ECE 6345

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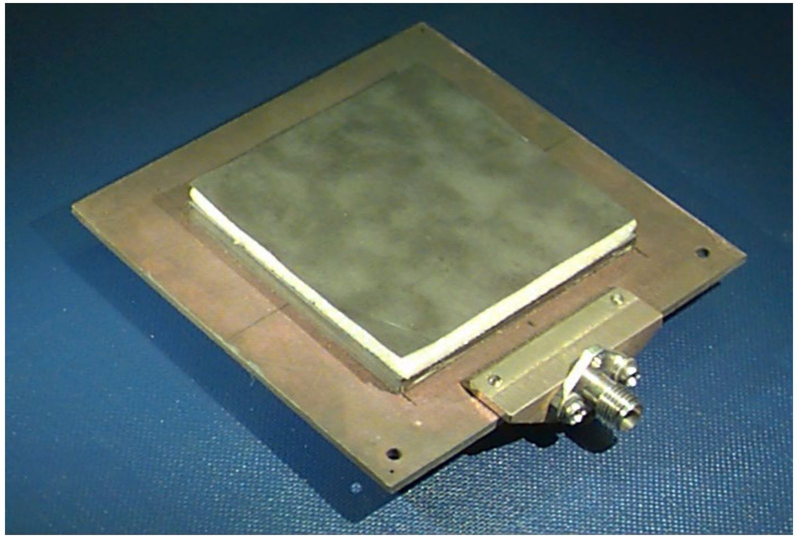
Prof. David R. Jackson
ECE Dept.



Notes 21

Overview

- In this set of notes we calculate a CAD formula for the directivity, gain, and efficiency of the circular patch, which is accurate for a thin substrate.

- The formulas are based on the CAD formula for Q_{sp} that was derived in Notes 19.

Directivity

We have:

$$D(0,0) \equiv \frac{4\pi r^2 S_r(r,0,0)}{P_{\text{sp}}}$$

where

$$S_r(r,\theta,\phi) = \frac{1}{2\eta_0}\left[|E_\theta|^2 + |E_\phi|^2\right]$$

From Notes 17:

$$E_\theta(r,\theta,\phi) = \frac{E_0}{\eta_0}(ah)\cos\phi\,\text{tanc}(k_{z1}h)\,Q(\theta)\;2\pi\,J_1'(k_0 a\sin\theta)$$

$$E_\phi(r,\theta,\phi) = -\frac{E_0}{\eta_0}(ah)\sin\phi\,\text{tanc}(k_{z1}h)\;P(\theta)\;2\pi J_{\text{inc}}(k_0 a\sin\theta)$$

$$|E_0| = \frac{\omega\mu_0}{4\pi r} = \frac{k_0\eta_0}{4\pi r}$$

Assumption:

$$E_z(a,\phi) = \cos\phi$$

Directivity (cont.)

We then have for the numerator:

$$4\pi r^2 S_r(r,\theta,\phi) = 4\pi r^2 \frac{1}{2\eta_0}\left(\frac{k_0\,\eta_0}{4\pi\, r}\frac{1}{\eta_0}\right)^2 (ah)^2 (2\pi)^2$$
$$\cdot \operatorname{tanc}^2(k_{z1}h)\left[\cos^2\phi\left|Q(\theta)\right|^2 J_1'^2(k_0 a\sin\theta) + \sin^2\phi\left|P(\theta)\right|^2 J_{\text{inc}}^2(k_0 a\sin\theta)\right]$$

Simplify:

$$4\pi r^2 S_r(r,\theta,\phi) = 4\pi \cancel{r}^2 \frac{1}{2\eta_0}\left(\frac{k_0\,\cancel{\eta_0}}{4\cancel{\pi}\,\cancel{r}}\frac{1}{\cancel{\eta_0}}\right)^2 (ah)^2 (2\cancel{\pi})^2$$
$$\cdot \operatorname{tanc}^2(k_{z1}h)\left[\cos^2\phi\left|Q(\theta)\right|^2 J_1'^2(k_0 a\sin\theta) + \sin^2\phi\left|P(\theta)\right|^2 J_{\text{inc}}^2(k_0 a\sin\theta)\right]$$

Directivity (cont.)

