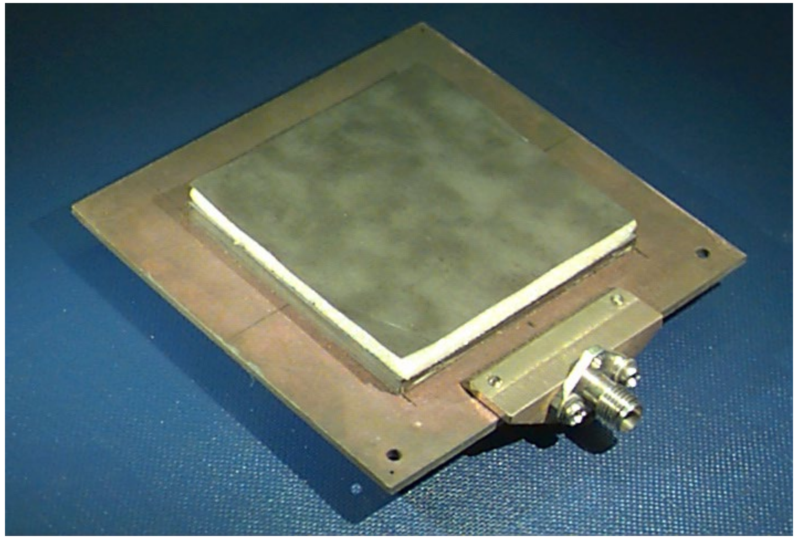
ECE 6345

Spring 2024

Prof. David R. Jackson
ECE Dept.



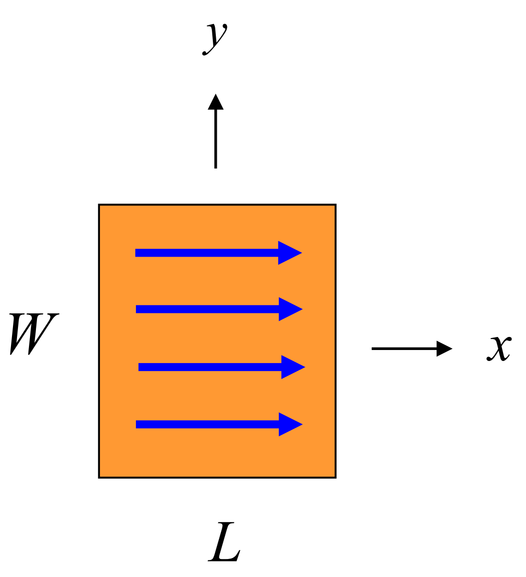
Notes 23

Overview

In this set of notes we apply the SDI method to investigate the fields produced by a patch current.

- We calculate the field due to a rectangular patch on top of a substrate.
- We examine the pole and branch point singularities in the complex plane.
- We examine the path of integration in the complex plane.

