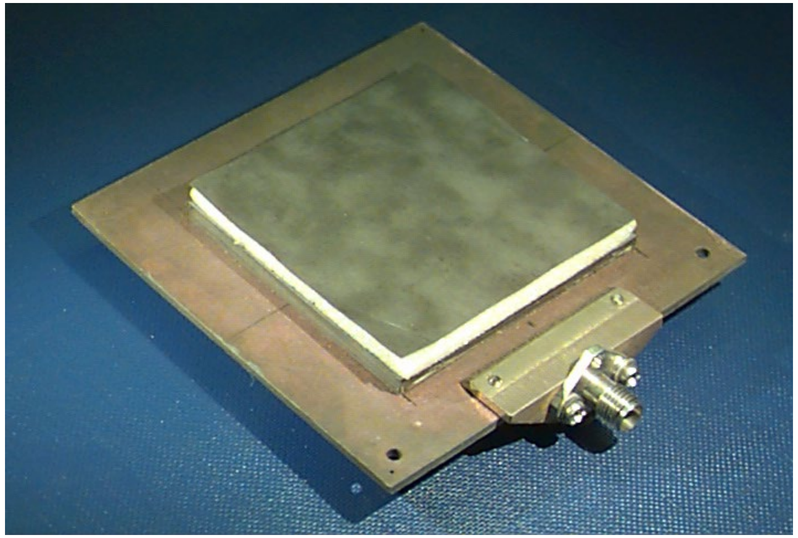
ECE 6345

Fall 2024

Prof. David R. Jackson
ECE Dept.



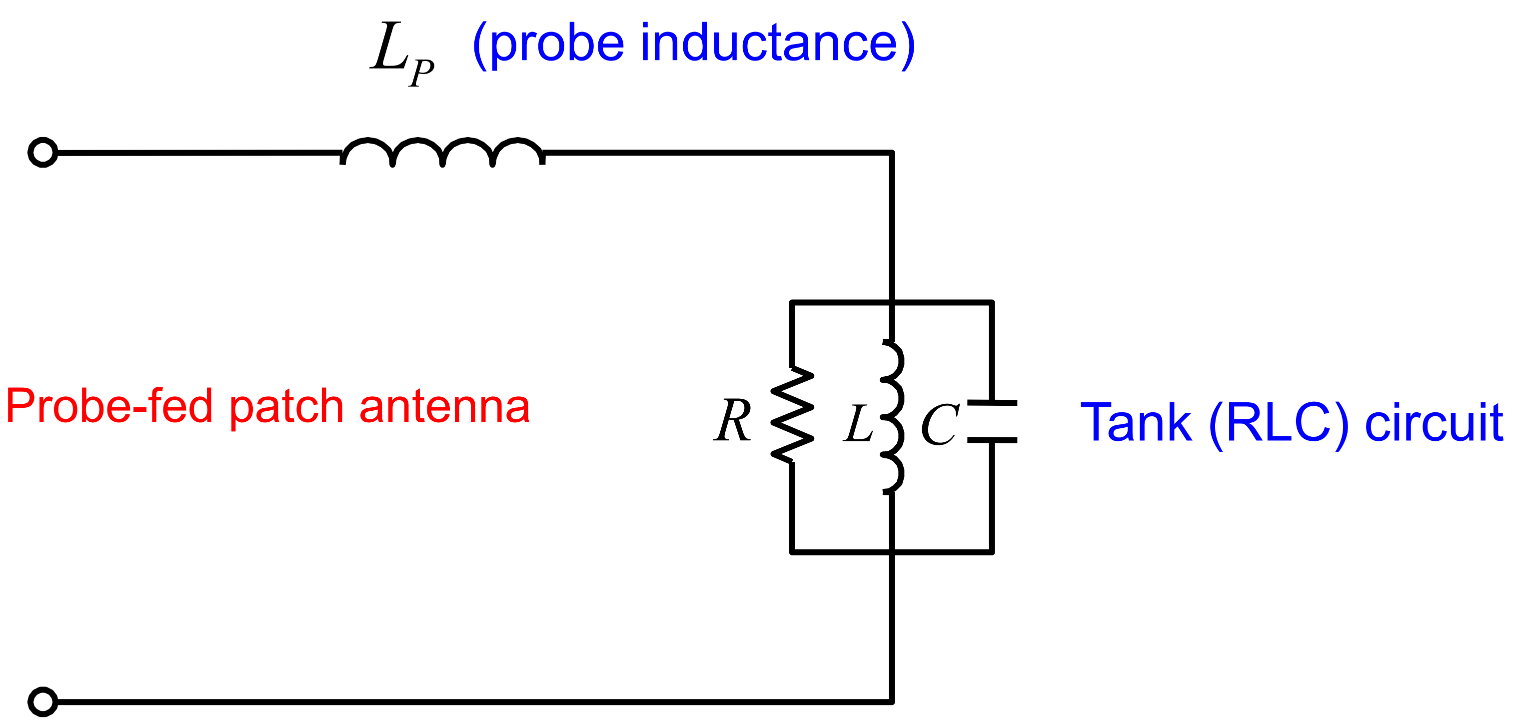
Notes 2

Overview

In this set of notes we discuss the CAD circuit model of the microstrip antenna.

