ECE 6345

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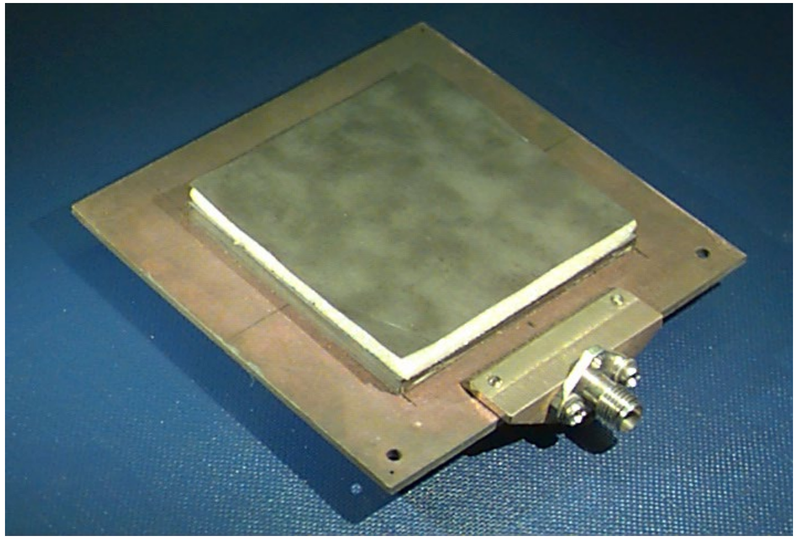
ECE Dept.



Notes 22

Overview

- In this set of notes we introduce the Spectral Domain Immitance (SDI) method, which is a powerful method for solving for the fields due to sources inside of layered media.

  - The basic idea is developed here by decomposing a finite current sheet (e.g. a patch antenna) into a set of infinite phased current sheets.

  - The fields are found from an infinite phased current sheet.

  - The fields from the infinite current sheets are added together (spectral integration) to recover the fields of the finite current sheet.

Overview

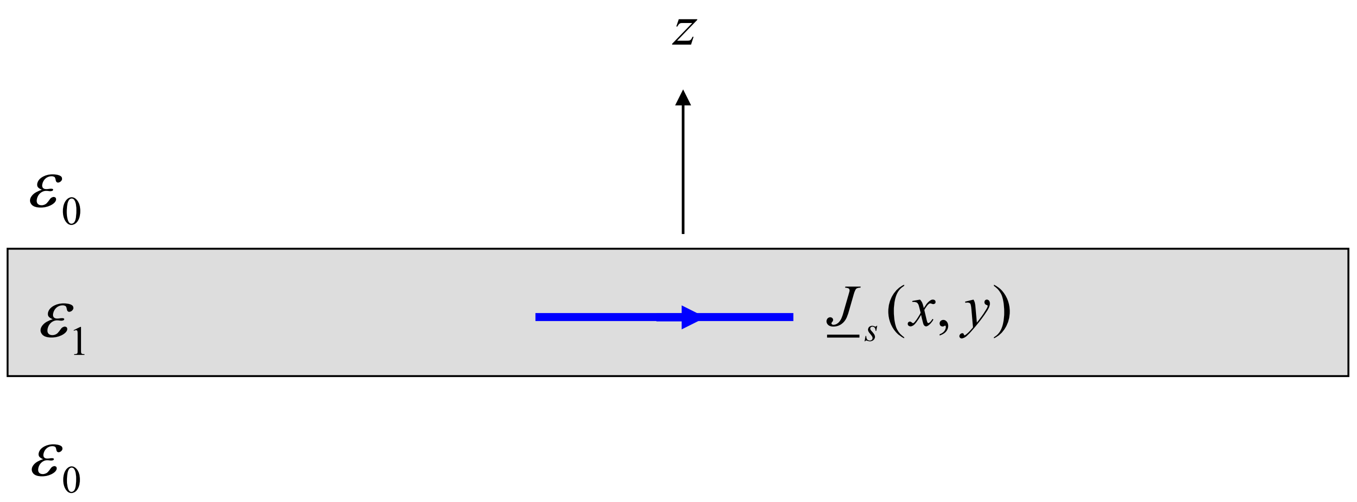
The SDI method is a powerful and systematic method for analyzing sources and structures in layered media.

- **Electric or magnetic dipole sources within layered media.**
- **Microstrip and printed antennas.**
- **Phased arrays, FSS structures, periodic leaky-wave antennas.**
- **Geophysical problems.**

The method was originally developed by Itoh and Menzel in the early 1980s.

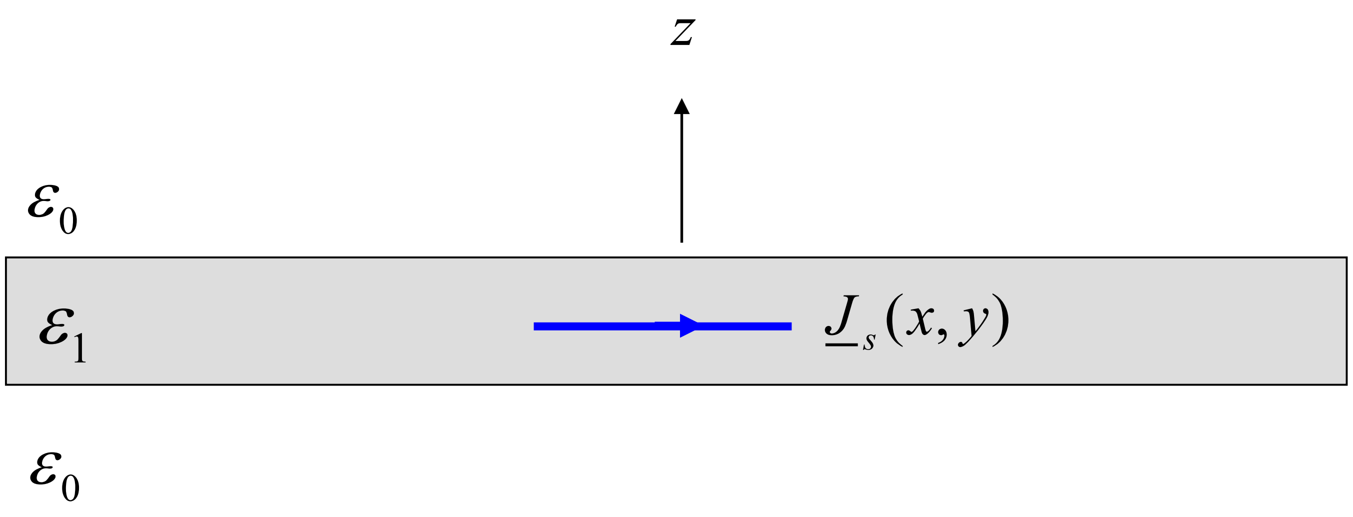
Spectral Domain Immitance Method

- We initially consider a planar source inside of a layered structure. (The method can be extended to include vertical sources as well.)
- The figure shows a single layer, but the method works for any number of layers.



Note: The layer structure is <u>infinite</u> horizontally.

SDI Method (cont.)



Introduce Fourier transform pair:

$$\underline{\tilde{J}}_s\left(k_x,k_y\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underline{J}_s(x,y)\,e^{j\left(k_x x+k_y y\right)}\,dxdy$$

$$x,y\in\left(-\infty,\infty\right)$$

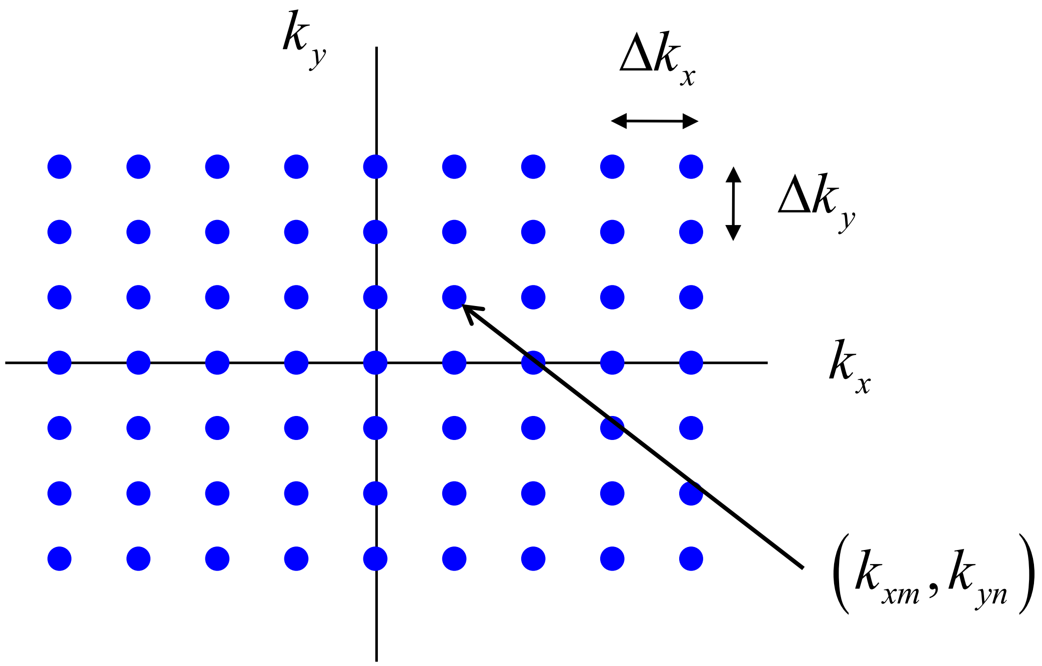
$$\underline{J}_s\left(x,y\right)=\frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underline{\tilde{J}}_s\left(k_x,k_y\right)\,e^{-j\left(k_x x+k_y y\right)}\,dk_x dk_y$$

Considering the integrals as limits of sums, we can write:

$$\underline{J}_s\left(x,y\right)\approx\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{\left(2\pi\right)^2}\underline{\tilde{J}}_s\left(k_{xm},k_{yn}\right)\Delta k_x\Delta k_y\,e^{-j\left(k_{xm}x+k_{yn}y\right)}$$

SDI Method (cont.)

$$\underline{J}_s(x,y) \approx \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^2} \underline{\tilde{J}}_s(k_{xm}, k_{yn}) \Delta k_x \Delta k_y \, e^{-j(k_{xm}x + k_{yn}y)}$$



SDI Method (cont.)

$$\underline{J}_s(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(2\pi)^2}\underline{\tilde{J}}_s(k_{xm},k_{yn})\Delta k_x\Delta k_y\, e^{-j(k_{xm}x+k_{yn}y)}$$

Denote: $$\underline{A}_{mn}\equiv\frac{1}{(2\pi)^2}\underline{\tilde{J}}_s(k_{xm},k_{yn})\,\Delta k_x\Delta k_y$$

Then we have: $$\underline{J}_s(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\underline{A}_{mn}\, e^{-j(k_{xm}x+k_{yn}y)}$$

The finite-size current sheet is thus expressed as a superposition of infinite phased current sheets.

SDI Method (cont.)