We thus have for the numerator:

$$4\pi r^2 S_r(r,\theta,\phi) = \frac{\pi}{2\eta_0}(k_0 a)^2 h^2 \operatorname{tanc}^2(k_{z1}h)$$

$$\cdot\left[\cos^2\phi\left|Q(\theta)\right|^2 J_1'^2(k_0 a\sin\theta) + \sin^2\phi\left|P(\theta)\right|^2 J_{\text{inc}}^2(k_0 a\sin\theta)\right]$$

We now let $\theta \to 0$ to calculate the numerator of the directivity expression.

$$\theta \to 0 \quad \Rightarrow \quad x \to 0$$

$$(x \equiv k_0 a \sin\theta)$$

As $x \to 0$:

$$J_1'(x) \to \frac{1}{2}$$

$$J_{\text{inc}}(x) = \frac{J_1(x)}{x} \to \frac{1}{2}$$

Directivity (cont.)

Hence, we have:

$$4\pi r^2 S_r(r,0,0)=\frac{\pi}{8\eta_0}(k_0 a)^2 h^2 \,\mathrm{tanc}^2(k_{z1}h)\left[(1)|Q(0)|^2+\sin^2\phi\,|P(0)|^2\right]$$

where

$$P(\theta)=\cos\theta(1-\Gamma^{\mathrm{TE}}(\theta))=\frac{2\cos\theta}{1+j\left(\dfrac{\mu_r\cos\theta}{N_1(\theta)}\right)\tan\left(k_0 h\,N_1(\theta)\right)}$$

$$Q(\theta)=1-\Gamma^{\mathrm{TM}}(\theta)=\frac{2}{1+j\left(\dfrac{N_1(\theta)\sec\theta}{\varepsilon_r}\right)\tan\left(k_0 h\,N_1(\theta)\right)}$$

$$N_1(\theta)=\sqrt{n_1^2-\sin^2\theta}$$

$$n_1=\sqrt{\varepsilon_r\mu_r}$$

We then see that

$$Q(0)=\frac{2}{1+j\sqrt{\dfrac{\mu_r}{\varepsilon_r}}\tan\left(k_1 h\right)}$$

Directivity (cont.)

We then have that:

$$4\pi r^2 \, S_r\left(r,0,0\right) = \frac{\pi}{8\eta_0}\left(k_0 a\right)^2 h^2 \,\mathrm{tanc}^2\left(k_{z1} h\right) \left|Q\left(0\right)\right|^2$$

where

$$\left|Q\left(0\right)\right|^2 = \frac{4}{1+\left(\dfrac{\mu_r}{\varepsilon_r}\right)\tan^2\left(k_1 h\right)}$$

Directivity (cont.)

Hence, we have:

$$4\pi r^2 \, S_r(r,0,\theta) = \frac{\pi}{8\eta_0}(k_0 a)^2 \, h^2 \, \mathrm{tanc}^2(k_1 h)\left[\frac{4}{1+\left(\dfrac{\mu_r}{\varepsilon_r}\right)\tan^2(k_1 h)}\right]$$

For the denominator in the directivity formula, we have from Notes 17:

$$P_{\mathrm{sp}} = \frac{\pi}{8\eta_0}(k_0 a)^2 \, h^2 \, I_c$$

$$= \frac{\pi}{8\eta_0}(k_0 a)^2 \, h^2 \, I_0 p_c$$

Directivity (cont.)

Hence, we have:

$$D(0,0)=\frac{\dfrac{\pi}{8\eta_0}(k_0a)^2 h^2 \,\text{tanc}^2(k_1h)\left[\dfrac{4}{1+\left(\dfrac{\mu_r}{\varepsilon_r}\right)\tan^2(k_1h)}\right]}{\dfrac{\pi}{8\eta_0}(k_0a)^2 h^2 \, I_0 p_c}$$

Simplifying, we have:

$$D(0,0)=\frac{4\,\text{tanc}^2(k_1h)}{p_c\, I_0\left[1+\left(\dfrac{\mu_r}{\varepsilon_r}\right)\tan^2(k_1h)\right]}$$

Directivity (cont.)