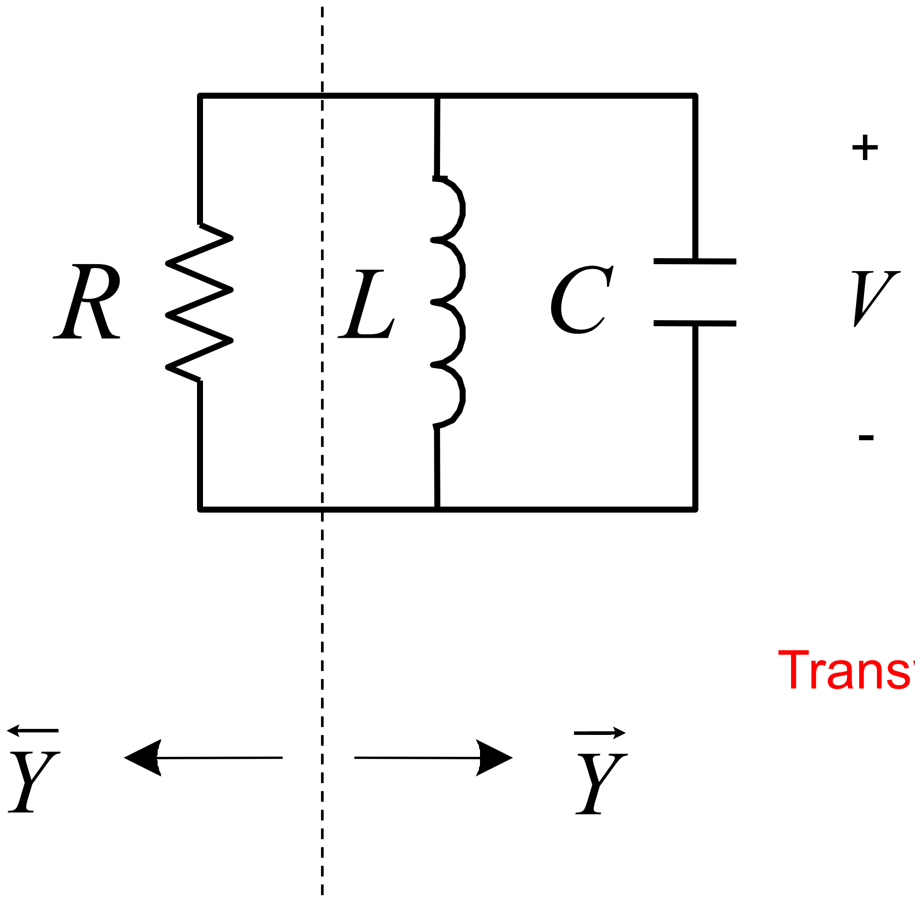
- Discuss complex resonance frequency
- Derive formula for Q
- Derive formula for input impedance
- Derive formula for impedance bandwidth

CAD Model of Microstrip Antennas



The circuit model is justified from the eigenfunction method in the cavity model, discussed later.

Tank Circuit: complex resonance frequency



$$G \equiv \frac{1}{R}$$

Transverse Resonance Equation (TRE):

$$\overleftarrow{Y} = -\overrightarrow{Y}$$

The complex resonance frequency is denoted as ω_0.

Complex Resonance Frequency (cont.)

TRE:

$$G = -\left[j\omega_0 C + \frac{1}{j\omega_0 L} \right]$$

$$\Rightarrow j\omega_0 LG = \omega_0^2 LC - 1$$

$$\Rightarrow \omega_0^2 \left(LC\right) + \omega_0 \left(-jLG\right) + \left(-1\right) = 0$$

$$\omega_0 = \frac{jLG \pm \sqrt{-L^2G^2 + 4LC}}{2LC}$$

$$G \to 0, \quad \omega_0 \to \frac{1}{\sqrt{LC}}$$

so choose + sign

Complex Resonance Frequency (cont.)

$$\omega_0 = j\frac{1}{2}\left(\frac{1}{RC}\right) + \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{1}{4}\frac{L}{R^2C}}$$

Denote: $\omega_0 = \omega_0' + j\omega_0''$

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{1}{4}\frac{L}{R^2C}}$$

$$\omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Complex Resonance Frequency (cont.)

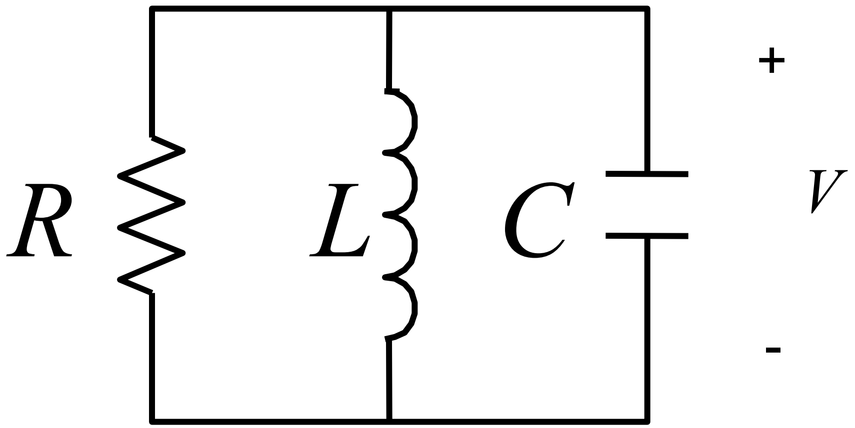
$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{1}{4}\frac{L}{R^2C}} \qquad \omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Assume $R >> \sqrt{\frac{L}{C}}$ (a good resonator)

We then have:

$$\omega_0' \approx \frac{1}{\sqrt{LC}} \qquad \omega_0'' = \frac{1}{2}\left(\frac{1}{RC}\right)$$

Natural Response (no source)



The complex resonance frequency is ω_0.