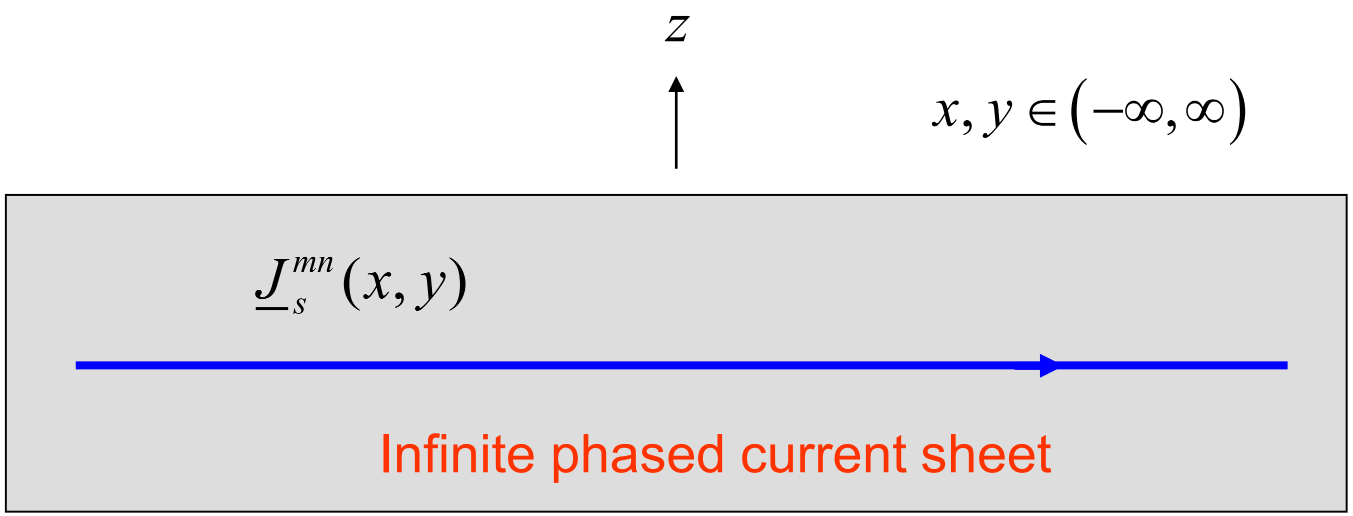
We can write this as

$$\underline{J}_s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \underline{J}_s^{mn}(x,y)$$

where

$$\underline{J}_s^{mn}(x,y) = \underline{A}_{mn}\, e^{-j\left(k_{xm}x + k_{yn}y\right)}$$

$$\left(\underline{A}_{mn} \equiv \frac{1}{(2\pi)^2} \underline{\tilde{J}}_s\left(k_{xm}, k_{yn}\right) \Delta k_x \Delta k_y \right)$$



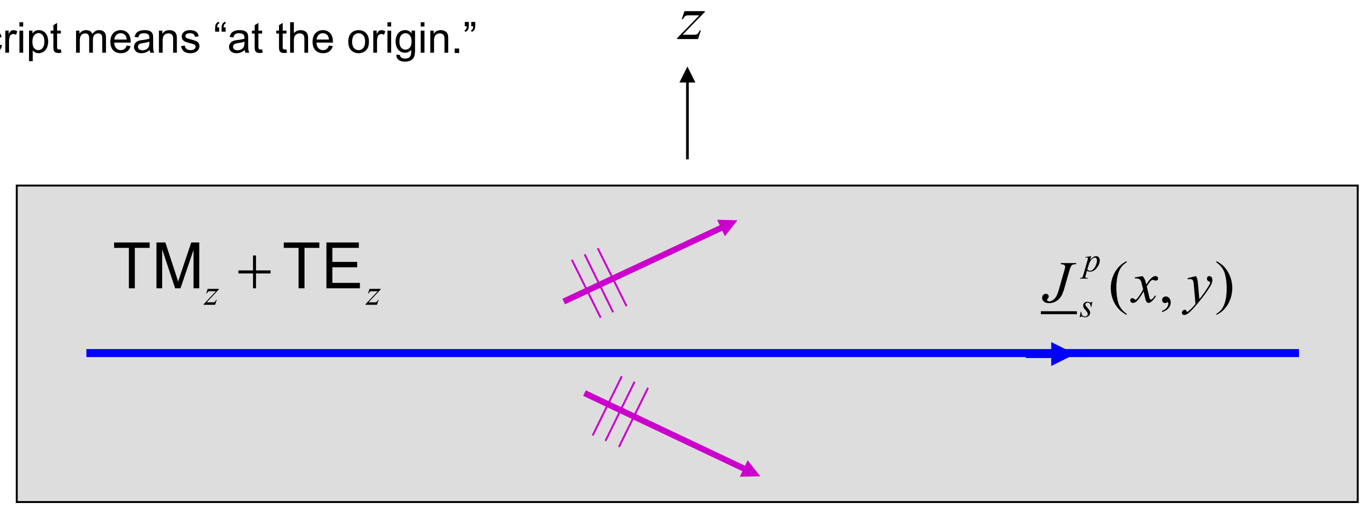
SDI Method (cont.)

Consider a single sheet of current of the form:

$$\underline{J}_s^p(x,y) = \underline{J}_{s0}^p \, e^{-j(k_x x + k_y y)}$$

The superscript p denotes "phased current sheet."

The zero subscript means "at the origin."



New Notation:

$$\underline{J}_s^p(x,y) = \underline{J}_s^{mn}(x,y)$$

$$\underline{J}_{s0}^p = \underline{A}_{mn}$$

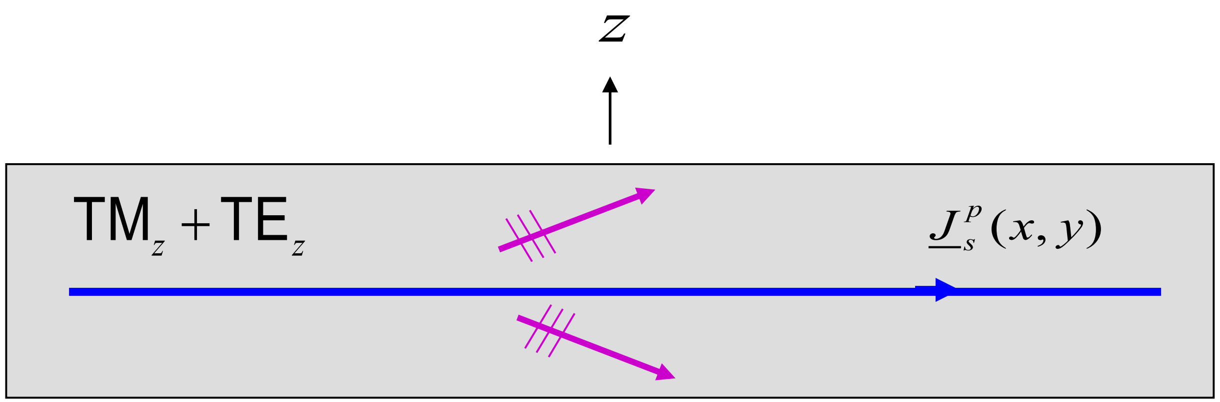
$$e^{-j(k_x x + k_y y)} = e^{-j(k_{xm} x + k_{yn} y)}$$

We wish to determine the amplitude of the plane waves that this current source launches, and the field at any point inside the structure.

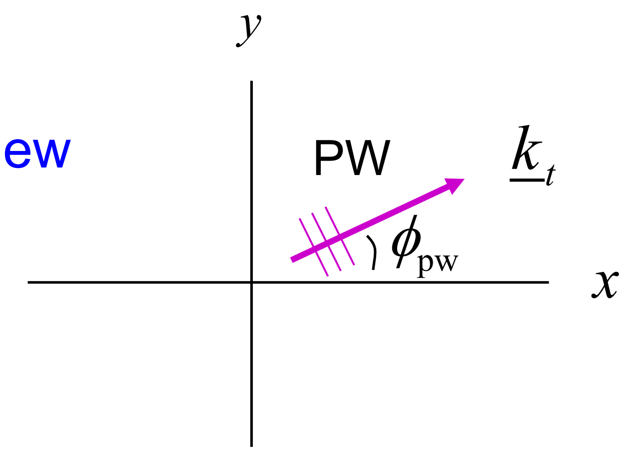
Note: TM_z and TE_z waves reflect from the boundaries and remain TM_z and TE_z, respectively.

SDI Method (cont.)

The current sheet launches a pair of plane waves that propagate up and down.



Top view



$$\tan\phi_{pw} = \frac{k_y}{k_x}$$

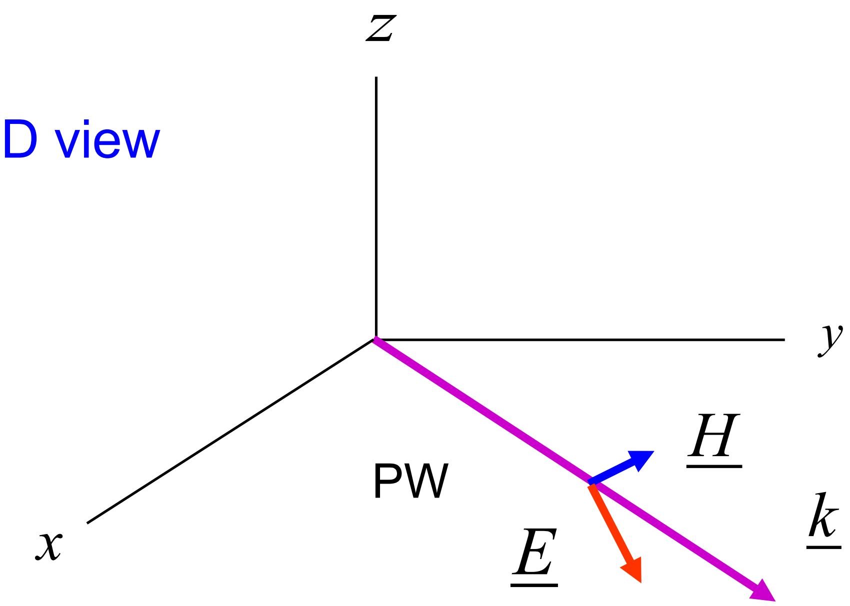
$$\underline{J}_s^p(x,y) = \underline{J}_{s0}^p \, e^{-j(k_x x + k_y y)}$$

The "t" subscript means "transverse" (perpendicular) to the z direction (i.e., horizontal).

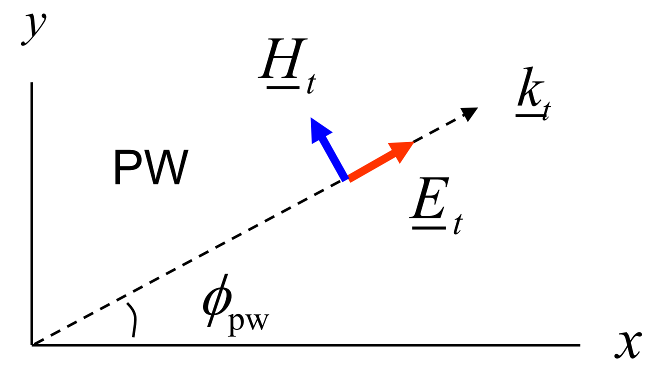
TM_z PW

The upward-going TM_z plane wave inside the layer is shown here.

