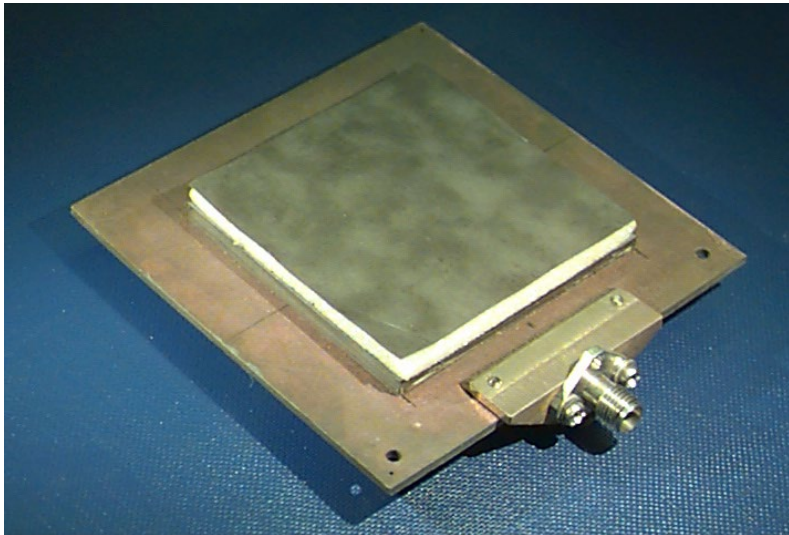


ECE 6345

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Notes 21

Overview

- ❖ In this set of notes we calculate CAD formulas for the **directivity, gain, efficiency, and input resistance of the circular patch**, which are accurate for a thin substrate.
- ❖ The formulas are based on the CAD formula for Q_{sp} that was derived in Notes 19.

Directivity

We have:

$$D(0,0) \equiv \frac{4\pi r^2 S_r(r,0,0)}{P_{\text{sp}}}$$

where

$$S_r(r,\theta,\phi) = \frac{1}{2\eta_0} \left[|E_\theta|^2 + |E_\phi|^2 \right]$$

From Notes 17:

$$E_\theta(r,\theta,\phi) = \frac{E_0}{\eta_0} (ah) \cos\phi \operatorname{tanc}(k_{z1}h) Q(\theta) 2\pi J'_1(k_0 a \sin\theta)$$

$$E_\phi(r,\theta,\phi) = -\frac{E_0}{\eta_0} (ah) \sin\phi \operatorname{tanc}(k_{z1}h) P(\theta) 2\pi J_{\text{inc}}(k_0 a \sin\theta)$$

$$|E_0| = \frac{\omega \mu_0}{4\pi r} = \frac{k_0 \eta_0}{4\pi r}$$

Assumption:

$$E_z(a,\phi) = \cos\phi$$

Directivity (cont.)

We then have for the numerator:

$$4\pi r^2 S_r(r, \theta, \phi) = 4\pi r^2 \frac{1}{2\eta_0} \left(\frac{k_0 \eta_0}{4\pi r} \frac{1}{\eta_0} \right)^2 (ah)^2 (2\pi)^2 \\ \cdot \tan^2(k_{z1}h) \left[\cos^2 \phi |Q(\theta)|^2 J_1'^2(k_0 a \sin \theta) + \sin^2 \phi |P(\theta)|^2 J_{\text{inc}}^2(k_0 a \sin \theta) \right]$$

Simplify:

$$4\pi r^2 S_r(r, \theta, \phi) = 4\cancel{\pi}^2 \frac{1}{2\eta_0} \left(\frac{k_0 \cancel{\eta_0}}{4\cancel{\pi} \cancel{r}} \frac{1}{\cancel{\eta_0}} \right)^2 (ah)^2 (2\cancel{\pi})^2 \\ \cdot \tan^2(k_{z1}h) \left[\cos^2 \phi |Q(\theta)|^2 J_1'^2(k_0 a \sin \theta) + \sin^2 \phi |P(\theta)|^2 J_{\text{inc}}^2(k_0 a \sin \theta) \right]$$

Directivity (cont.)