$$\omega_0 = \omega_0' + j\omega_0''$$

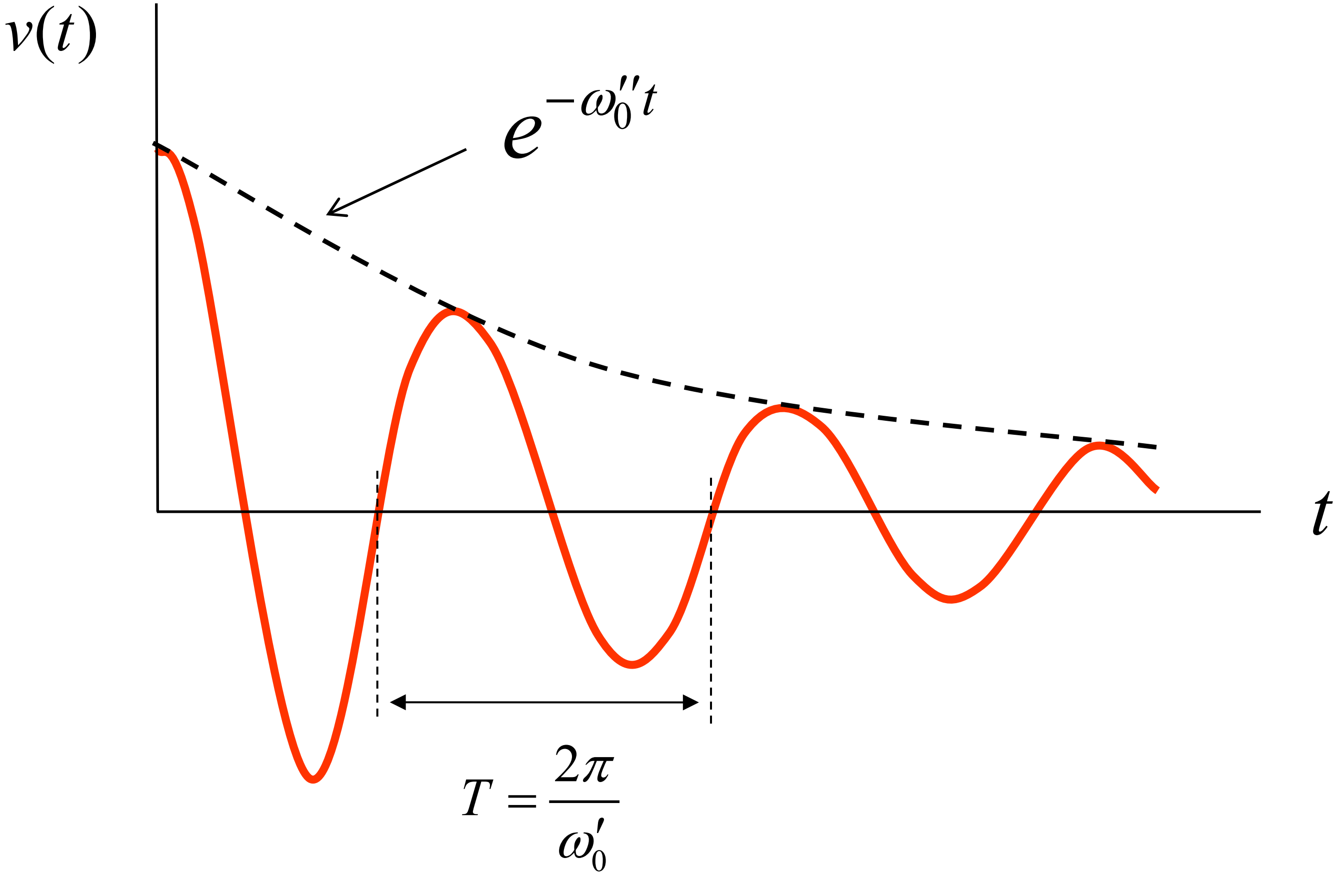
In the time domain: $v(t) = \mathrm{Re}\left(V\, e^{j\omega_0 t}\right)$ (Assume phasor voltage $V = 1$)

so

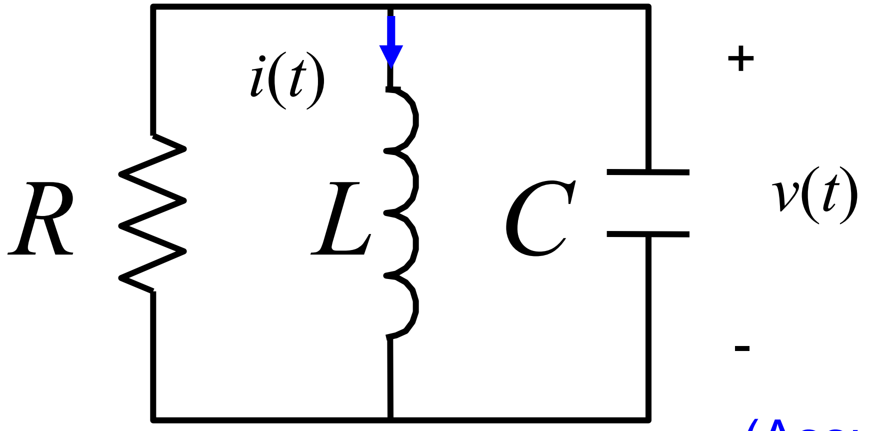
$$v(t) = \mathrm{Re}\left(e^{j\omega_0' t} e^{-\omega_0'' t}\right) = e^{-\omega_0'' t}\cos\left(\omega_0' t\right)$$

Natural Response (cont.)

$$v(t) = e^{-\omega_0'' t}\cos\left(\omega_0' t\right)$$



Stored Energy



For the capacitor:

$$U_E(t) = \frac{1}{2} C v^2(t)$$

$$= \frac{1}{2} C e^{-2\omega_0'' t} \cos^2(\omega_0' t)$$

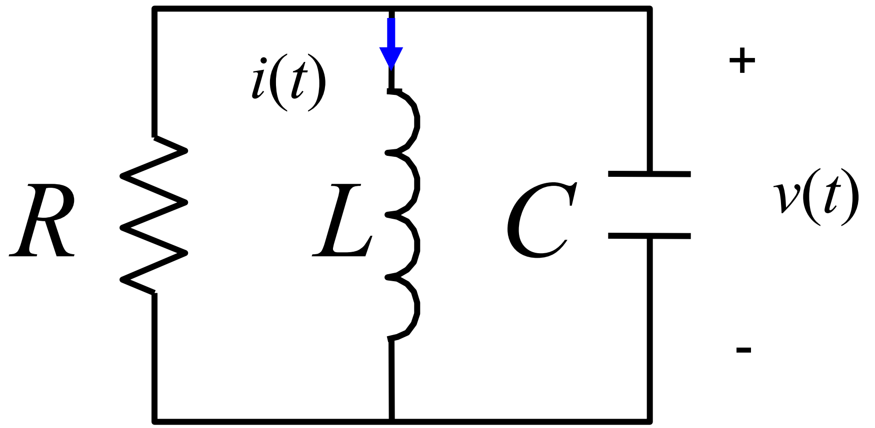
(Assume $V = 1$)

For the inductor: $$I = \frac{V}{j\omega_0 L} = \frac{1}{j\omega_0 L} \approx \frac{1}{j\omega_0' L}$$

Therefore,

$$i(t) = \mathrm{Re}\left(I\, e^{j\omega_0 t}\right) \approx \frac{1}{\omega_0' L} \mathrm{Re}\left(\frac{1}{j} e^{-\omega_0'' t} e^{+j\omega_0' t}\right) = \frac{1}{\omega_0' L} e^{-\omega_0'' t} \sin(\omega_0' t)$$

Stored Energy (cont.)