3D view



Top view



$$\underline{k} = \underline{k}_t + \underline{\hat{z}} k_z$$

The "t" subscript means "transverse" (perpendicular) to the z direction (i.e., horizontal).

TM_z PW (cont.)

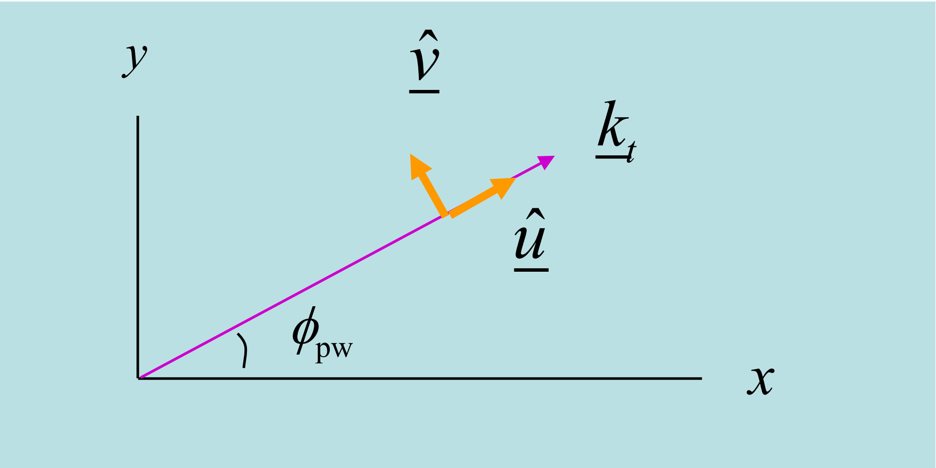
Denote:

$$\underline{\hat{u}} = \underline{\hat{k}}_t = \underline{\hat{x}}\cos\phi_{\text{pw}} + \underline{\hat{y}}\sin\phi_{\text{pw}} = \underline{\hat{x}}\left(\frac{k_x}{k_t}\right) + \underline{\hat{y}}\left(\frac{k_y}{k_t}\right)$$

and

$$\underline{\hat{v}} = \underline{\hat{z}} \times \underline{\hat{u}}$$

Note:
These unit vectors depend on the values of (k_x, k_y).

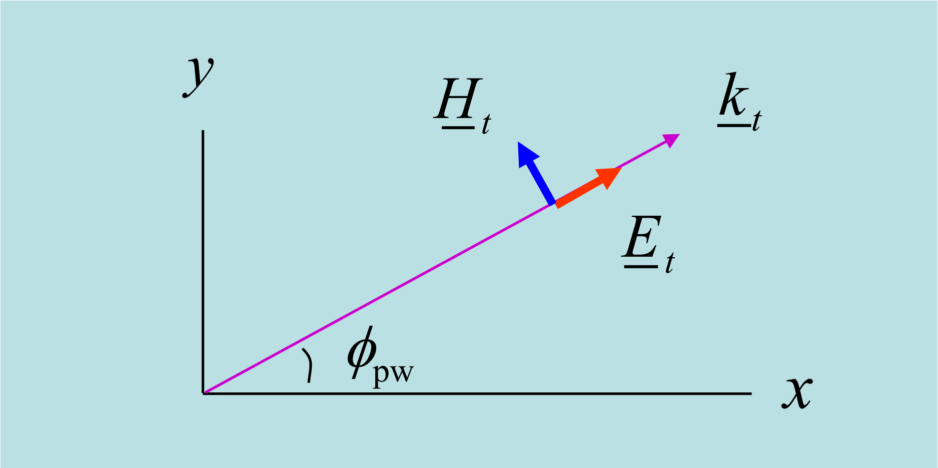


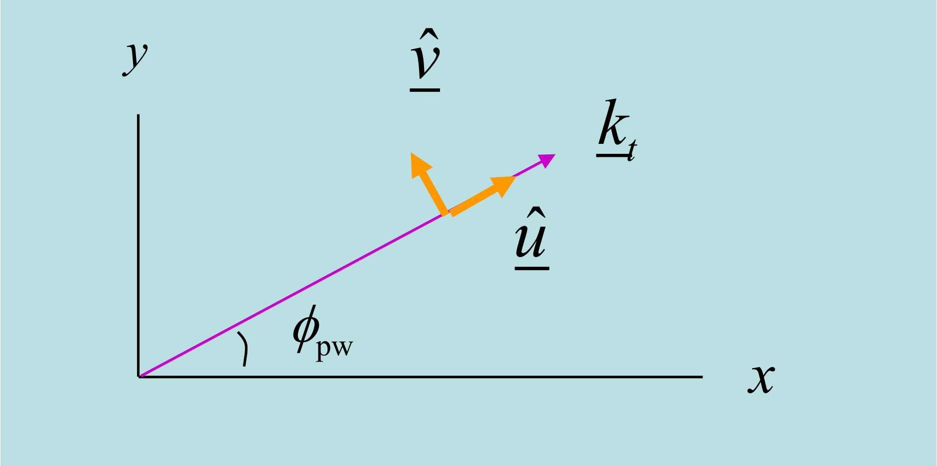
TM$_z$ PW (cont.)

For the TM$_z$ plane wave, we then have:

$$\underline{E}_t = \underline{\hat{u}}\, E_u$$
$$\underline{H}_t = \underline{\hat{v}}\, H_v$$

The "t" subscript means "transverse: (perpendicular) to the z direction.

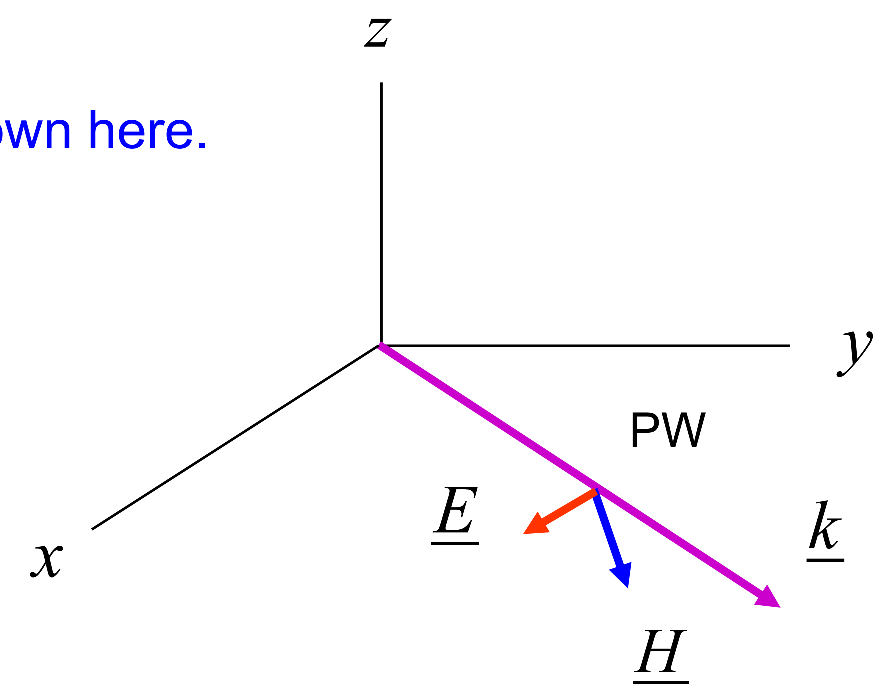


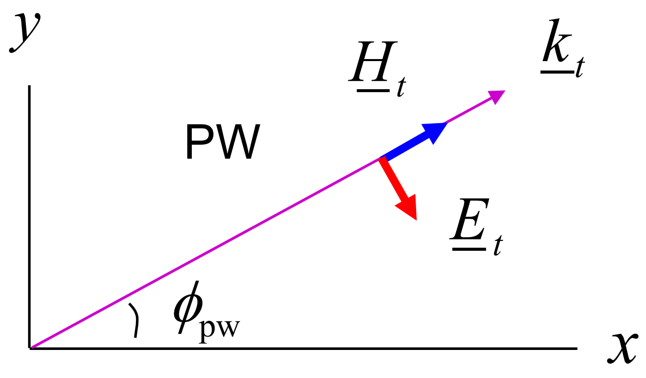


TE$_z$ PW

The upward-going TE$_z$ plane wave is polarized as shown here.

$$\underline{H}_t = \underline{\hat{u}}\, H_u$$
$$\underline{E}_t = \underline{\hat{v}}\, E_v$$



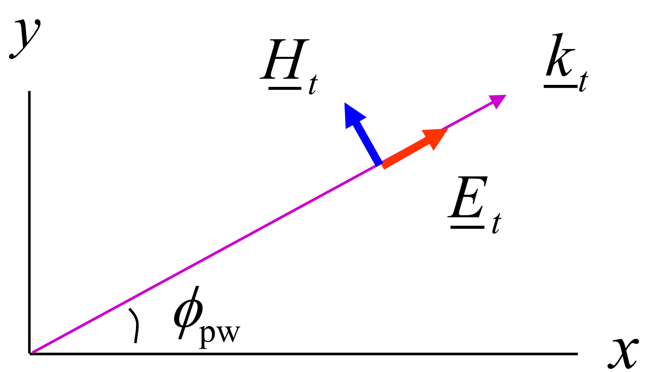


Wave Impedances

The wave impedances are defined for waves traveling upward* (in the net $+z$ direction).

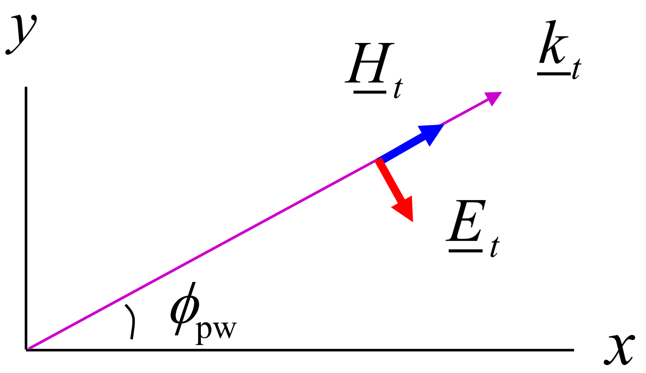
TM$_z$:

$$\frac{E_u}{H_v} = Z^{\text{TM}} = \frac{k_z}{\omega\varepsilon}$$



TE$_z$:

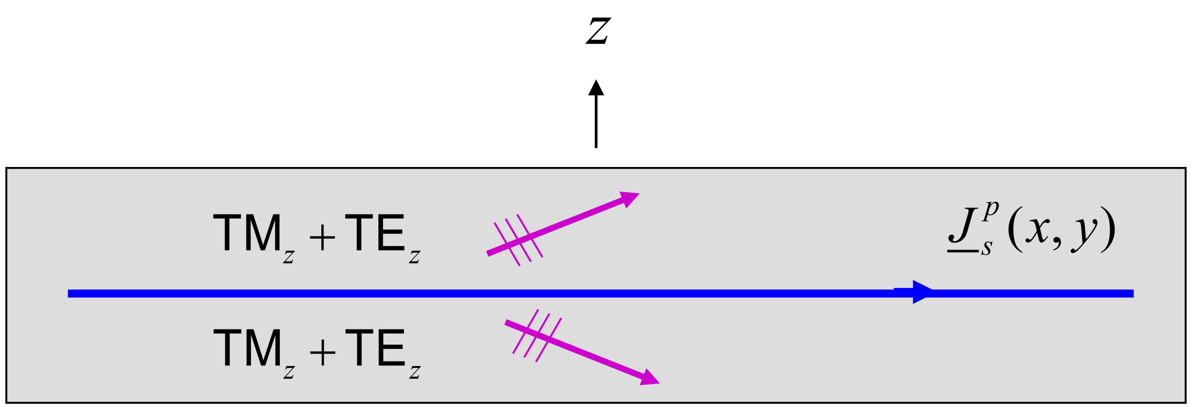
$$\frac{-E_v}{H_u} = Z^{\text{TE}} = \frac{\omega\mu}{k_z}$$



*This is analogous to how Z_0 is defined for a transmission line.