We thus have for the numerator:

$$4\pi r^2 S_r(r, \theta, \phi) = \frac{\pi}{2\eta_0} (k_0 a)^2 h^2 \tan^2(k_{z1} h) \\ \cdot \left[\cos^2 \phi |Q(\theta)|^2 J_1'^2(k_0 a \sin \theta) + \sin^2 \phi |P(\theta)|^2 J_{\text{inc}}^2(k_0 a \sin \theta) \right]$$

We now let $\theta \rightarrow 0$ to calculate the **numerator** of the directivity expression.

$$\theta \rightarrow 0 \Rightarrow x \rightarrow 0 \\ (x \equiv k_0 a \sin \theta)$$

As $x \rightarrow 0$:

$$J_1'(x) \rightarrow \frac{1}{2}$$

$$J_{\text{inc}}(x) = \frac{J_1(x)}{x} \rightarrow \frac{1}{2}$$

Directivity (cont.)

Hence, we have:

$$4\pi r^2 S_r(r, 0, 0) = \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 \tan^2(k_{z1} h) \left[(1) |Q(0)|^2 + \cancel{\sin^2 \phi} |P(0)|^2 \right]$$

where

$$P(\theta) = \cos \theta (1 - \Gamma^{\text{TE}}(\theta)) = \frac{2 \cos \theta}{1 + j \left(\frac{\mu_r \cos \theta}{N_1(\theta)} \right) \tan(k_0 h N_1(\theta))}$$

$$Q(\theta) = 1 - \Gamma^{\text{TM}}(\theta) = \frac{2}{1 + j \left(\frac{N_1(\theta) \sec \theta}{\epsilon_r} \right) \tan(k_0 h N_1(\theta))}$$

We then see that

$$Q(0) = \frac{2}{1 + j \sqrt{\frac{\mu_r}{\epsilon_r}} \tan(k_1 h)}$$

$$N_1(\theta) = \sqrt{n_1^2 - \sin^2 \theta}$$

$$n_1 = \sqrt{\epsilon_r \mu_r}$$

Directivity (cont.)

We then have that:

$$4\pi r^2 S_r(r, 0, 0) = \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 \tan^2(k_{z1} h) |Q(0)|^2$$

where

$$|Q(0)|^2 = \frac{4}{1 + \left(\frac{\mu_r}{\epsilon_r} \right) \tan^2(k_1 h)}$$

Directivity (cont.)

Hence, we have:

$$4 \pi r^2 S_r(r, 0, \theta) = \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 \tan^2(k_1 h) \left[\frac{4}{1 + \left(\frac{\mu_r}{\epsilon_r} \right) \tan^2(k_1 h)} \right]$$

For the **denominator** in the directivity formula, we have from Notes 17:

$$\begin{aligned} P_{\text{sp}} &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_c \\ &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_0 p_c \end{aligned}$$

Directivity (cont.)

Hence, we have:

$$D(0,0) = \frac{\frac{\pi}{8\eta_0} (k_0 a)^2 h^2 \operatorname{tanc}^2(k_1 h) \left[\frac{4}{1 + \left(\frac{\mu_r}{\epsilon_r} \right) \tan^2(k_1 h)} \right]}{\frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_0 p_c}$$

Simplifying, we have:

$$D(0,0) = \frac{4 \operatorname{tanc}^2(k_1 h)}{p_c I_0 \left[1 + \left(\frac{\mu_r}{\epsilon_r} \right) \tan^2(k_1 h) \right]}$$

Directivity (cont.)

Since we are assuming that $h / \lambda_0 \ll 1$

$$D(0,0) \approx \frac{4}{p_c I_0}$$

For the I_0 term, we have from Notes 18 that

$$I_0 \approx \frac{4}{3}$$

Hence
$$D(0,0) \approx \frac{3}{p_c}$$

Directivity (cont.)

We then have:

$$D(0,0) \approx \frac{3}{p_c}$$

where

$$p_c \approx \sum_{k=0}^6 (k_0 a)^{2k} e_{2k}$$

with

$$e_0 = 1$$

$$e_2 = -0.400000$$

$$e_4 = 0.0785710$$

$$e_6 = -7.27509 \times 10^{-3}$$

$$e_8 = 3.81786 \times 10^{-4}$$

$$e_{10} = -1.09839 \times 10^{-5}$$

$$e_{12} = 1.47731 \times 10^{-7}$$

Gain

The gain of the patch is related to the directivity as:

$$G(0,0) = D(0,0) e_r$$

where

$$e_r = \frac{Q}{Q_{\text{sp}}}$$

Note:

CAD formulas for all Q 's have now been derived, except for Q_{sw} .

and

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\text{sp}}} + \frac{1}{Q_{\text{sw}}}$$

CAD formulas for all of the Q factors were presented in Notes 3.