$$U_H(t) = \frac{1}{2} L i^2(t)$$

$$= \frac{1}{2} L \left(\frac{1}{\omega_0' L} \right)^2 e^{-2\omega_0'' t} \sin^2(\omega_0' t)$$

$$= \frac{1}{2} C e^{-2\omega_0'' t} \sin^2(\omega_0' t)$$

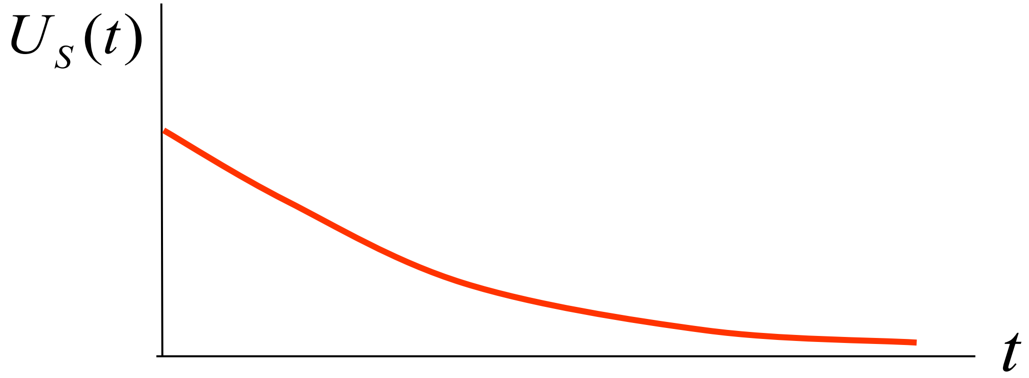
Note: $$< U_E(t) > = < U_H(t) > = \frac{1}{4} C e^{-2\omega_0'' t}$$

Also, note that $$U_S(t) = U_E(t) + U_H(t) = \frac{1}{2} C e^{-2\omega_0'' t}$$

Stored Energy (cont.)

Hence, we have:

$$U_S\left(t\right) = U_S(0)\; e^{-2\omega_0'' t}$$



Q of Cavity

$$Q \equiv 2\pi \left(\frac{U_S}{U_D^T} \right)$$

U_S = stored energy

U_D^T = energy dissipated per cycle

$$Q = \frac{2\pi}{T} \left(\frac{U_S}{U_D^T / T} \right)$$

or

$$Q \equiv \omega_0' \left(\frac{U_S}{P_D^{\text{ave}}} \right)$$

P_D^{ave} = average power dissipated

(This includes radiation loss.)

Q of Cavity (cont.)

Power dissipation in circuit: $P_D(t) = G\,v(t)^2$

$$Q = \omega_0' \left(\frac{\frac{1}{2} C e^{-2\omega_0'' t}}{< G\,v(t)^2 >} \right) = \omega_0' \left(\frac{\frac{1}{2} C\, e^{-2\omega'' t}}{G < e^{-2\omega_0'' t} \cos^2\left(\omega_0' t\right) >} \right)$$

$$= \omega_0' \left(\frac{\frac{1}{2} C\, e^{-2\omega_0'' t}}{\frac{1}{2} G\, e^{-2\omega_0'' t}} \right)$$

$$= \frac{\omega_0' C}{G}$$

$$= \omega_0' RC$$

Note :

$$< e^{-2\omega_0'' t} \cos^2\left(\omega_0' t\right) >$$
$$\approx \; e^{-2\omega_0'' t} < \cos^2\left(\omega_0' t\right) >$$
$$= e^{-2\omega_0'' t} \left(\frac{1}{2} \right)$$

Q of Cavity (cont.)

We then have:

$$Q = \omega_0' RC = \frac{1}{\omega_0'}\left(\frac{R}{L}\right) = R\sqrt{\frac{C}{L}}$$

Recall that $\omega_0' \approx \dfrac{1}{\sqrt{LC}}$ $\qquad \omega_0'' = \dfrac{1}{2}\left(\dfrac{1}{RC}\right)$

Hence

$$\omega_0 = \omega_0' + j\omega_0'' \approx \omega_0'\left(1 + j\frac{1}{2}\left(\frac{1}{RC}\right)\frac{1}{\omega_0'}\right)$$

Next, put this in terms of Q.

Q of Cavity (cont.)

$$\omega_0 = \omega_0' + j\omega_0'' = \omega_0'\left(1 + j\frac{1}{2}\left(\frac{1}{RC}\right)\frac{1}{\omega_0'}\right) \approx \omega_0'\left(1 + j\frac{1}{2}\left(\frac{1}{RC}\right)\sqrt{LC}\right) = \omega_0'\left(1 + j\frac{1}{2}\left(\frac{1}{R}\right)\sqrt{\frac{L}{C}}\right)$$

Hence

$$\omega_0 \approx \omega_0'\left(1 + j\frac{1}{2Q}\right)$$

$\left(\textbf{Recall:}\ Q = \omega_0' RC = \frac{1}{\omega_0'}\left(\frac{R}{L}\right) = R\sqrt{\frac{C}{L}}\right)$

so

$$\omega_0'' = \frac{1}{2Q}\omega_0'$$

or

$$Q = \frac{1}{2}\frac{\omega_0'}{\omega_0''}$$

Q of Cavity (Cont.)

We can thus write

$$v(t) = e^{-\left(\frac{\omega_0'}{2Q}\right)t} \cos\left(\omega_0' t\right)$$

$$U_S(t) = U_S(0)\, e^{-\left(\frac{\omega_0'}{Q}\right)t}$$

Input Impedance

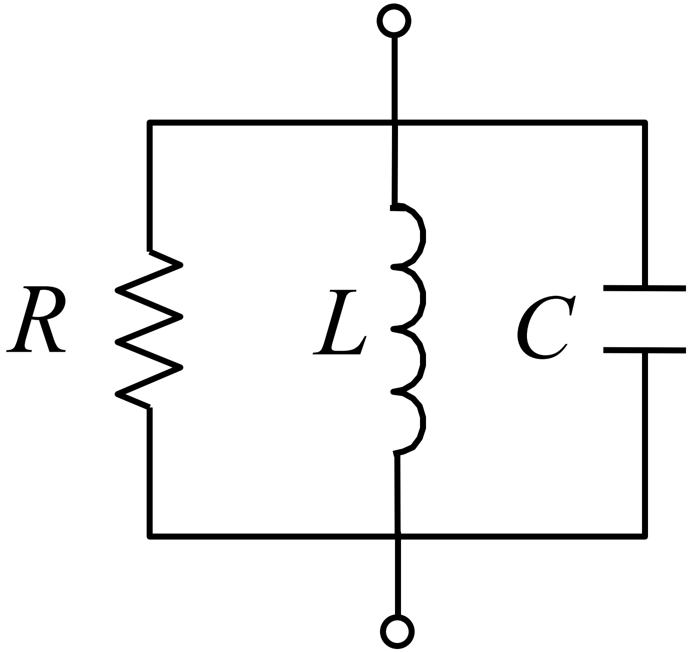
$$Y_{RLC} = G + j\omega C + \frac{1}{j\omega L}$$

$$Z_{RLC} = \frac{1}{G + j\omega C + \dfrac{1}{j\omega L}}$$

$$= \frac{R}{1 + j\omega RC + \dfrac{R}{j\omega L}}$$

$$= \frac{R}{1 + j\left(\omega RC - \dfrac{R}{\omega L}\right)}$$

The probe inductance is neglected here.