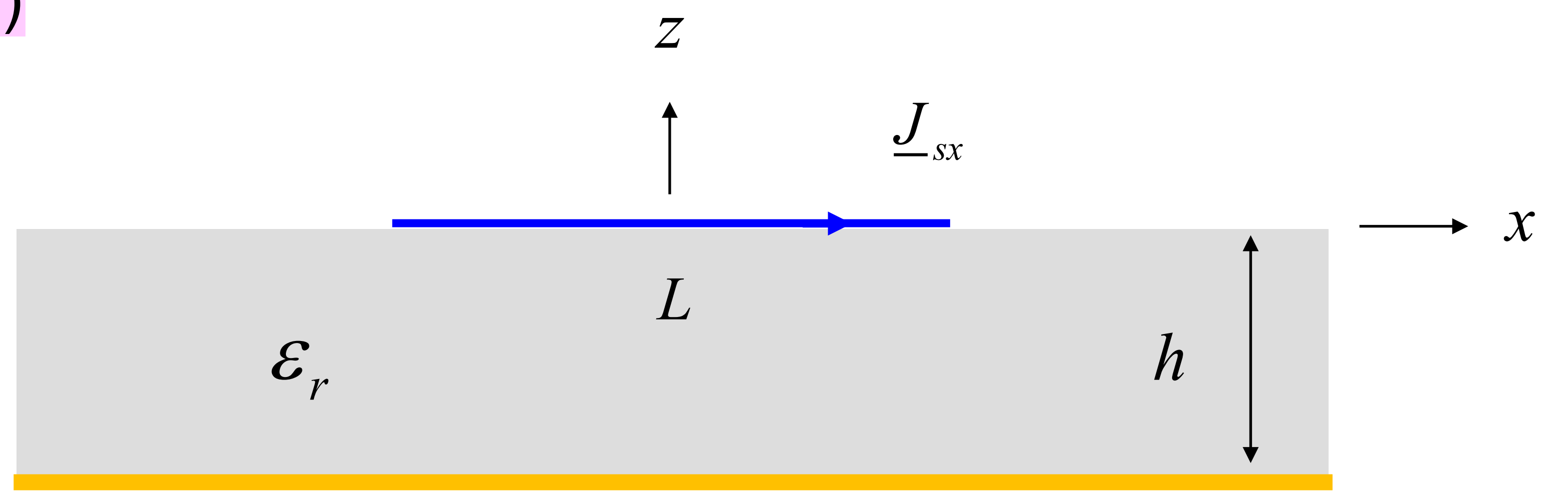
Patch Fields



$$\underline{J}_{sx}^{(1,0)} = \underline{\hat{x}} \cos\left(\frac{\pi x}{L}\right)$$

Note: The origin is at the center of the patch.

Find $E_x\,(x,y,0)$



Patch Fields (cont.)

From Notes 22 we have:

$$E_x = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{G}_{xx}\left(k_x, k_y, z\right) \tilde{J}_{sx}\left(k_x, k_y\right) e^{-j\left(k_x x + k_y y\right)} \, dk_x \, dk_y$$

where

$$\tilde{G}_{xx} = -\left[\left(\frac{k_x}{k_t} \right)^2 V_i^{\mathrm{TM}}(z) + \left(\frac{k_y}{k_t} \right)^2 V_i^{\mathrm{TE}}(z) \right]$$

For the patch current, we have:

$$\tilde{J}_{sx}\left(k_x, k_y\right) = \left(\frac{\pi}{2} LW \right) \operatorname{sinc}\left(k_y \frac{W}{2} \right) \left[\frac{\cos\left(k_x \frac{L}{2} \right)}{\left(\frac{\pi}{2} \right)^2 - \left(k_x \frac{L}{2} \right)^2} \right]$$

Patch Fields (cont.)

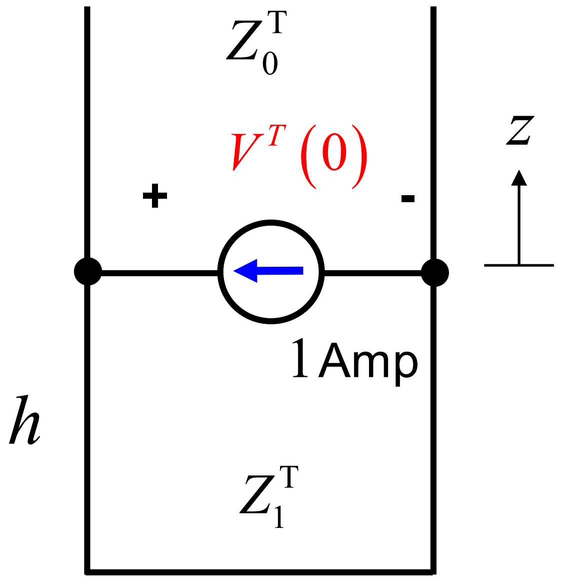
From the TEN we have:

$$V_i^{\mathrm{T}}(0) = (1)\, Z_{\mathrm{in}}^{\mathrm{T}}(0)$$

$$= \frac{1}{Y_{\mathrm{in}}^{\mathrm{T}}(0)}$$

$$= \frac{1}{Y_0^{\mathrm{T}} - jY_1^{\mathrm{T}} \cot\left(k_{z1} h\right)}$$

T denotes TM or TE



(TM$_z$ or TE$_z$)

$$Y_0^{\mathrm{TM}} = \frac{\omega \varepsilon_0}{k_{z0}}$$

$$Y_1^{\mathrm{TM}} = \frac{\omega \varepsilon_1}{k_{z1}}$$

$$Y_0^{\mathrm{TE}} = \frac{k_{z0}}{\omega \mu_0}$$

$$Y_1^{\mathrm{TE}} = \frac{k_{z1}}{\omega \mu_1}$$

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

$$k_t^2 = \left(k_x^2 + k_y^2\right)^{1/2}$$

(The branch choice for k_{z1} is arbitrary.)