Since we are assuming that $h / \lambda_0 << 1$

$$D(0,0) \approx \frac{4}{p_c \, I_0}$$

For the I_0 term, we have from Notes 18 that

$$I_0 \approx \frac{4}{3}$$

Hence $D(0,0) \approx \dfrac{3}{p_c}$

Directivity (cont.)

We then have:

$$D(0,0) \approx \frac{3}{p_c}$$

where

$$p_c \approx \sum_{k=0}^{6} (k_0 a)^{2k} \; e_{2k}$$

with

$$\begin{aligned} e_0 &= 1 \\ e_2 &= -0.400000 \\ e_4 &= 0.0785710 \\ e_6 &= -7.27509 \times 10^{-3} \\ e_8 &= 3.81786 \times 10^{-4} \\ e_{10} &= -1.09839 \times 10^{-5} \\ e_{12} &= 1.47731 \times 10^{-7} \end{aligned}$$

Gain

The gain of the patch is related to the directivity as:

$$G(0,0) = D(0,0)\, e_r$$

where

$$e_r = \frac{Q}{Q_{\mathrm{sp}}}$$

Note:
CAD formulas for all Q's have now been derived, except for Q_{sw}.

and

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\mathrm{sp}}} + \frac{1}{Q_{\mathrm{sw}}}$$

CAD formulas for all of the Q factors were presented in Notes 3.

Gain (cont.)

From Notes 1:

$$Q_{\text{sw}} = Q_{\text{sp}} \left(\frac{e_r^{\text{sw}}}{1 - e_r^{\text{sw}}} \right)$$

$$e_r^{\text{sw}} \approx e_r^{\text{hed}}$$

$$e_r^{\text{hed}} = \frac{1}{1 + (k_0 h) \left(\frac{3\pi}{4} \right) \mu_r \frac{1}{c_1} \left(1 - \frac{1}{n_1^2} \right)^3}$$

(This will be derived later from the spectral-domain method.)

Summary

$$G(0,0) = D(0,0)\, e_r$$

$$D(0,0) \approx \frac{3}{p_c}$$

$$e_r = \frac{Q}{Q_{\text{sp}}}$$

$$p_c \approx \sum_{k=0}^{6} (k_0 a)^{2k}\, e_{2k}$$

$$\begin{aligned}
e_0 &= 1 \\
e_2 &= -0.400000 \\
e_4 &= 0.0785710 \\
e_6 &= -7.27509 \times 10^{-3} \\
e_8 &= 3.81786 \times 10^{-4} \\
e_{10} &= -1.09839 \times 10^{-5} \\
e_{12} &= 1.47731 \times 10^{-7}
\end{aligned}$$

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\text{sp}}} + \frac{1}{Q_{\text{sw}}}$$

$$Q_{\text{sp}} = \frac{3}{2}\frac{1}{x_{11}'^2}\left(x_{11}'^2 - 1\right)\left(\frac{1}{p}\right)\left(\frac{1}{k_0 h}\right)\varepsilon_r$$

$$x_{11}' = 1.84118$$

$$Q_c = \left(\frac{\eta_0}{2}\right)\mu_r \left[\frac{(k_0 h)}{R_s^{\text{ave}}}\right]$$

$$Q_d = \frac{1}{\tan\delta}$$

$$Q_{\text{sw}} = Q_{\text{sp}}\left(\frac{e_r^{\text{sw}}}{1 - e_r^{\text{sw}}}\right)$$

$$e_r^{\text{sw}} \approx e_r^{\text{hed}}$$

$$e_r^{\text{hed}} = \frac{1}{1 + (k_0 h)\left(\frac{3\pi}{4}\right)\mu_r \frac{1}{c_1}\left(1 - \frac{1}{n_1^2}\right)^3}$$