Input Impedance (cont.)

We can write this as:

$$Z_{RLC} = \frac{R}{1 + j\left(\frac{\omega}{\omega_0'}\left(\omega_0' RC\right) - \frac{R}{\omega_0' L}\left(\frac{\omega_0'}{\omega}\right)\right)}$$

$$= \frac{R}{1 + j\left(\frac{\omega}{\omega_0'} Q - Q\left(\frac{\omega_0'}{\omega}\right)\right)}$$

Define $f_r \equiv \frac{f}{f_0} = \frac{\omega}{\omega_0'}$ where $f_0 \equiv \frac{\omega_0'}{2\pi}$

(real resonance frequency)

Then we have: $Z_{RCL} = \frac{R}{1 + jQ\left(f_r - \frac{1}{f_r}\right)}$

Input Impedance (cont.)

Define:

$$F \equiv f_r - \frac{1}{f_r}$$

$$F = \frac{1}{f_r}\left(f_r^2 - 1\right)$$

$$= \frac{1}{f_r}\left(f_r - 1\right)\left(f_r + 1\right)$$

$$\approx 2\left(f_r - 1\right) \text{ for } f_r \approx 1$$

Hence, we have:

$$Z_{RLC} = \frac{R}{1 + jQF} = \frac{R}{1 + jQ\left(f_r - \dfrac{1}{f_r}\right)} \approx \frac{R}{1 + j2Q\left(f_r - 1\right)}$$

Input Impedance (cont.)

$$Z_{RLC} = R\left[\frac{1}{1+jQF}\right]$$

Define:

$$x \equiv QF = Q\left(f_r - \frac{1}{f_r}\right) \approx 2Q\left(f_r - 1\right)$$

$$\bar{Z}_{RLC} \equiv \frac{Z_{RLC}}{R}$$

We then have:

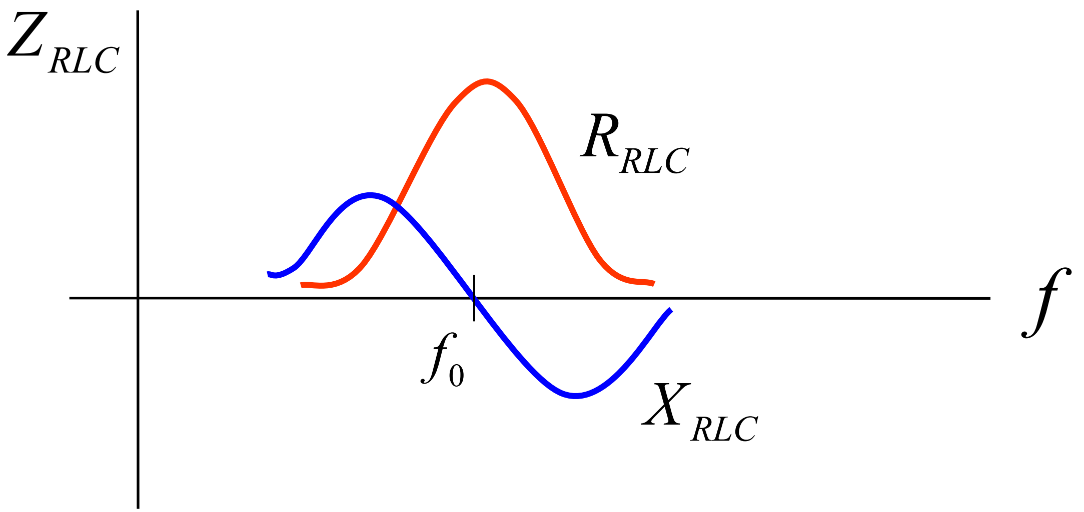
$$\bar{Z}_{RLC} = \frac{1}{1+jx}$$

$$\bar{R}_{RLC} = \frac{1}{1+x^2}$$

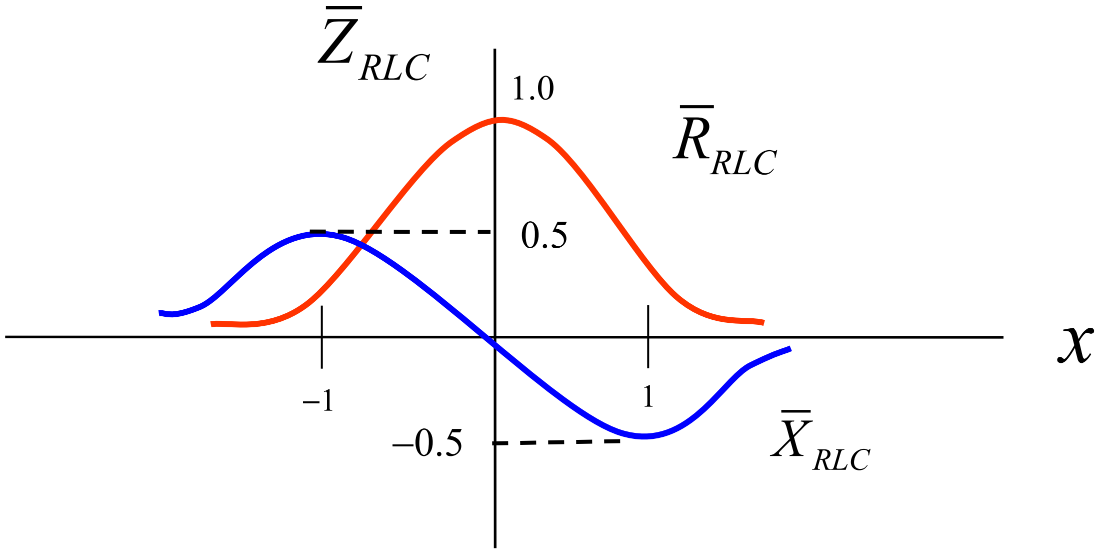
$$\bar{X}_{RLC} = \frac{-x}{1+x^2}$$

Input Impedance (cont.)