Note: For k_{z0} we choose a positive real number or a negative imaginary number: $k_{z0} = -j\sqrt{k_t^2 - k_0^2}$ (works for any k_t)

Patch Fields (cont.)

Define the denominator term as:

$$D^{\mathrm{T}}(k_t) \equiv Y_{\mathrm{in}}^{\mathrm{T}}(0)$$

so that

$$D^{\mathrm{TM}}(k_t) = Y_0^{\mathrm{TM}} - jY_1^{\mathrm{TM}} \cot\left(k_{z1}h\right)$$

$$D^{\mathrm{TE}}(k_t) = Y_0^{\mathrm{TE}} - jY_1^{\mathrm{TE}} \cot\left(k_{z1}h\right)$$

Patch Fields (cont.)

We then have:

$$\tilde{G}_{xx} = -\left[\left(\frac{k_x}{k_t} \right)^2 \frac{1}{D^{\mathrm{TM}}} + \left(\frac{k_y}{k_t} \right)^2 \frac{1}{D^{\mathrm{TE}}} \right]$$

The final form of the electric field at the interface is then:

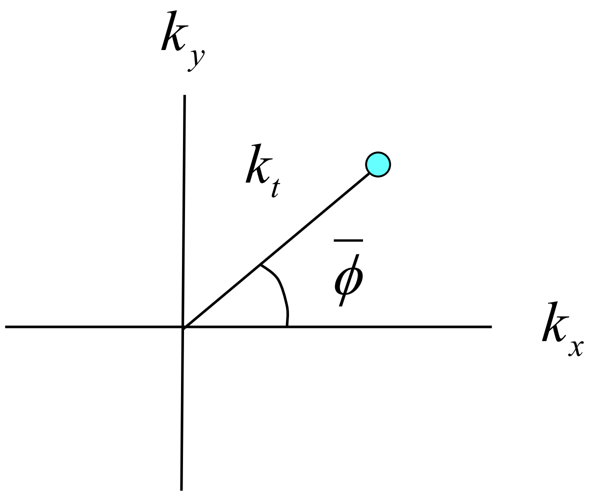
$$E_x(x, y, 0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} -\left[\left(\frac{k_x}{k_t} \right)^2 \frac{1}{D^{\mathrm{TM}}} + \left(\frac{k_y}{k_t} \right)^2 \frac{1}{D^{\mathrm{TE}}} \right] \tilde{J}_{sx}(k_x, k_y) e^{-j(k_x x + k_y y)} \, dk_x \, dk_y$$

Polar Coordinates

Use the following change of variables:

$$dk_x \, dk_y = k_t \, dk_t \, d\bar{\phi}$$

(k_t is also often called k_ρ)



$$\int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} (\ldots) e^{-j(k_x x + k_y y)} dk_x \, dk_y = \int_{0}^{2\pi} \int_{0}^{+\infty} (\ldots) e^{-j(k_x x + k_y y)} k_t \, dk_t \, d\bar{\phi}$$

$$= 4 \int_{0}^{\pi/2} \int_{0}^{+\infty} (\ldots) \cos(k_x x) \cos(k_y y) k_t \, dk_t \, d\bar{\phi}$$

$$\cos\bar{\phi} = \frac{k_x}{k_t}$$

$$\sin\bar{\phi} = \frac{k_y}{k_t}$$

Advantage of polar coordinates: The poles and branch points are located at a fixed position in the complex k_t plane.

$$k_{z0} = -j\sqrt{k_t^2 - k_0^2} \qquad k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

Polar Coordinates (cont.)

Hence, we have:

$$E_x(x,y,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}-\left[\left(\frac{k_x}{k_t}\right)^2\frac{1}{D^{\mathrm{TM}}}+\left(\frac{k_y}{k_t}\right)^2\frac{1}{D^{\mathrm{TE}}}\right]\tilde{J}_{sx}(k_x,k_y)\,e^{-j(k_x x+k_y y)}\,dk_x\,dk_y$$

⇓

$$E_x(x,y,0)=-\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\infty}\tilde{J}_{sx}(k_t,\overline{\phi})\left[\cos^2\overline{\phi}\,\frac{1}{D^{\mathrm{TM}}(k_t)}+\sin^2\overline{\phi}\,\frac{1}{D^{\mathrm{TE}}(k_t)}\right]\cdot\cos(k_x x)\cos(k_y y)\,k_t\,dk_t\,d\overline{\phi}$$