TEN: TM_z PW

Consider the plane waves that gets launched by the current sheet:



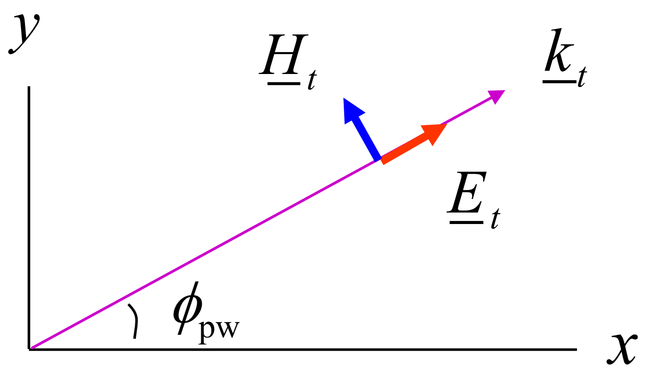
We wish to use a TEN model to find the plane-wave field inside the layered structure.

TEN: TM_z PW (cont.)

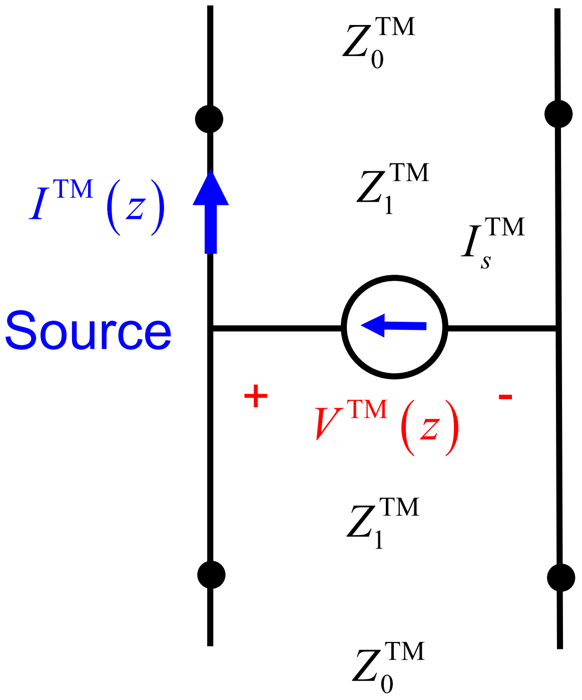
We introduce the following modeling equations:

$$V^{\text{TM}}(z) = E_{u0}(z) \equiv E_u(0,0,z)$$
$$I^{\text{TM}}(z) = H_{v0}(z) \equiv H_v(0,0,z)$$

$$E_u(x,y,z) = E_{u0}(z)\, e^{-j(k_x x + k_y y)}$$
$$H_v(x,y,z) = H_{v0}(z)\, e^{-j(k_x x + k_y y)}$$



The zero subscript indicates that the field has the exponential phase term suppressed.



TEN model

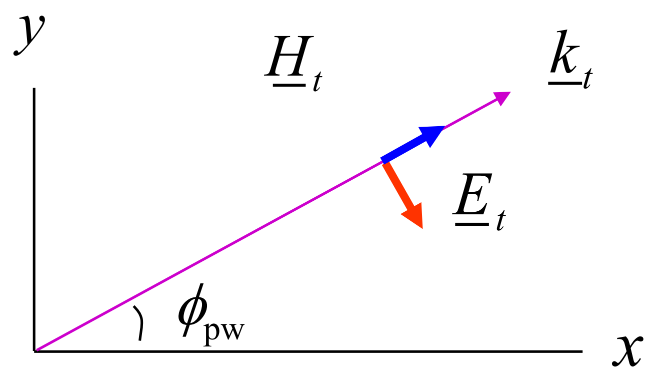
Note that the voltage (tangential electric field) must be continuous at the source location, so the source model is a parallel element.

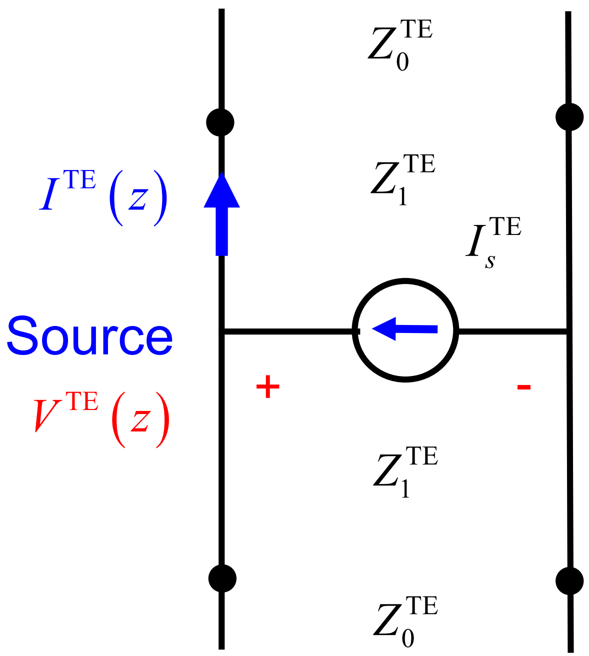
$$k_{zi} = \left(k_i^2 - k_x^2 - k_y^2\right)^{1/2}$$
$$= \left(k_i^2 - k_t^2\right)^{1/2}$$

TEN: TE$_z$ PW

We introduce similar modeling equations for the TE$_z$ case:

$$V^{\text{TE}}(z) = -E_{v0}(z) = -E_v(0,0,z)$$
$$I^{\text{TE}}(z) = H_{u0}(z) = H_u(0,0,z)$$





TEN model

$$k_{zi} = \left(k_i^2 - k_x^2 - k_y^2\right)^{1/2}$$
$$= \left(k_i^2 - k_t^2\right)^{1/2}$$

Source Model

Denote

$$\underline{J}_s^p(x,y) = \underline{J}_s^{p^{\mathrm{TM}}}(x,y) + \underline{J}_s^{p^{\mathrm{TE}}}(x,y)$$

The TM surface current is that part of the total surface current that launches only a TM_z plane wave, while the TE current launches only a TE_z plane wave.

$$\begin{aligned}
\underline{J}_s^{p^{\mathrm{TM}}}(x,y) &= \underline{\hat{z}} \times \left(\underline{H}_t^{+^{\mathrm{TM}}}(x,y,0) - \underline{H}_t^{-^{\mathrm{TM}}}(x,y,0) \right) \\
&= \underline{\hat{z}} \times \left(\underline{\hat{v}} H_v^{+}(x,y,0) - \underline{\hat{v}} H_v^{-}(x,y,0) \right) \\
&= -\underline{\hat{u}} \left(H_v^{+}(x,y,0) - H_v^{-}(x,y,0) \right) \\
&= -\underline{\hat{u}}\, e^{-j(k_x x + k_y y)} \left(H_v^{+}(0,0,0) - H_v^{-}(0,0,0) \right) \\
&= -\underline{\hat{u}}\, e^{-j(k_x x + k_y y)} \left(I^{\mathrm{TM}}(0^+) - I^{\mathrm{TM}}(0^-) \right) \\
&= -\underline{\hat{u}}\, e^{-j(k_x x + k_y y)} I_s^{\mathrm{TM}}
\end{aligned}$$

The source is assumed to be at $z = 0$.

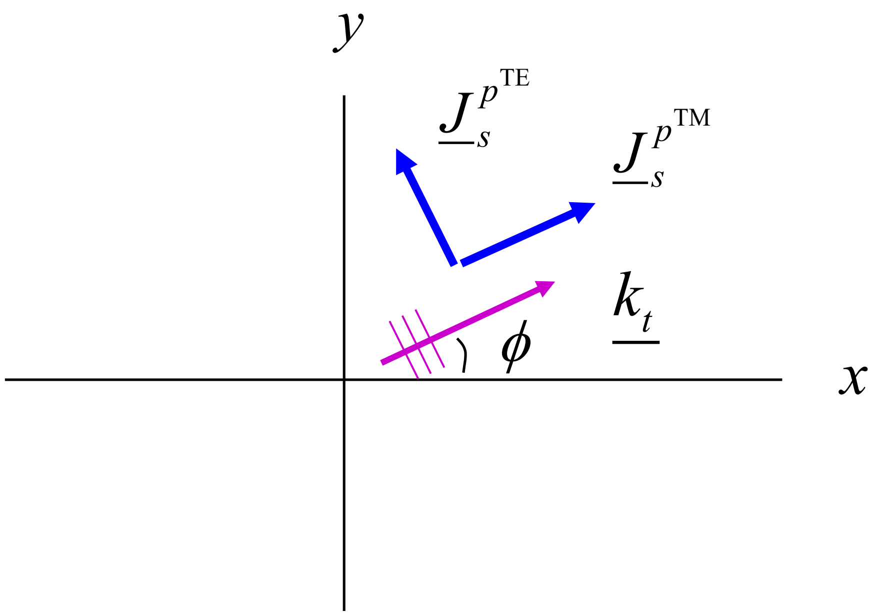
Conclusion: The current that launches the TM_z plane wave is polarized in the u direction.

Source Model (cont.)

Hence $$\underline{J}_s^{p^{\mathrm{TM}}}(x,y)=\underline{\hat{u}}\left(\underline{\hat{u}}\cdot\underline{J}_s^{p}(x,y)\right)$$

Similarly, $$\underline{J}_s^{p^{\mathrm{TE}}}(x,y)=\underline{\hat{v}}\left(\underline{\hat{v}}\cdot\underline{J}_s^{p}(x,y)\right)$$

$$\underline{J}_s^{p}(x,y)=\underline{J}_s^{p^{\mathrm{TM}}}(x,y)+\underline{J}_s^{p^{\mathrm{TE}}}(x,y)$$



Source Model (cont.)