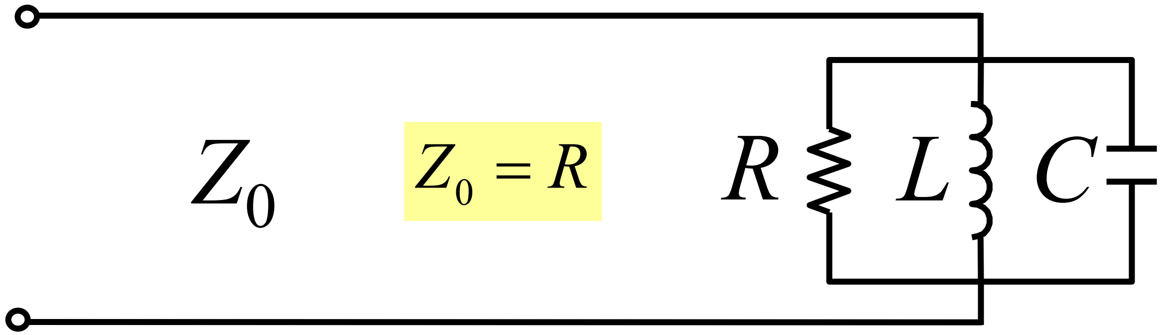
Frequency domain:



Normalized frequency (x) domain:



Reflection Coefficient



$$\Gamma = \frac{Z_{RLC} - Z_0}{Z_{RLC} + Z_0} = \frac{\bar{Z}_{RLC} - 1}{\bar{Z}_{RLC} + 1}$$

$$= \frac{1 - \bar{Y}_{RLC}}{1 + \bar{Y}_{RLC}} = \frac{1 - (1 + jx)}{1 + (1 + jx)}$$

$$= \frac{-jx}{2 + jx}$$

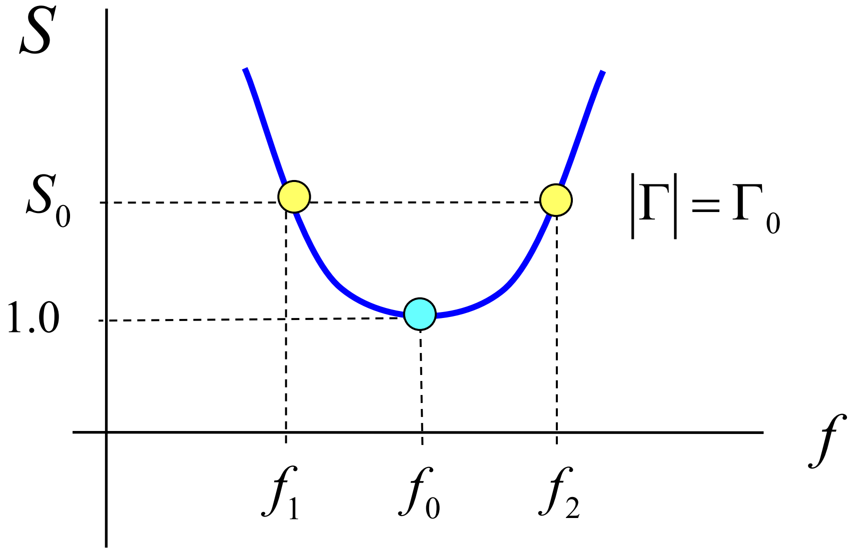
Bandwidth

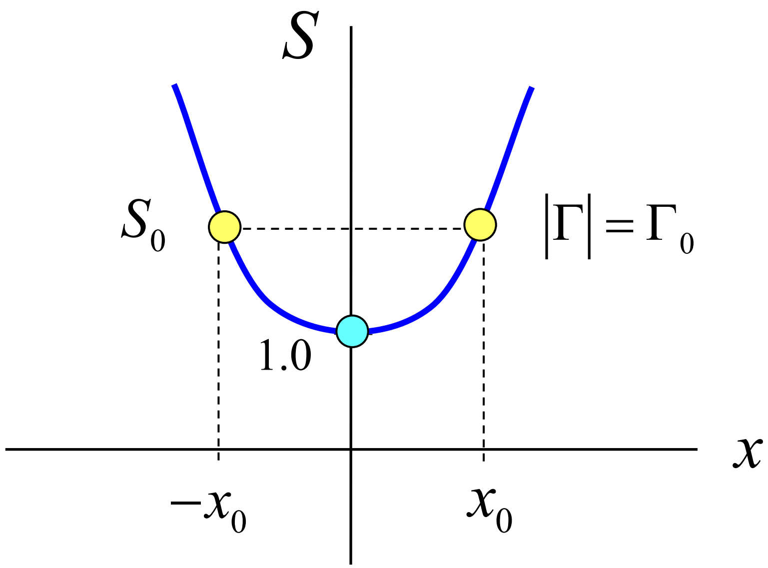
$$|\Gamma| = \frac{|x|}{\sqrt{4+x^2}}$$

$$S = \mathrm{SWR} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Bandwidth definition is based on SWR $< S_0$

(The value S_0 is often chosen as 2.0.)





Bandwidth (cont.)

Fractional (relative) bandwidth: $$\mathrm{BW} = \frac{f_2 - f_1}{f_0} = f_{r2} - f_{r1}$$

Recall that $$x \equiv QF = Q\left(f_r - \frac{1}{f_r}\right)$$

We can solve for f_r in terms of x:

$$f_r - \frac{1}{f_r} = \frac{x}{Q} \quad \Rightarrow \quad f_r^2 - f_r\left(\frac{x}{Q}\right) - 1 = 0$$

so

$$f_r = \frac{\dfrac{x}{Q} \pm \sqrt{\dfrac{x^2}{Q^2} + 4}}{2}$$

Bandwidth (cont.)

To determine correct sign, enforce that $x \to 0, \quad f_r \to 1$

(Therefore, choose the plus sign.)