This is in the following general form:

$$E_x=\int_0^{\pi/2}\int_0^{\infty}F(k_t,\overline{\phi})\,dk_t\,d\overline{\phi}$$

Poles

Poles occur when either of the following conditions are satisfied:

$$D^{\mathrm{TM}}(k_t) = 0 \qquad \left(k_t = k_{tp}^{\mathrm{TM}}\right)$$

$$D^{\mathrm{TE}}(k_t) = 0 \qquad \left(k_t = k_{tp}^{\mathrm{TE}}\right)$$

TM$_z$:

$$D^{\mathrm{TM}} = 0$$

$$\Rightarrow \; Y_0^{\mathrm{TM}} - jY_1^{\mathrm{TM}} \cot\left(k_{z1}h\right) = 0$$

Poles (cont.)

This equation coincides with the well-known Transverse Resonance Equation (TRE) in microwave engineering for determining the characteristic equation of a guided mode (e.g, TM_0 SW mode).

$$\vec{Z} = \frac{V(0^+)}{I(0^+)} \qquad \overleftarrow{Z} = \frac{V(0^-)}{-I(0^-)}$$

$$V(0^+) = V(0^-)$$
$$I(0^+) = I(0^-)$$

(Kirchhoff's laws)

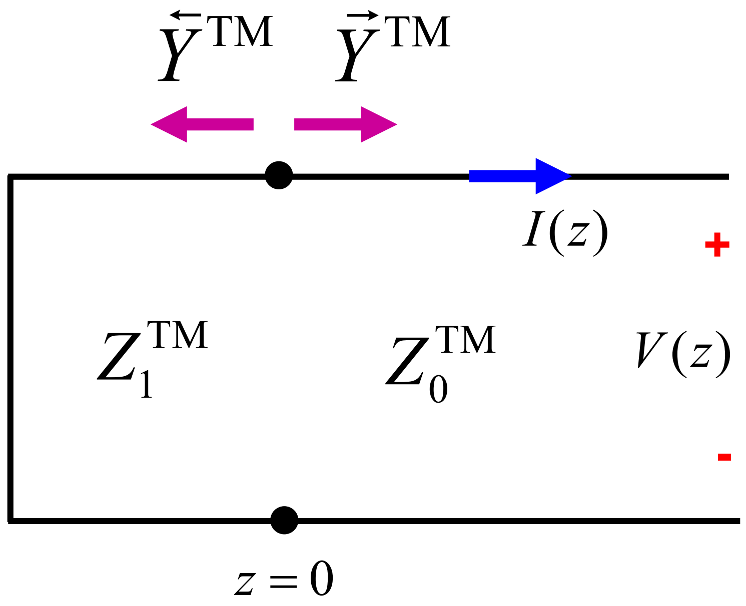
Hence, we have:

$$\overleftarrow{Z} = -\vec{Z} \quad \Rightarrow \quad \overleftarrow{Y} = -\vec{Y}$$

This gives us:

$$-jY_1^{\mathrm{TM}} \cot(k_{z1}h) = -Y_0^{\mathrm{TM}}$$

TM_0 SW:
$$\begin{cases} k_{z0} = -j\sqrt{\beta_{\mathrm{TM}_0}^2 - k_0^2} \\ k_{z1} = \left(k_1^2 - \beta_{\mathrm{TM}_0}^2\right)^{1/2} \end{cases}$$



Poles (cont.)

Comparison:

Poles in k_t plane

$$Y_0^{\mathrm{TM}} - jY_1^{\mathrm{TM}} \cot\left(k_{z1} h\right) = 0$$

$$Y_0^{\mathrm{TM}} = \frac{\omega \varepsilon_0}{k_{z0}}$$

$$Y_1^{\mathrm{TM}} = \frac{\omega \varepsilon_1}{k_{z1}}$$

$$k_{z0} = -j\sqrt{k_{tp}^2 - k_0^2}$$

$$k_{z1} = \left(k_1^2 - k_{tp}^2\right)^{1/2}$$

TRE (surface-wave mode)

$$Y_0^{\mathrm{TM}} - jY_1^{\mathrm{TM}} \cot\left(k_{z1} h\right) = 0$$

$$Y_0^{\mathrm{TM}} = \frac{\omega \varepsilon_0}{k_{z0}}$$

$$Y_1^{\mathrm{TM}} = \frac{\omega \varepsilon_1}{k_{z1}}$$

$$k_{z0} = -j\sqrt{\beta_{\mathrm{TM}_0}^2 - k_0^2}$$

$$k_{z1} = \left(k_1^2 - \beta_{\mathrm{TM}_0}^2\right)^{1/2}$$

(A similar comparison holds for the TE case.)