We can now determine the source amplitude in the TEN model:

Recall: $$\underline{J}_s^{p^{\text{TM}}}(x,y) = -\underline{\hat{u}}\, e^{-j\left(k_x x + k_y y\right)} I_s^{\text{TM}}$$

$$\underline{\hat{u}} \cdot \underline{J}_s^{p^{\text{TM}}} = \underline{\hat{u}} \cdot \underline{J}_s^{p} = \underline{\hat{u}} \cdot \left(-\underline{\hat{u}}\, e^{-j\left(k_x x + k_y y\right)} I_s^{\text{TM}} \right)$$

so $$I_s^{\text{TM}} = -\underline{\hat{u}} \cdot \underline{J}_s^{p}(x,y)\, e^{j\left(k_x x + k_y y\right)}$$

or $$I_s^{\text{TM}} = -\underline{\hat{u}} \cdot \underline{J}_{s0}^{p}$$

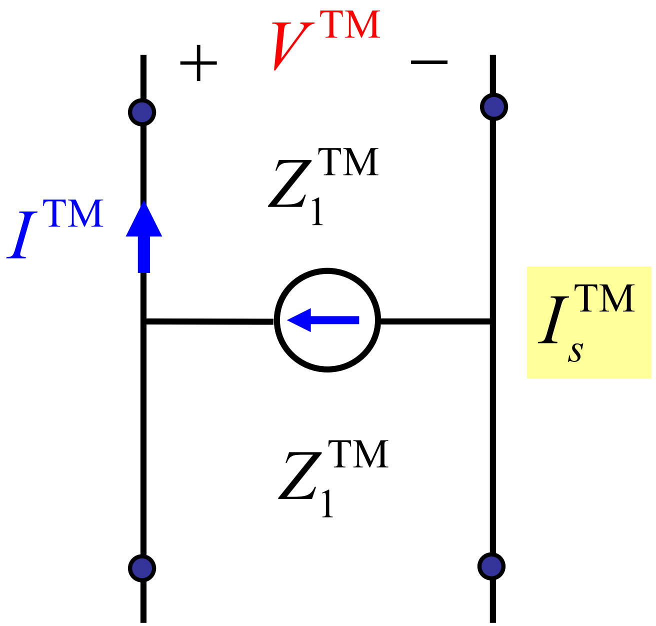
The zero subscript indicates that the current has the exponential phase term suppressed.

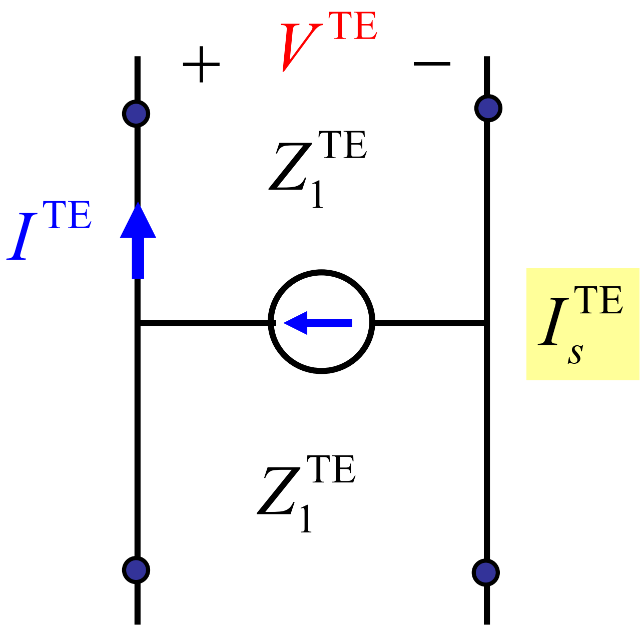
Similarly,

$$I_s^{\text{TE}} = \underline{\hat{v}} \cdot \underline{J}_{s0}^{p}$$

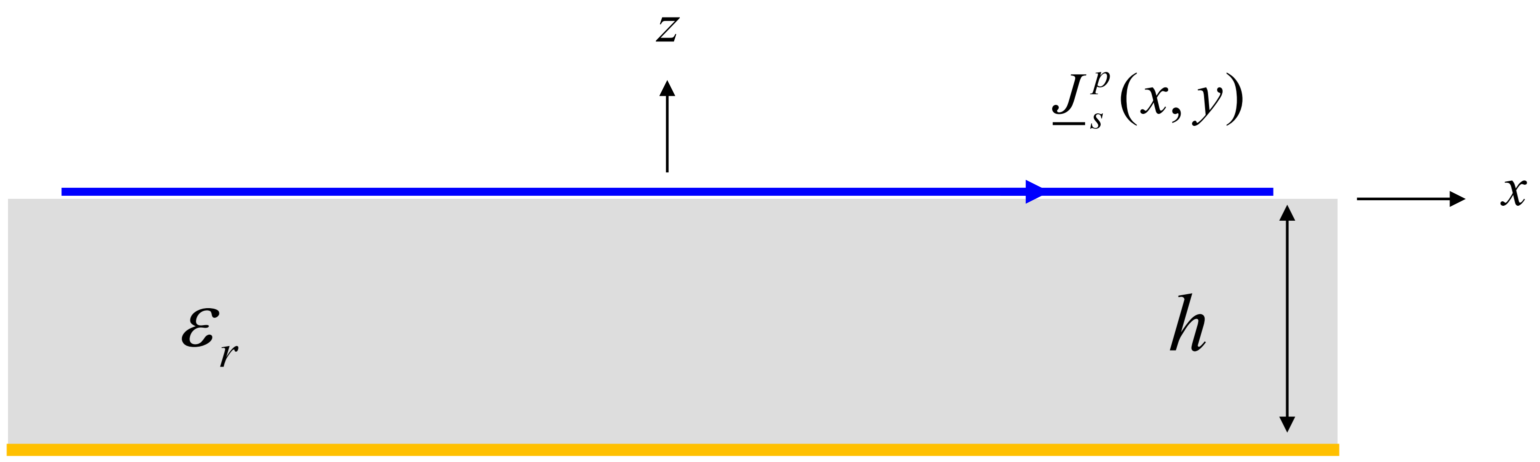
TEN

The TEN models are shown below.





Example



$\varepsilon_r = 2.1$

Assume $\underline{J}_s^p(x,y) = \underline{\hat{x}}\, e^{-jk_0(x+y)}$

$k_x = k_0$

$k_y = k_0$

Find $\underline{E}_t(x,y)$ for $z \geq 0$

Note: If we wanted to find E_z, we would need to find the transverse magnetic field first, and then apply Ampere's law.

Example (cont.)

$$\underline{J}_s^p(x,y) = \underline{\hat{x}}\, e^{-jk_0(x+y)}$$

In this example we have

$$\left.\begin{aligned} k_x &= k_0 \\ k_y &= k_0 \end{aligned}\right\} \quad \underline{k}_t = k_0\left(\underline{\hat{x}} + \underline{\hat{y}}\right)$$

The unit vectors are

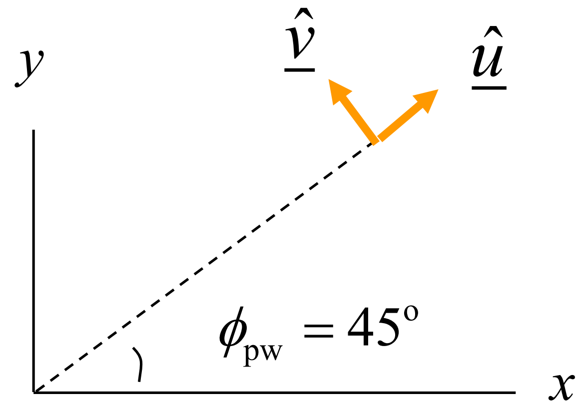
$$\underline{\hat{u}} = \frac{1}{\sqrt{2}}\left(\underline{\hat{x}} + \underline{\hat{y}}\right)$$

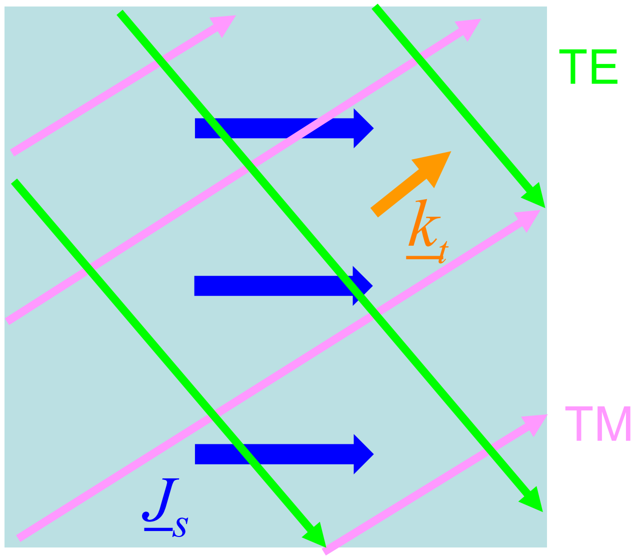
$$\underline{\hat{v}} = \underline{\hat{z}} \times \underline{\hat{u}} = \frac{1}{\sqrt{2}}\left(-\underline{\hat{x}} + \underline{\hat{y}}\right)$$

Example (cont.)

$$\underline{J}_s^{p\,\mathrm{TM}} = \left(\underline{J}_s^{p} \cdot \underline{\hat{u}}\right) \underline{\hat{u}}$$
$$= \left(\underline{\hat{x}} \cdot \underline{\hat{u}}\right) e^{-jk_0(x+y)} \underline{\hat{u}}$$
$$= \cos\phi_{\mathrm{pw}} \, e^{-jk_0(x+y)} \underline{\hat{u}}$$
$$= \frac{1}{\sqrt{2}} e^{-jk_0(x+y)} \underline{\hat{u}}$$
$$= \frac{1}{2}\left(\underline{\hat{x}} + \underline{\hat{y}}\right) e^{-jk_0(x+y)}$$

$$\underline{J}_s^{p\,\mathrm{TE}} = \left(\underline{J}_s^{p} \cdot \underline{\hat{v}}\right) \underline{\hat{v}}$$
$$= \left(\underline{\hat{x}} \cdot \underline{\hat{v}}\right) e^{-jk_0(x+y)} \underline{\hat{v}}$$
$$= -\sin\phi_{\mathrm{pw}} \, e^{-jk_0(x+y)} \underline{\hat{v}}$$
$$= -\frac{1}{\sqrt{2}} e^{-jk_0(x+y)} \underline{\hat{v}}$$
$$= -\frac{1}{2}\left(-\underline{\hat{x}} + \underline{\hat{y}}\right) e^{-jk_0(x+y)}$$





$$\underline{J}_s^{p}(x,y) = \underline{\hat{x}}\, e^{-jk_0(x+y)}$$

Example (cont.)