Hence

$$f_r = \frac{\frac{x}{Q} + \sqrt{\frac{x^2}{Q^2} + 4}}{2}$$

Therefore

$$f_{r2} = \frac{\frac{x_0}{Q} + \sqrt{\frac{x_0^2}{Q^2} + 4}}{2}$$

$$f_{r1} = \frac{-\frac{x_0}{Q} + \sqrt{\frac{x_0^2}{Q^2} + 4}}{2}$$

Bandwidth (cont.)

Hence, $\mathrm{BW} = f_{r2} - f_{r1} = \dfrac{x_0}{Q}$

Now we need to solve for x_0:

$$S_0 = \frac{1+|\Gamma_0|}{1-|\Gamma_0|} \quad \text{so} \quad |\Gamma_0| = \frac{S_0 - 1}{S_0 + 1}$$

Also, $$|\Gamma| = \frac{|x|}{\sqrt{4+x^2}} \Rightarrow |\Gamma_0| = \frac{|x_0|}{\sqrt{4+x_0^{\,2}}} = \frac{x_0}{\sqrt{4+x_0^{\,2}}}$$

Bandwidth (cont.)

Therefore $$\frac{x_0}{\sqrt{4+x_0^2}} = \frac{S_0 - 1}{S_0 + 1}$$

so

$$\frac{x_0^2}{4+x_0^2} = \left(\frac{S_0 - 1}{S_0 + 1}\right)^2 \equiv A$$

Thus, we have $$4A + x_0^2 A = x_0^2$$

or

$$x_0^2\left(1 - A\right) = 4A$$

Bandwidth (cont.)

The solution is:

$$x_0 = 2\sqrt{\frac{A}{1-A}}$$

$$= 2\frac{\left(\frac{S_0 - 1}{S_0 + 1}\right)}{\sqrt{1-\left(\frac{S_0 - 1}{S_0 + 1}\right)^2}}$$

$$= 2\frac{S_0 - 1}{\sqrt{(S_0+1)^2-(S_0-1)^2}}$$

$$= 2\left(\frac{S_0 - 1}{\sqrt{4S_0}}\right)$$

Recall : $A \equiv \left(\frac{S_0 - 1}{S_0 + 1}\right)^2$

Bandwidth (cont.)

Hence, $$x_0 = \frac{S_0 - 1}{\sqrt{S_0}}$$

We then have $$\mathrm{BW} = \frac{1}{Q}\left(\frac{S_0 - 1}{\sqrt{S_0}}\right)$$

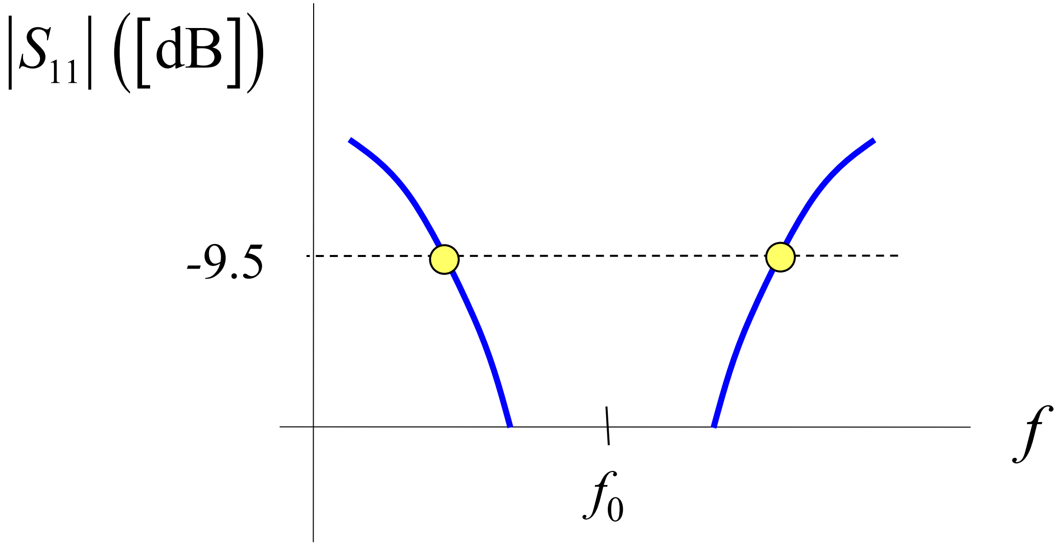
For $S_0 = 2$ we have: $$\mathrm{BW} = \frac{1}{\sqrt{2}Q}$$

Bandwidth (cont.)