Poles (cont.)

Hence, we have the conclusion that

$$k_{tp}^{\mathrm{TM}} = \beta_{\mathrm{TM}}$$

$$k_{tp}^{\mathrm{TE}} = \beta_{\mathrm{TE}}$$

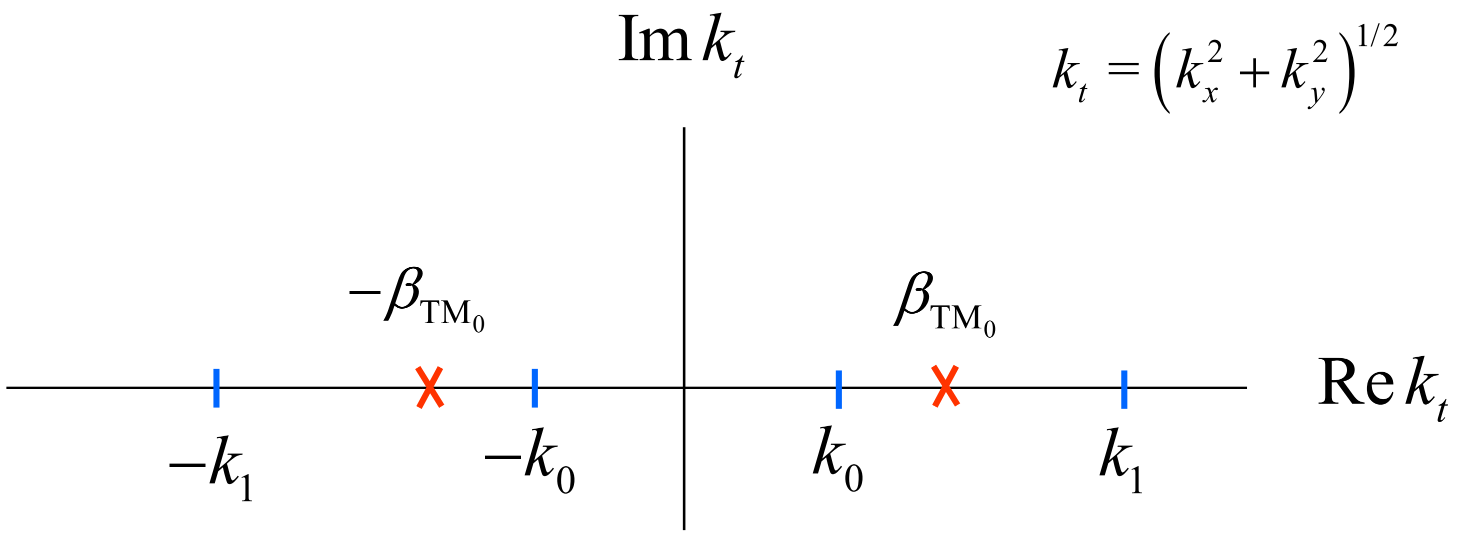
That is, the poles are located at the wavenumbers of the guided modes (the surface-wave modes).

Note: In most practical substrate cases, there is only a single TM_0 surface-wave mode.

Poles (cont.)

The complex plane thus has poles on the real axis at the wavenumbers of the surface waves.