For the sources we have:

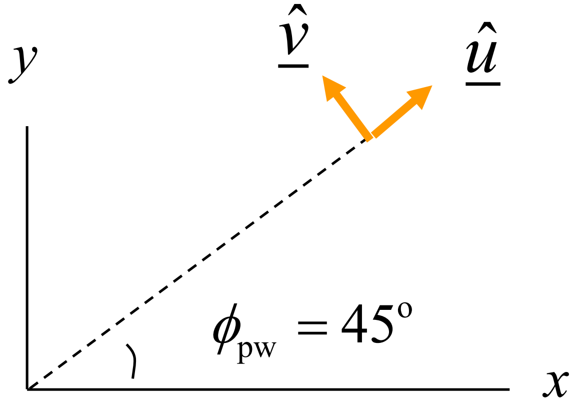
$$
\begin{aligned}
I_s^{\text{TM}} &= -\underline{J}_{s0}^{p} \cdot \underline{\hat{u}} \\
&= -\underline{\hat{x}} \cdot \underline{\hat{u}} \\
&= -\cos\phi_{\text{pw}} \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
I_s^{\text{TE}} &= \underline{J}_{s0}^{p} \cdot \underline{\hat{v}} \\
&= \underline{\hat{x}} \cdot \underline{\hat{v}} \\
&= -\sin\phi_{\text{pw}} \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}
$$

Recall:

$$\underline{J}_s^{p}(x,y) = \underline{\hat{x}}\, e^{-jk_0(x+y)}$$

$$\underline{J}_{s0}^{p}(x,y) = \underline{\hat{x}}$$

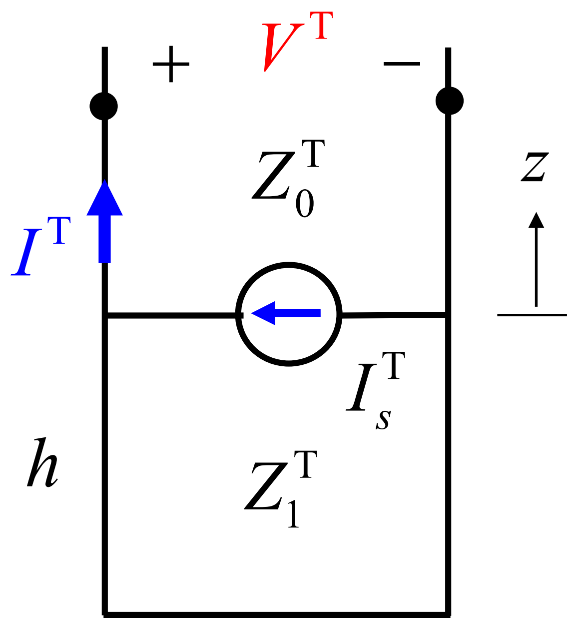


Example (cont.)

$$I_s^{\mathrm{TM}} = -\frac{1}{\sqrt{2}}$$

$$I_s^{\mathrm{TE}} = -\frac{1}{\sqrt{2}}$$

T denotes TM or TE



(TM$_z$ or TE$_z$)

$$k_{z0} = \left(k_0^2 - k_0^2 - k_0^2\right)^{1/2} = -jk_0$$

$$k_{z1} = \left(k_1^2 - k_0^2 - k_0^2\right)^{1/2} = \left(k_1^2 - 2k_0^2\right)^{1/2} = k_0\sqrt{\varepsilon_r - 1}$$

$$Z_0^{\mathrm{TM}} = \frac{k_{z0}}{\omega\varepsilon_0}$$

$$Z_1^{\mathrm{TM}} = \frac{k_{z1}}{\omega\varepsilon_1}$$

$$Z_0^{\mathrm{TE}} = \frac{\omega\mu_0}{k_{z0}}$$

$$Z_1^{\mathrm{TE}} = \frac{\omega\mu_1}{k_{z1}}$$

From TL theory,

$$Z_{\mathrm{in}}^{\mathrm{T}} = Z_0^{\mathrm{T}} \parallel jZ_1^{\mathrm{T}} \tan(k_{z1}h)$$

$$Z_{\mathrm{in}}^{\mathrm{T}} = \frac{jZ_0^{\mathrm{T}} Z_1^{\mathrm{T}} \tan(k_{z1}h)}{Z_0^{\mathrm{T}} + jZ_1^{\mathrm{T}} \tan(k_{z1}h)}$$

Example (cont.)

$$V^{\mathrm{T}}(0) = Z_{\mathrm{in}}^{\mathrm{T}} I_s^{\mathrm{T}}$$

$z \geq 0$

$$V^{\mathrm{T}}(z) = V^{\mathrm{T}}(0)\, e^{-jk_{z0}z} = Z_{\mathrm{in}}^{\mathrm{T}} I_s^{\mathrm{T}} e^{-jk_{z0}z}$$

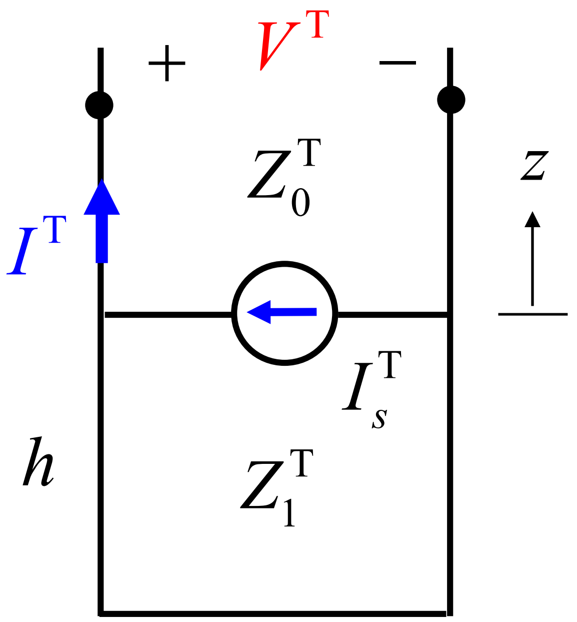
Hence

$$V^{\mathrm{TM}}(z) = Z_{\mathrm{in}}^{\mathrm{TM}} I_s^{\mathrm{TM}} e^{-jk_{z0}z} = Z_{\mathrm{in}}^{\mathrm{TM}} \left(-\frac{1}{\sqrt{2}}\right) e^{-jk_{z0}z}$$

Therefore,

$$E_u^p(0,0,z) = Z_{\mathrm{in}}^{\mathrm{TM}} \left(-\frac{1}{\sqrt{2}}\right) e^{-jk_{z0}z}$$

(TM$_z$ or TE$_z$)



T denotes TM or TE

Example (cont.)

Similarly,

$$V^{\mathrm{TE}}(z) = Z_{\mathrm{in}}^{\mathrm{TE}} I_s^{\mathrm{TE}} e^{-jk_{z0}z}$$

$$= Z_{\mathrm{in}}^{\mathrm{TE}} \left(-\frac{1}{\sqrt{2}} \right) e^{-jk_{z0}z}$$

Therefore,

$$-E_v^p(0,0,z) = Z_{\mathrm{in}}^{\mathrm{TE}} \left(-\frac{1}{\sqrt{2}} \right) e^{-jk_{z0}z}$$

Example (cont.)

We then have:

$$\underline{E}_t(x,y,z) = \underline{\hat{u}}\left[Z_{\text{in}}^{\text{TM}}\left(-\frac{1}{\sqrt{2}}\right) e^{-jk_{z0}z} e^{-jk_0(x+y)} \right] + \underline{\hat{v}}\left[-Z_{\text{in}}^{\text{TE}}\left(-\frac{1}{\sqrt{2}}\right) e^{-jk_{z0}z} e^{-jk_0(x+y)} \right]$$

$$\underline{\hat{u}} = \frac{1}{\sqrt{2}}\left(\underline{\hat{x}} + \underline{\hat{y}}\right)$$

$$\underline{\hat{v}} = \underline{\hat{z}} \times \underline{\hat{u}} = \frac{1}{\sqrt{2}}\left(-\underline{\hat{x}} + \underline{\hat{y}}\right)$$

where

$$Z_{\text{in}}^{\text{T}} = \frac{jZ_0^{\text{T}} Z_1^{\text{T}} \tan(k_{z1}h)}{Z_0^{\text{T}} + jZ_1^{\text{T}} \tan(k_{z1}h)}$$

with

$$k_{z0} = \left(k_0^2 - k_0^2 - k_0^2\right)^{1/2} = -jk_0$$

$$k_{z1} = \left(k_1^2 - k_0^2 - k_0^2\right)^{1/2} = k_0\sqrt{\varepsilon_r - 2}$$

$$Z_0^{\text{TM}} = \frac{k_{z0}}{\omega\varepsilon_0} \qquad Z_1^{\text{TM}} = \frac{k_{z1}}{\omega\varepsilon_1}$$

$$Z_0^{\text{TE}} = \frac{\omega\mu_0}{k_{z0}} \qquad Z_1^{\text{TE}} = \frac{\omega\mu_1}{k_{z1}}$$

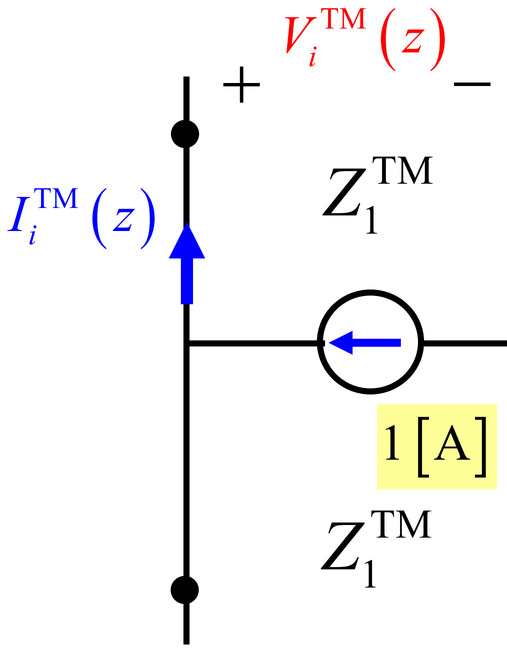
Michalski Notation

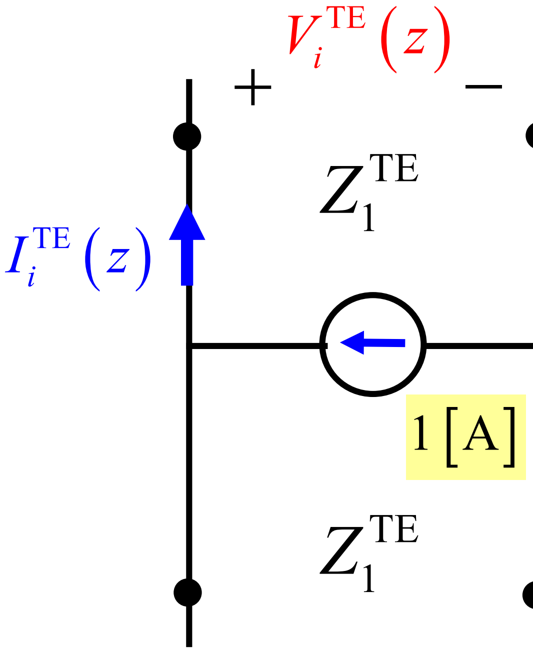
The infinite current sheet produces a field

$$\underline{E}_t^p(x,y) = \underline{E}_{t0}^p e^{-j(k_x x + k_y y)}$$

where

$$\begin{aligned}\underline{E}_{t0}^p &= \underline{\hat{u}}\, E_{u0} + \underline{\hat{v}}\, E_{v0} \\ &= \underline{\hat{u}}\, V^{\text{TM}}(z) + \underline{\hat{v}}\left(-V^{\text{TE}}(z)\right) \\ &= \underline{\hat{u}}\, V_i^{\text{TM}}(z)\left[-\underline{J}_{s0}^p \cdot \hat{u}\right] + \underline{\hat{v}}\left(-V_i^{\text{TE}}(z)\right)\left[+\underline{J}_{s0}^p \cdot \hat{v}\right]\end{aligned}$$





Note:
The Michalski functions are calculated by using only transmission line theory (no EM calculations).

The "i" subscript on the voltage functions denotes the voltage due to a *one-Amp parallel current source*.

