 + b_{10} x^{10} + b_{12} x^{12}$$

$$b_0 = 0.5$$

$$b_2 = -0.062499983$$

$$b_4 = 0.0026041448$$

$$b_6 = -5.424265 \times 10^{-5}$$

$$b_8 = 6.75688 \times 10^{-7}$$

$$b_{10} = -5.3788 \times 10^{-9}$$

$$b_{12} = 2.087 \times 10^{-11}$$

error $< 1.3 \times 10^{-3}, \quad -3 \le x \le 3$

We keep $\left(b_0, b_2, b_4, b_6\right)$ error < 0.001 for $x \le x'_{11} = 1.8412$

Approximation of "p_c" (cont.)

Define $c_n = a_n - b_n$

We then have:

$$p_c = 3\int_0^{\pi/2} \sin\theta \left(\sum_{n=0}^{3} c_{2n} A^{2n} \sin^{2n}\theta \right)^2 d\theta + 3\int_0^{\pi/2} \left(\sin\theta - \sin^3\theta\right) \left(\sum_{n=0}^{3} b_{2n} A^{2n} \sin^{2n}\theta \right)^2 d\theta$$

This comes from $\sin\theta\cos^2\theta$.

where

$$A \equiv k_0 a$$

Approximation of "p_c" (cont.)

In order to evaluate the integrals that appear, we use:

$$\left(\sum_{n=0}^{3} c_{2n} A^{2n} \sin^{2n}\theta\right)^2 = \left(\sum_{n=0}^{3} c_{2n} A^{2n} \sin^{2n}\theta\right)\left(\sum_{m=0}^{3} c_{2m} A^{2m} \sin^{2m}\theta\right)$$

$$= \sum_{n=0}^{3}\sum_{m=0}^{3} c_{2n} c_{2m} A^{2n+2m} \sin^{2n+2m}\theta$$

Similarly,

$$\left(\sum_{n=0}^{3} b_{2n} A^{2n} \sin^{2n}\theta\right)^2 = \left(\sum_{n=0}^{3} b_{2n} A^{2n} \sin^{2n}\theta\right)\left(\sum_{m=0}^{3} b_{2m} A^{2m} \sin^{2m}\theta\right)$$

$$= \sum_{n=0}^{3}\sum_{m=0}^{3} b_{2n} b_{2m} A^{2n+2m} \sin^{2n+2m}\theta$$

Approximation of “p_c” (cont.)

We had:

$$p_c = 3\int_0^{\pi/2} \sin\theta \left(\sum_{n=0}^{3} c_{2n} A^{2n} \sin^{2n}\theta \right)^2 d\theta + 3\int_0^{\pi/2} \left(\sin\theta - \sin^3\theta\right) \left(\sum_{n=0}^{3} b_{2n} A^{2n} \sin^{2n}\theta \right)^2 d\theta$$

Hence, we now have:

$$p_c = 3\sum_{n=0}^{3}\sum_{m=0}^{3} c_{2n}c_{2m}A^{2(n+m)} \int_0^{\pi/2} \sin^{2(n+m)+1}\theta \, d\theta$$

$$+3\sum_{n=0}^{3}\sum_{m=0}^{3} b_{2n}b_{2m}A^{2(n+m)} \left\{ \int_0^{\pi/2} \sin^{2(n+m)+1}\theta \, d\theta - \int_0^{\pi/2} \sin^{2(n+m)+3}\theta \, d\theta \right\}$$

Approximation of "p_c" (cont.)

Define: $$S_n \equiv \int_0^{\pi/2} \sin^n \theta \, d\theta$$

General formula: $$S_{2m+1} = \frac{\left(2^m \, m!\right)^2}{(2m+1)!} \qquad m \geq 1$$

Values:

$$S_0 = \pi / 2$$
$$S_1 = 1.0$$
$$S_3 = 2/3$$
$$S_5 = 8/15$$
$$S_7 = 16/35$$
$$S_9 = 128/315$$
$$S_{11} = 256/693$$
$$S_{13} = 1024/3003$$
$$S_{15} = 2048/6435$$

Approximation of "p_c" (cont.)

We then have:

$$p_c = 3\sum_{n=0}^{3}\sum_{m=0}^{3} c_{2n}c_{2m}A^{2(n+m)}S_{2(n+m)+1} + 3\sum_{n=0}^{3}\sum_{m=0}^{3} b_{2n}b_{2m}A^{2(n+m)}\left(S_{2(n+m)+1} - S_{2(n+m)+3}\right)$$

Collecting the terms of the series, we have $p_c = \sum_{k=0}^{6} A^{2k} e_{2k} \qquad \left(A = k_0 a\right)$

where

$$e_0 = 3c_0^2 S_1 + 3b_0^2\left(S_1 - S_3\right)$$
$$e_2 = 6\,c_0 c_2 S_3 + 6\,b_0 b_2\left(S_3 - S_5\right)$$
$$e_4 = 3\left(c_2^2 + 2c_0 c_4\right)S_5 + 3\left(b_2^2 + 2b_0 b_4\right)\left(S_5 - S_7\right)$$
$$e_6 = 3\left(2c_0 c_6 + 2c_2 c_4\right)S_7 + 3\left(2b_0 b_6 + 2b_2 b_4\right)\left(S_7 - S_9\right)$$
$$e_8 = 3\left(c_4^2 + 2c_2 c_6\right)S_9 + 3\left(b_4^2 + 2b_2 b_6\right)\left(S_9 - S_{11}\right)$$
$$e_{10} = 6\,c_4 c_6 S_{11} + 6\,b_4 b_6\left(S_{11} - S_{13}\right)$$
$$e_{12} = 3c_6^2 S_{13} + 3b_6^2\left(S_{13} - S_{15}\right)$$

Approximation of "p_c" (cont.)

Hence, we have:

$$p_c \approx \sum_{k=0}^{6} \left(k_0 a\right)^{2k} e_{2k}$$

where

$$\begin{aligned}
e_0 &= 1 \\
e_2 &= -0.400000 \\
e_4 &= 0.0785710 \\
e_6 &= -7.27509 \times 10^{-3} \\
e_8 &= 3.81786 \times 10^{-4} \\
e_{10} &= -1.09839 \times 10^{-5} \\
e_{12} &= 1.47731 \times 10^{-7}
\end{aligned}$$