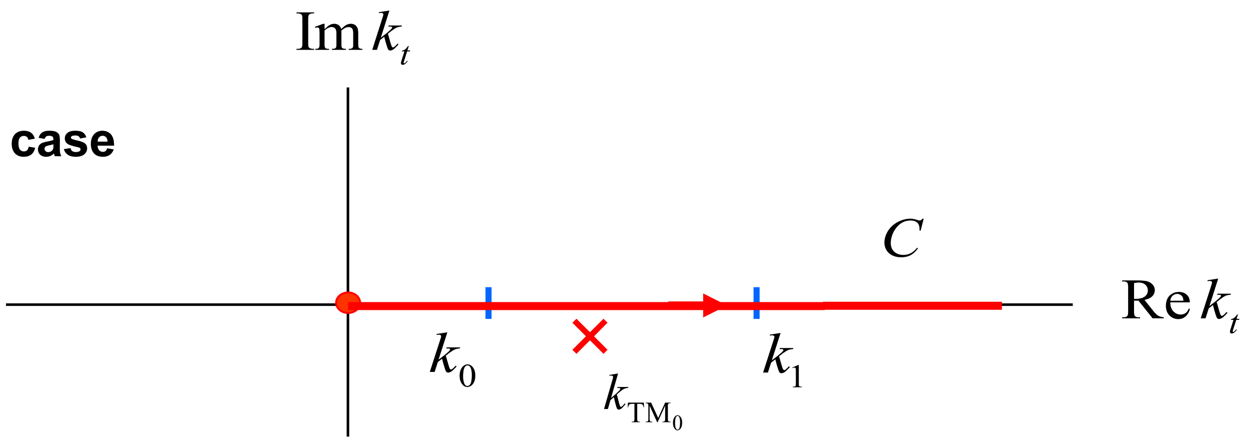
$$k_t = \left(k_x^2 + k_y^2\right)^{1/2}$$



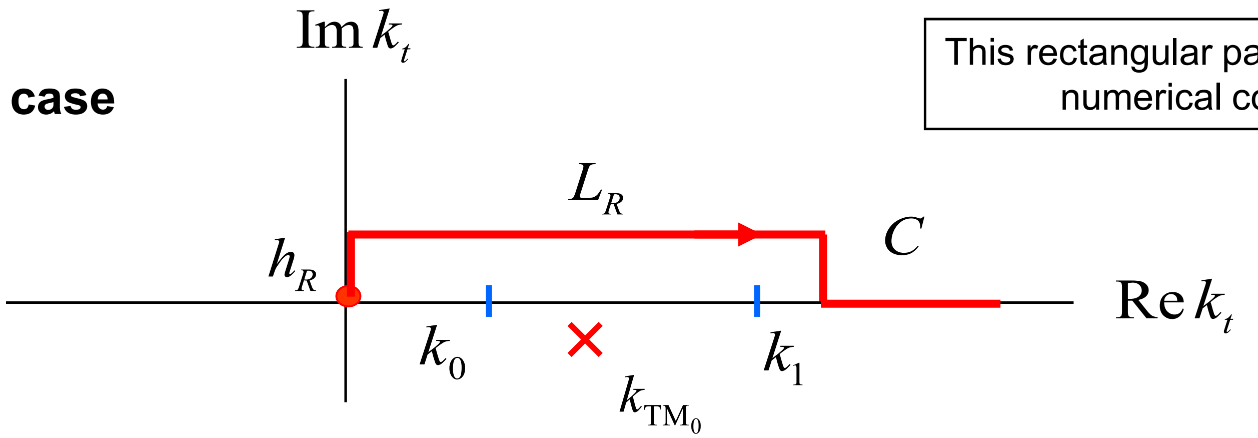
Path of Integration

The path avoids the poles by going above them.

Lossy case



Lossy case

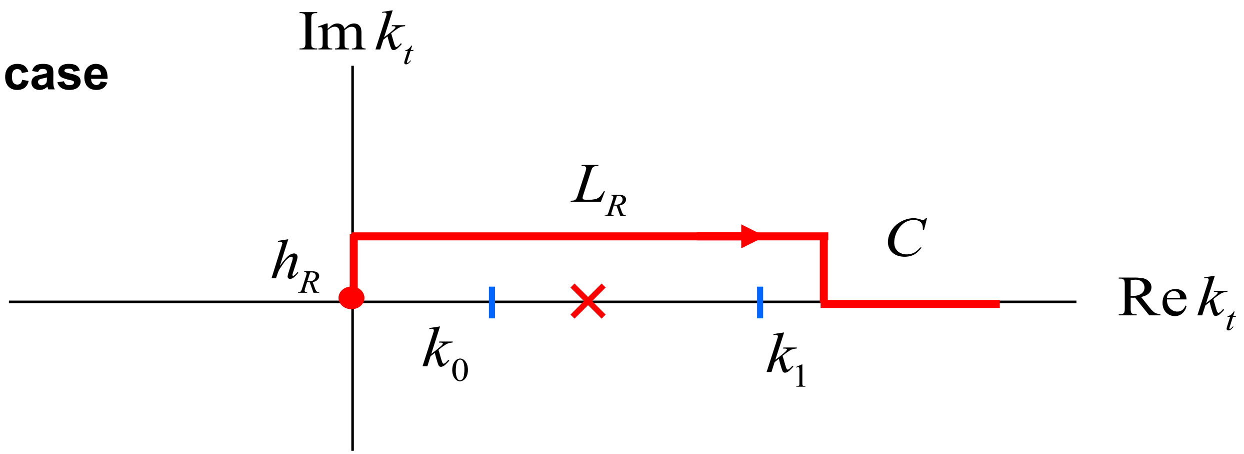


This rectangular path can be used for numerical computation.