Finite Source

Recall that

$$\underline{J}_{s0}^{p} = \underline{A}_{mn} = \frac{1}{(2\pi)^2}\underline{\tilde{J}}_s\left(k_{xm}, k_{yn}\right)\Delta k_x \Delta k_y$$

Hence, from the last slide we have:

$$\underline{E}_{t0}^{p} = \underline{\hat{u}}\, V_i^{\mathrm{TM}}(z)\left[\frac{1}{(2\pi)^2}\left[-\underline{\tilde{J}}_s\left(k_{xm}, k_{yn}\right)\cdot \hat{u}\right]\Delta k_x \Delta k_y\right]$$
$$-\underline{\hat{v}}\, V_i^{\mathrm{TE}}(z)\left[\frac{1}{(2\pi)^2}\left[\underline{\tilde{J}}_s\left(k_{xm}, k_{yn}\right)\cdot \underline{\hat{v}}\right]\Delta k_x \Delta k_y\right]$$

where

$$\underline{\hat{u}} = \underline{\hat{u}}\left(k_{xm}, k_{yn}\right) \qquad V_i^{\mathrm{TM}}(z) = V_i^{\mathrm{TM}}\left(k_t^{mn}, z\right)$$

$$\underline{\hat{v}} = \underline{\hat{v}}\left(k_{xm}, k_{yn}\right) \qquad V_i^{\mathrm{TE}}(z) = V_i^{\mathrm{TE}}\left(k_t^{mn}, z\right)$$

$$k_t^{mn} = \left(k_{xm}^2 + k_{yn}^2\right)^{1/2}$$

Finite Source (cont.)

Adding the contributions from all the phased current sheets, we have:

$$\underline{E}_t(x,y)=\frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\begin{bmatrix}\underline{\hat{u}}\,V_i^{\mathrm{TM}}(z)\left(-\underline{\tilde{J}}_s\left(k_{xm},k_{yn}\right)\cdot\hat{\underline{u}}\right)\\-\underline{\hat{v}}\,V_i^{\mathrm{TE}}(z)\left(\underline{\tilde{J}}_s\left(k_{xm},k_{yn}\right)\cdot\underline{\hat{v}}\right)\end{bmatrix}e^{-j\left(k_{xm}x+k_{yn}y\right)}\,\Delta k_x\,\Delta k_y$$

We then take the limit as

$$\Delta k_x \to dk_x$$
$$\Delta k_y \to dk_y$$

$$\underline{E}_t(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix}\underline{\hat{u}}\,V_i^{TM}(z)\left(-\underline{\tilde{J}}_s\left(k_x,k_y\right)\cdot\hat{\underline{u}}\right)\\-\underline{\hat{v}}\,V_i^{TE}(z)\left(\underline{\tilde{J}}_s\left(k_x,k_y\right)\cdot\underline{\hat{v}}\right)\end{bmatrix}e^{-j\left(k_x x+k_y y\right)}\,dk_x\,dk_y$$

TEN for Transform of Fields

From this we can identify:

$$\underline{\tilde{E}}_t\left(k_x,k_y,z\right)=\underline{\hat{u}}\,V_i^{\mathrm{TM}}(z)\left(-\underline{\tilde{J}}_s\cdot\underline{\hat{u}}\right)-\underline{\hat{v}}\,V_i^{\mathrm{TE}}(z)\left(\underline{\tilde{J}}_s\cdot\underline{\hat{v}}\right)$$

Taking the u and v components, we have:

$$\underline{\hat{u}}\cdot\underline{\tilde{E}}_t=V_i^{\mathrm{TM}}(z)\left(-\tilde{J}_s\cdot\underline{\hat{u}}\right)$$

$$-\underline{\hat{v}}\cdot\underline{\tilde{E}}_t=V_i^{\mathrm{TE}}(z)\left(+\tilde{J}_s\cdot\underline{\hat{v}}\right)$$

From this we can make the following TEN identifications:

$$V^{\mathrm{TM}}(z)=\underline{\hat{u}}\cdot\underline{\tilde{E}}_t\left(k_x,k_y,z\right)\qquad I_s^{\mathrm{TM}}(z)=-\underline{\hat{u}}\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

$$V^{\mathrm{TE}}(z)=-\underline{\hat{v}}\cdot\underline{\tilde{E}}_t\left(k_x,k_y,z\right)\qquad I_s^{\mathrm{TE}}(z)=\underline{\hat{v}}\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

TEN for Transform of Fields (cont.)

Similarly, we have:

$$\underline{\tilde{H}}_t\left(k_x,k_y,z\right)=\underline{\hat{v}}\;V_i^{\mathrm{TM}}(z)\left(-\underline{\tilde{J}}_s\cdot\underline{\hat{u}}\right)+\underline{\hat{u}}\,V_i^{\mathrm{TE}}(z)\left(\underline{\tilde{J}}_s\cdot\underline{\hat{v}}\right)$$

Taking the u and v components, we have:

$$\underline{\hat{v}}\cdot\underline{\tilde{H}}_t=I_i^{\mathrm{TM}}(z)\left(-\tilde{J}_s\cdot\underline{\hat{u}}\right)$$

$$\underline{\hat{u}}\cdot\underline{\tilde{H}}_t=I_i^{\mathrm{TE}}(z)\left(+\tilde{J}_s\cdot\underline{\hat{v}}\right)$$

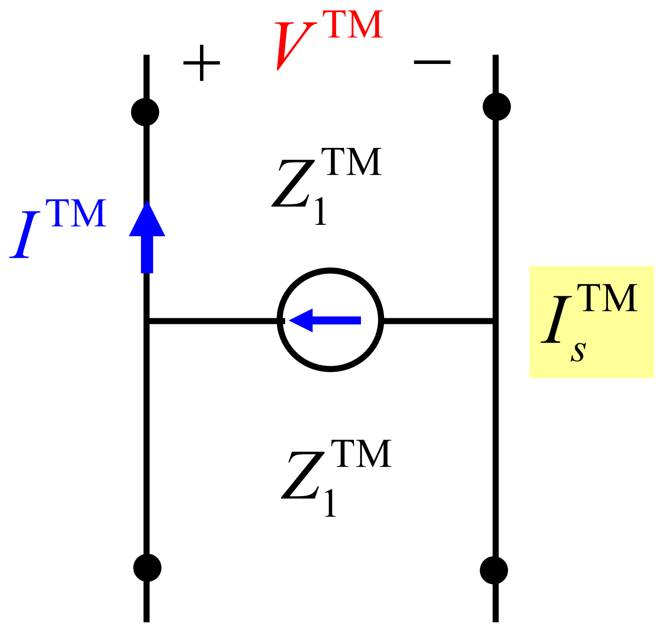
From this we can make the following TEN identifications:

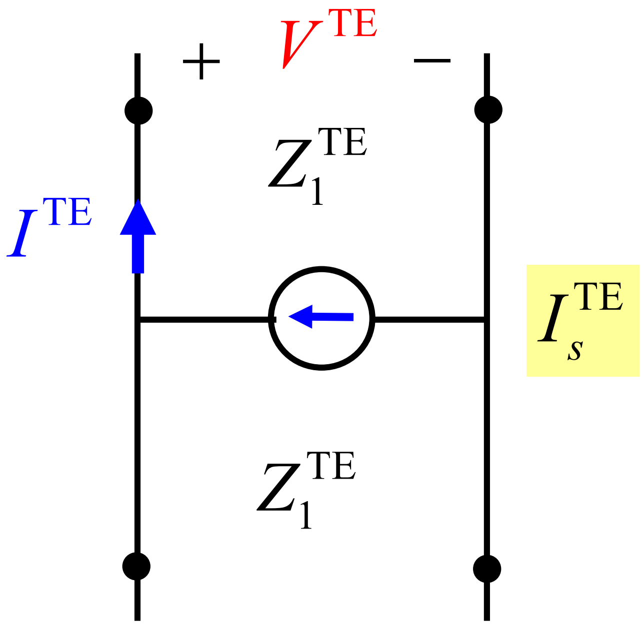
$$I^{\mathrm{TM}}(z)=\underline{\hat{v}}\cdot\underline{\tilde{H}}_t\left(k_x,k_y,z\right)\qquad I_s^{\mathrm{TM}}(z)=-\underline{\hat{u}}\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

$$I^{\mathrm{TE}}(z)=\underline{\hat{u}}\cdot\underline{\tilde{H}}_t\left(k_x,k_y,z\right)\qquad I_s^{\mathrm{TE}}(z)=\underline{\hat{v}}\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

TEN for Transform of Fields (cont.)

The TEN models are shown below.





$$I_s^{\text{TM}}(z) = -\underline{\hat{u}} \cdot \underline{\tilde{J}}_s(k_x, k_y)$$

$$I_s^{\text{TE}}(z) = \underline{\hat{v}} \cdot \underline{\tilde{J}}_s(k_x, k_y)$$

Summary

(A summary for modeling the transform of the field.)

$$V^{\mathrm{TM}}(z) = \underline{\hat{u}} \cdot \underline{\tilde{E}}_t(k_x, k_y, z)$$

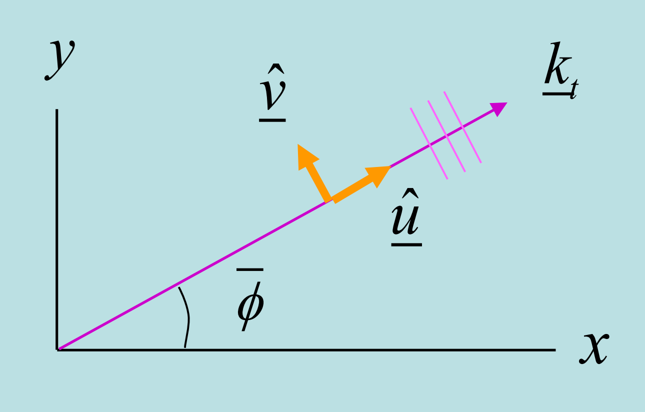
$$I^{\mathrm{TM}}(z) = \underline{\hat{v}} \cdot \underline{\tilde{H}}_t(k_x, k_y, z)$$

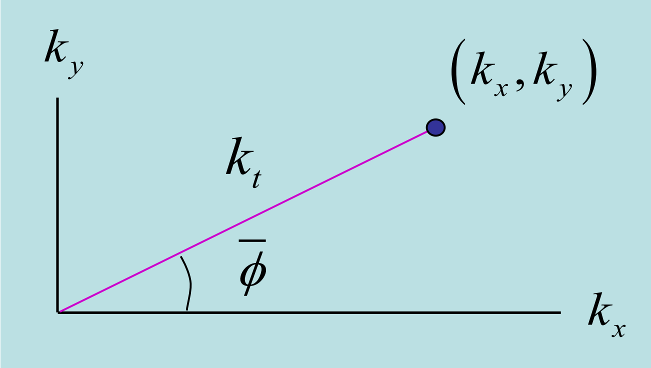
$$I_s^{\mathrm{TM}}(z) = -\underline{\hat{u}} \cdot \underline{\tilde{J}}_s(k_x, k_y)$$

$$V^{\mathrm{TE}}(z) = -\underline{\hat{v}} \cdot \underline{\tilde{E}}_t(k_x, k_y, z)$$

$$I^{\mathrm{TE}}(z) = \underline{\hat{u}} \cdot \underline{\tilde{H}}_t(k_x, k_y, z)$$

$$I_s^{\mathrm{TE}}(z) = \underline{\hat{v}} \cdot \underline{\tilde{J}}_s(k_x, k_y)$$



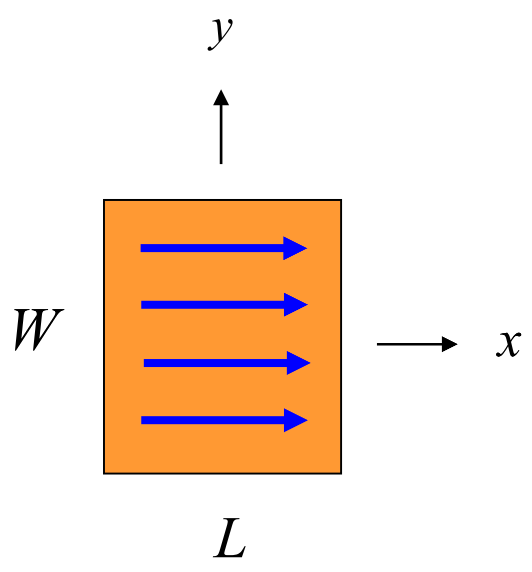


◊ The angle of propagation in the horizontal plane is denoted as $\bar{\phi}$.