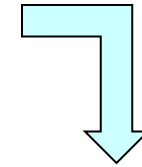
Gain (cont.)

From Notes 1:

$$Q_{\text{sw}} = Q_{\text{sp}} \left(\frac{e_r^{\text{sw}}}{1 - e_r^{\text{sw}}} \right)$$

$$e_r^{\text{sw}} \approx e_r^{\text{hed}}$$

$$e_r^{\text{hed}} = \frac{1}{1 + (k_0 h) \left(\frac{3\pi}{4} \right) \mu_r \frac{1}{c_1} \left(1 - \frac{1}{n_1^2} \right)^3}$$



(This will be derived later from the spectral-domain method.)

Input Resistance

Reminder:

The input resistance at resonance is given by:

$$R_{\text{in}} \approx R_{\text{edge}} \left[\frac{J_1^2(k_1 \rho_0)}{J_1^2(k_1 a)} \right]$$

$$\begin{aligned} P_{\text{sp}} &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_c \\ &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_0 p_c \end{aligned}$$

$$I_0 \approx \frac{4}{3}$$

We have:

$$P_{\text{tot}} = \frac{1}{2} \frac{|V_{\text{edge}}|^2}{R_{\text{edge}}} \quad \text{We assume here that } V_{\text{edge}} = 1 \text{ [V]}$$

$$\Rightarrow R_{\text{edge}} = \left[\frac{1}{2P_{\text{tot}}} \right] e_r$$

Also, we have:

$$P_{\text{tot}} = P_{\text{sp}} / e_r$$

$$\Rightarrow R_{\text{edge}} = \left[\frac{1}{2P_{\text{sp}}} \right] e_r$$

Summary

$$G(0,0) = D(0,0) e_r$$

$$D(0,0) \approx \frac{3}{p_c}$$

$$e_r = \frac{Q}{Q_{\text{sp}}}$$

$$p_c \approx \sum_{k=0}^6 (k_0 a)^{2k} e_{2k}$$

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\text{sp}}} + \frac{1}{Q_{\text{sw}}}$$

$$Q_{\text{sw}} = Q_{\text{sp}} \left(\frac{e_r^{\text{sw}}}{1 - e_r^{\text{sw}}} \right)$$

$$\begin{aligned} e_0 &= 1 \\ e_2 &= -0.400000 \\ e_4 &= 0.0785710 \end{aligned}$$

$$Q_{\text{sp}} = \frac{3}{2} \frac{1}{x'_{11}} (x'^2_{11} - 1) \left(\frac{1}{p} \right) \left(\frac{1}{k_0 h} \right) \varepsilon_r$$

$$e_r^{\text{sw}} \approx e_r^{\text{hed}}$$

$$\begin{aligned} e_6 &= -7.27509 \times 10^{-3} \\ e_8 &= 3.81786 \times 10^{-4} \\ e_{10} &= -1.09839 \times 10^{-5} \\ e_{12} &= 1.47731 \times 10^{-7} \end{aligned}$$

$$x'_{11} = 1.84118$$

$$Q_c = \left(\frac{\eta_0}{2} \right) \mu_r \left[\frac{(k_0 h)}{R_s^{\text{ave}}} \right]$$

$$Q_d = \frac{1}{\tan \delta}$$

$$e_r^{\text{hed}} = \frac{1}{1 + (k_0 h) \left(\frac{3\pi}{4} \right) \mu_r \frac{1}{c_1} \left(1 - \frac{1}{n_1^2} \right)^3}$$

Summary (cont.)

$$R_{\text{in}} \approx R_{\text{edge}} \left[\frac{J_1^2(k_1 \rho_0)}{J_1^2(k_1 a)} \right]$$

$$R_{\text{edge}} = \left[\frac{1}{2P_{\text{sp}}} \right] e_r$$

$$\begin{aligned} P_{\text{sp}} &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_c \\ &= \frac{\pi}{8\eta_0} (k_0 a)^2 h^2 I_0 p_c \end{aligned}$$

$$I_0 \approx \frac{4}{3}$$

$$p_c \approx \sum_{k=0}^6 (k_0 a)^{2k} e_{2k}$$

$$e_0 = 1$$

$$e_2 = -0.400000$$

$$e_4 = 0.0785710$$

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