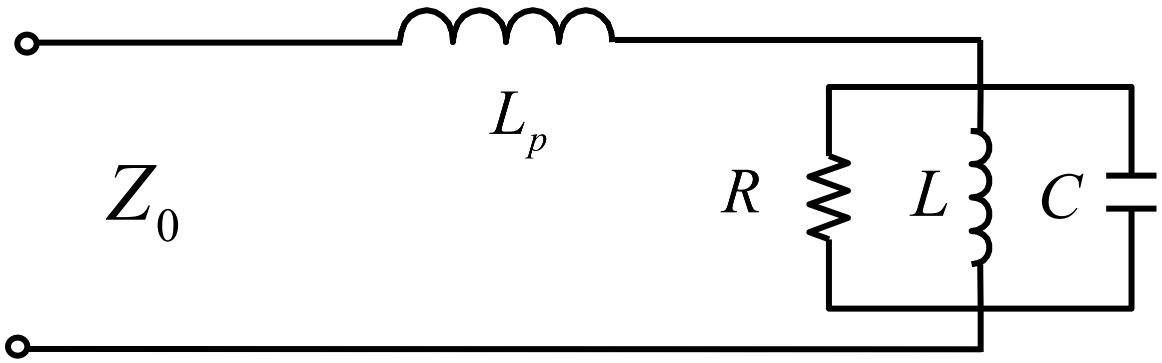
Assume $S_0 = \text{SWR} = 2.0$

$$|\Gamma| = |S_{11}| = \frac{\text{SWR}-1}{\text{SWR}+1} = \frac{1}{3}$$

$$20\log_{10}|S_{11}| = -9.5\ [\text{dB}]$$



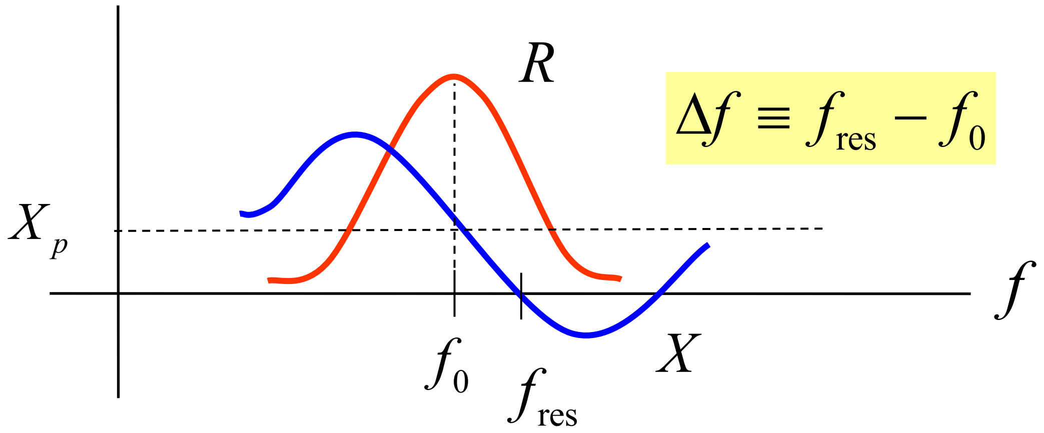
Complete Model (with Probe Inductance)



$$X_p = \omega L_p \approx \omega_0' L_p$$
$$Z_{\text{in}} = jX_p + Z_{RLC}$$

$$\frac{\Delta f}{f_0} \approx (\text{BW})\left(\frac{1}{\sqrt{2}}\right)\left(\frac{X_p}{R}\right)$$

(This will be derived in a HW problem.)



$$\Delta f \equiv f_{\text{res}} - f_0$$

Complete Model (with Probe Inductance) (cont.)

Outline of Derivation of Formula for Δf

Define $\bar{X}_p \equiv X_p / R$

In terms of the normalized frequency variable x, the resonance frequency f_{res} where the input impedance is purely real, corresponds to

$$x_{\text{res}} = \frac{1-\sqrt{1-4\bar{X}_p^2}}{2\bar{X}_p}$$

(This will be derived in a HW problem.)

If $X_p << R$ then $x_{\text{res}} \approx \bar{X}_p$

(This follows from a binomial expansion of the square-root term in the numerator.)

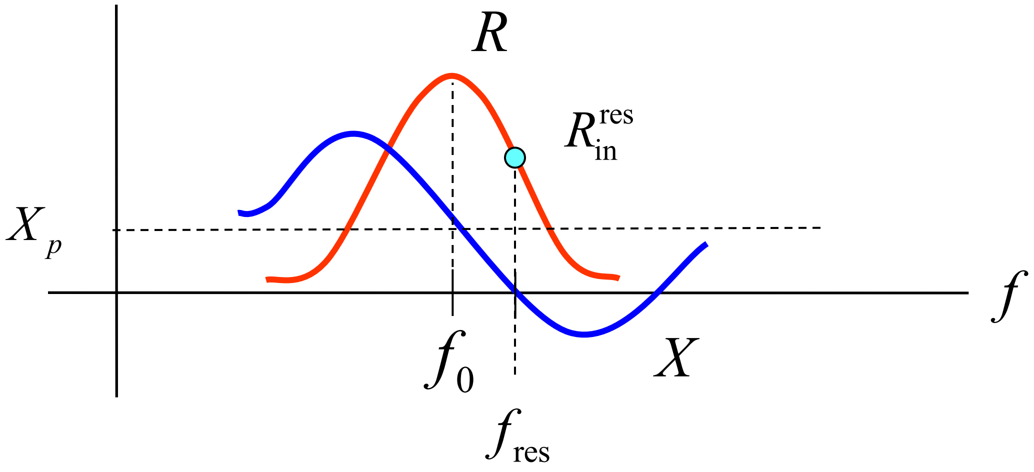
Recall: $f_r = \dfrac{\dfrac{x}{Q} + \sqrt{\dfrac{x^2}{Q^2} + 4}}{2}, \quad f_r \equiv f / f_0$

$$f_r^{\text{res}} = \frac{\dfrac{x_{\text{res}}}{Q} + \sqrt{\dfrac{x_{\text{res}}^2}{Q^2} + 4}}{2} \approx 1 + \frac{x_{\text{res}}}{2Q} \approx 1 + \frac{\bar{X}_p}{2Q} \qquad (x_{\text{res}} \ll Q)$$

Complete Model (with Probe Inductance) (cont.)

At the resonance frequency, the input resistance is then:

$$\bar{R}_{RLC} = \frac{1}{1+x^2} \Rightarrow R_{\text{in}}^{\text{res}} = \frac{R}{1+x_{\text{res}}^2} \Rightarrow R_{\text{in}}^{\text{res}} \approx \frac{R}{1+\left(\frac{X_p}{R}\right)^2}$$



Complete Model (with Probe Inductance) (cont.)

$$R_{\text{in}}^{\text{res}} \approx \frac{R}{1+\left(\dfrac{X_p}{R}\right)^2}$$

Note that the probe reactance changes the input resistance at resonance.

Given a specified value of the input resistance at resonance (e.g., $R_{\text{in}}^{\text{res}} = 50\ \Omega$), we wish to solve for the corresponding value of R.

Note that the CAD formula for resonant input resistance (in the short-course notes) gives us the value of R in terms of the feed location.

Complete Model (with Probe Inductance) (cont.)

To solve for R, use

$$R = R_{\text{in}}^{\text{res}} + R_{\text{in}}^{\text{res}}\left(\frac{X_p^2}{R^2}\right)$$

and solve iteratively:

$$R^{(i)} = R_{\text{in}}^{\text{res}} + R_{\text{in}}^{\text{res}}\left(\frac{X_p^2}{\left(R^{(i-1)}\right)^2}\right)$$

Zero iteration:

$$R = R_{\text{in}}^{\text{res}}$$

First iteration:

$$R = R_{\text{in}}^{\text{res}} + \left(\frac{X_p^2}{R_{\text{in}}^{\text{res}}}\right)$$

Second iteration:

$$R = R_{\text{in}}^{\text{res}} + R_{\text{in}}^{\text{res}}\left(\frac{X_p^2}{\left(R_{\text{in}}^{\text{res}} + \dfrac{X_p^2}{R_{\text{in}}^{\text{res}}}\right)^2}\right)$$