Path of Integration (cont.)

The path avoids the poles by going above them.

Lossless case



$$h_R = 0.05\, k_0$$

$$L_R = k_1 (1.1)$$

(typical choices)

Practical note:
If h_R is too small, we are too close to the pole. If h_R is too large, there is too much round-off error due to exponential growth in the sin and cos functions.

Branch Points

To explain why we have branch points, consider the TM function:

$$D^{\mathrm{TM}}(k_t) = Y_0^{\mathrm{TM}} - jY_1^{\mathrm{TM}} \cot(k_{z1}h) = \left(\frac{\omega\varepsilon_0}{k_{z0}}\right) - j\left(\frac{\omega\varepsilon_1}{k_{z1}}\right)\cot(k_{z1}h)$$

with

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_t^2\right)^{1/2}$$

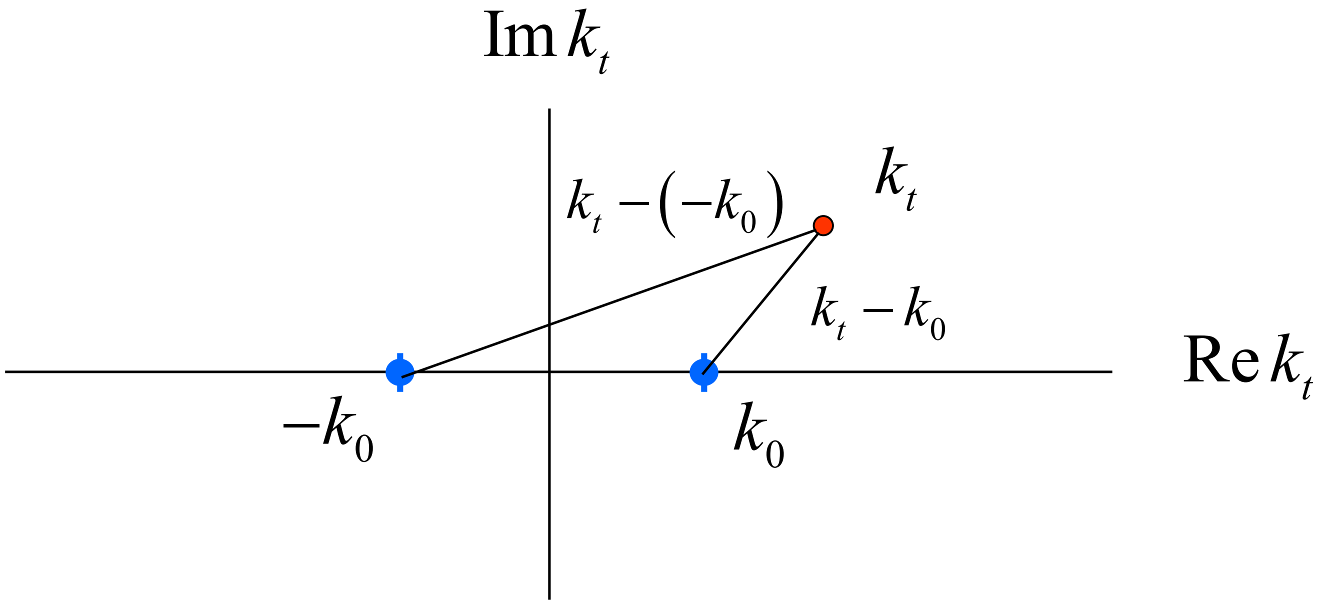
> **Note:**
> There are no branch cuts for k_{z1}
> (the function D^{TM} is an even function of k_{z1}).

If $k_{z0} \to -k_{z0}$ $\quad D^{\mathrm{TM}}(k_t)$ changes (We need branch cuts for k_{z0}.)