◊ This is also the angle in the spectral wavenumber plane.

Spectral-Domain Green's Function

- Assume an x-directed surface current J_{sx}.
- Assume that we wish to find E_x.



Goal: Find E_x

This is the most useful case for the rectangular microstrip antenna.

Spectral-Domain Green's Function

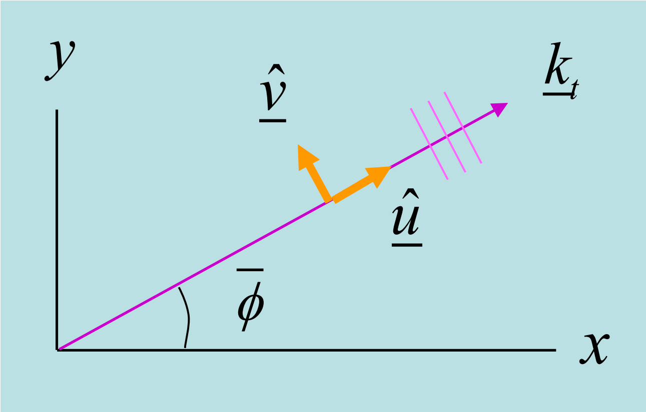
This is a typical "spectral-domain calculation."

$$
\begin{aligned}
\tilde{E}_x &= \underline{\tilde{E}}_t \cdot \underline{\hat{x}} \\
&= \left[\left(\underline{\tilde{E}}_t \cdot \underline{\hat{u}} \right) \underline{\hat{u}} + \left(\underline{\tilde{E}}_t \cdot \underline{\hat{v}} \right) \underline{\hat{v}} \right] \cdot \underline{\hat{x}} \\
&= \left[\left(V^{\mathrm{TM}}(z) \right) \underline{\hat{u}} + \left(-V^{\mathrm{TE}}(z) \right) \underline{\hat{v}} \right] \cdot \underline{\hat{x}} \\
&= V^{\mathrm{TM}}(z) \left(\frac{k_x}{k_t} \right) - V^{\mathrm{TE}}(z) \left(-\frac{k_y}{k_t} \right) \\
&= \left(\frac{k_x}{k_t} \right) V_i^{\mathrm{TM}}(z) \left[I_s^{\mathrm{TM}} \right] + \left(\frac{k_y}{k_t} \right) \left(V_i^{\mathrm{TE}}(z) \right) \left[I_s^{\mathrm{TE}} \right] \\
&= \left(\frac{k_x}{k_t} \right) V_i^{\mathrm{TM}}(z) \left[-\underline{\tilde{J}}_s \cdot \underline{\hat{u}} \right] + \left(\frac{k_y}{k_t} \right) \left(V_i^{\mathrm{TE}}(z) \right) \left[\underline{\tilde{J}}_s \cdot \underline{\hat{v}} \right] \\
&= \left(\frac{k_x}{k_t} \right) V_i^{\mathrm{TM}}(z) \left[-\underline{\hat{x}} \cdot \underline{\hat{u}} \right] \tilde{J}_{sx} + \left(\frac{k_y}{k_t} \right) \left(V_i^{\mathrm{TE}}(z) \right) \left[\underline{\hat{x}} \cdot \underline{\hat{v}} \right] \tilde{J}_{sx} \\
&= -\left(\frac{k_x}{k_t} \right)^2 V_i^{\mathrm{TM}}(z)\, \tilde{J}_{sx} - \left(\frac{k_y}{k_t} \right)^2 V_i^{\mathrm{TE}}(z)\, \tilde{J}_{sx} \\
&= \left[-\left(\frac{k_x}{k_t} \right)^2 V_i^{\mathrm{TM}}(z) - \left(\frac{k_y}{k_t} \right)^2 V_i^{\mathrm{TE}}(z) \right] \tilde{J}_{sx}
\end{aligned}
$$

Recall:

$$\underline{\hat{u}} \cdot \underline{\hat{x}} = \cos \overline{\phi} = k_x / k_t$$

$$\underline{\hat{v}} \cdot \underline{\hat{x}} = -\sin \overline{\phi} = -k_y / k_t$$



$$\tilde{E}_x = -\left[\left(\frac{k_x}{k_t} \right)^2 V_i^{\mathrm{TM}}(z) + \left(\frac{k_y}{k_t} \right)^2 V_i^{\mathrm{TE}}(z) \right] \tilde{J}_{sx}$$

Spectral-Domain Green's Function (cont.)

Define:

$$\tilde{G}_{xx}\left(k_x,k_y,z\right) \equiv -\left(\frac{k_x}{k_t}\right)^2 V_i^{\mathrm{TM}}(z) - \left(\frac{k_y}{k_t}\right)^2 V_i^{\mathrm{TE}}(z)$$

Then

$$\tilde{E}_x = \tilde{G}_{xx}\,\tilde{J}_{sx}$$

This is the "xx component of the spectral-domain Green's function."

Taking the inverse transform, we have:

$$E_x = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{G}_{xx}\left(k_x,k_y,z\right)\tilde{J}_{sx}\left(k_x,k_y\right)e^{-j\left(k_x x + k_y y\right)}\,dk_x\,dk_y$$

Spectral-Domain Green's Function (cont.)

The spectral-domain Green's function is the
Fourier transform of the space-domain Green's function.

To see this:

$$E_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{xx}\left(x-x', y-y'; z, z'\right) J_{sx}\left(x', y'\right) dx'\, dy'$$

$$= G_{xx} * J_{sx}$$ (2D convolution)

The space-domain Green's function $G_{xx}(x, y; z, z')$ is the field $E_x(x, y, z)$ due to a unit-amplitude dipole located at $(0, 0, z')$.

From the convolution property of Fourier transforms: $\tilde{E}_x = F\left\{G_{xx}\right\}\tilde{J}_{sx}$

Hence $\tilde{G}_{xx} = F\left\{G_{xx}\right\}$

Spectral-Domain Green's Function (cont.)

More generally,

$$\begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = \begin{bmatrix} \tilde{G}_{xx} & \tilde{G}_{xy} \\ \tilde{G}_{yx} & \tilde{G}_{yy} \end{bmatrix} \begin{bmatrix} \tilde{J}_{sx} \\ \tilde{J}_{sy} \end{bmatrix}$$

The other three components can be found in a similar manner:

$$\tilde{G}_{xy}, \tilde{G}_{yx}, \tilde{G}_{yy}$$

Note : $\tilde{G}_{xy} = \tilde{G}_{yx}$ from reciprocity.

Summary

(Summary for the E_x field from an x-directed electric current.)

$$\tilde{E}_x = \tilde{G}_{xx}\,\tilde{J}_{sx}$$

where

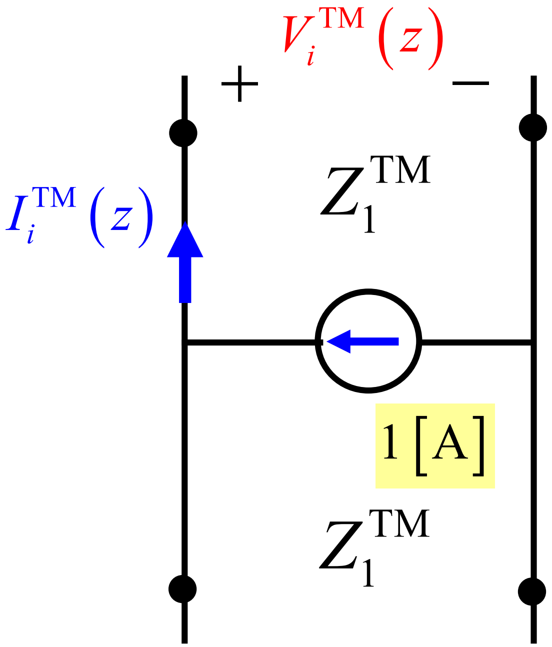
$$\tilde{G}_{xx}\left(k_x, k_y, z\right) \equiv -\left[\left(\frac{k_x}{k_t}\right)^2 V_i^{\mathrm{TM}}(z) + \left(\frac{k_y}{k_t}\right)^2 V_i^{\mathrm{TE}}(z)\right]$$

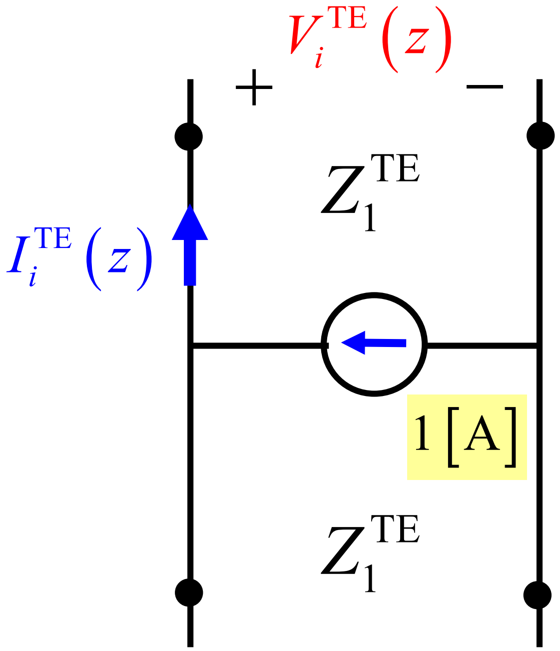
Taking the inverse transform, we have:

$$E_x = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{G}_{xx}\left(k_x, k_y, z\right)\tilde{J}_{sx}\left(k_x, k_y\right) e^{-j\left(k_x x + k_y y\right)}\, dk_x\, dk_y$$

Summary (cont.)

The TEN models are shown below.





$$Z_0^{\text{TM}} = \frac{k_{z0}}{\omega\varepsilon_0}$$

$$Z_1^{\text{TM}} = \frac{k_{z1}}{\omega\varepsilon_1}$$

$$Z_0^{\text{TE}} = \frac{\omega\mu_0}{k_{z0}}$$

$$Z_1^{\text{TE}} = \frac{\omega\mu_1}{k_{z1}}$$