Branch Points (cont.)

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$
$$= \left(k_0 - k_t\right)^{1/2} \left(k_0 + k_t\right)^{1/2}$$
$$= -j\left(k_t - k_0\right)^{1/2} \left(k_t + k_0\right)^{1/2}$$
$$= -j\left(k_t - k_0\right)^{1/2} \left(k_t - (-k_0)\right)^{1/2}$$

Note:
The representation of the square root of -1 as $-j$ is arbitrary here.



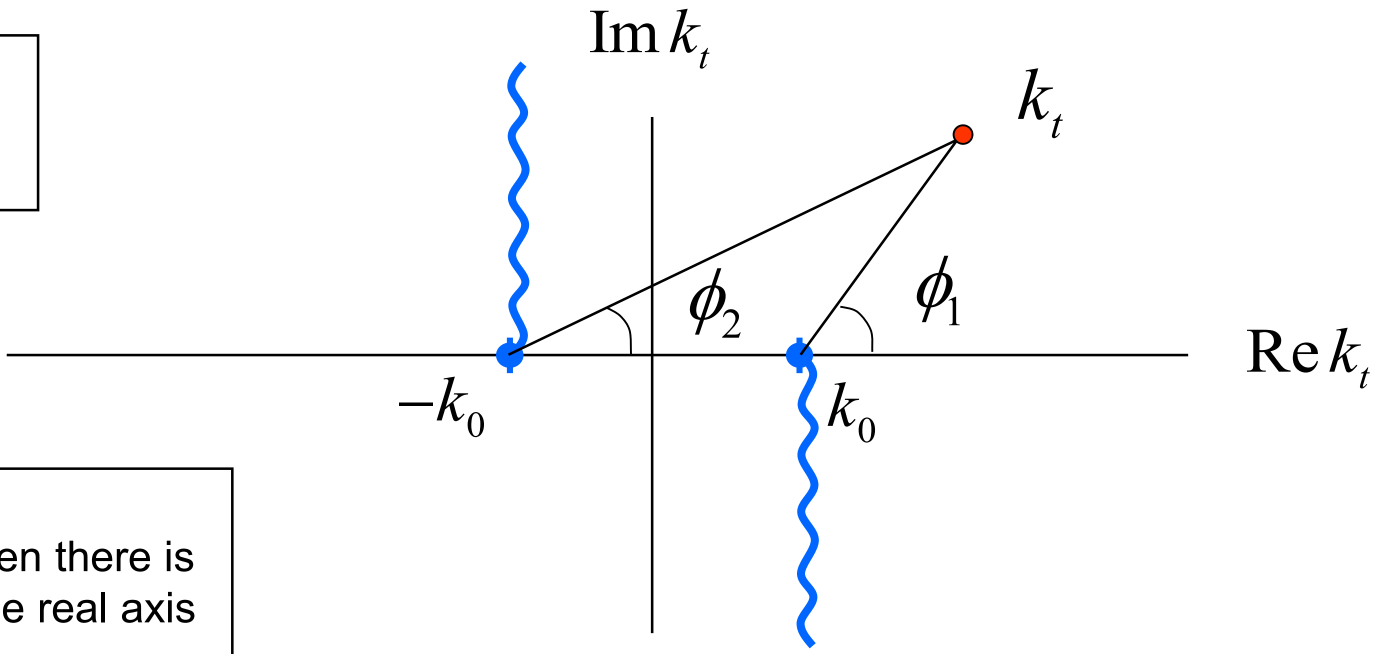
Branch Points (cont.)

$$k_{z0} = -j\left(k_t - k_0\right)^{1/2}\left(k_t - \left(-k_0\right)\right)^{1/2}$$
$$= -j\sqrt{\left|k_t - k_0\right|}\ \sqrt{\left|k_t - \left(-k_0\right)\right|}\ e^{j\phi_1/2}\ e^{j\phi_2/2}$$

$$\phi_1 = \text{Arg}\left(k_t - k_0\right)$$
$$\phi_2 = \text{Arg}\left(k_t - \left(-k_0\right)\right)$$

Branch cuts are necessary to prevent the angles from changing by 2π:

Note:
The shape of the branch cuts is arbitrary, but vertical cuts are shown here.



Note:
The branch cuts should not cross the real axis when there is loss in the air and the branch points move off of the real axis (the integrand must be continuous).

Branch Points (cont.)

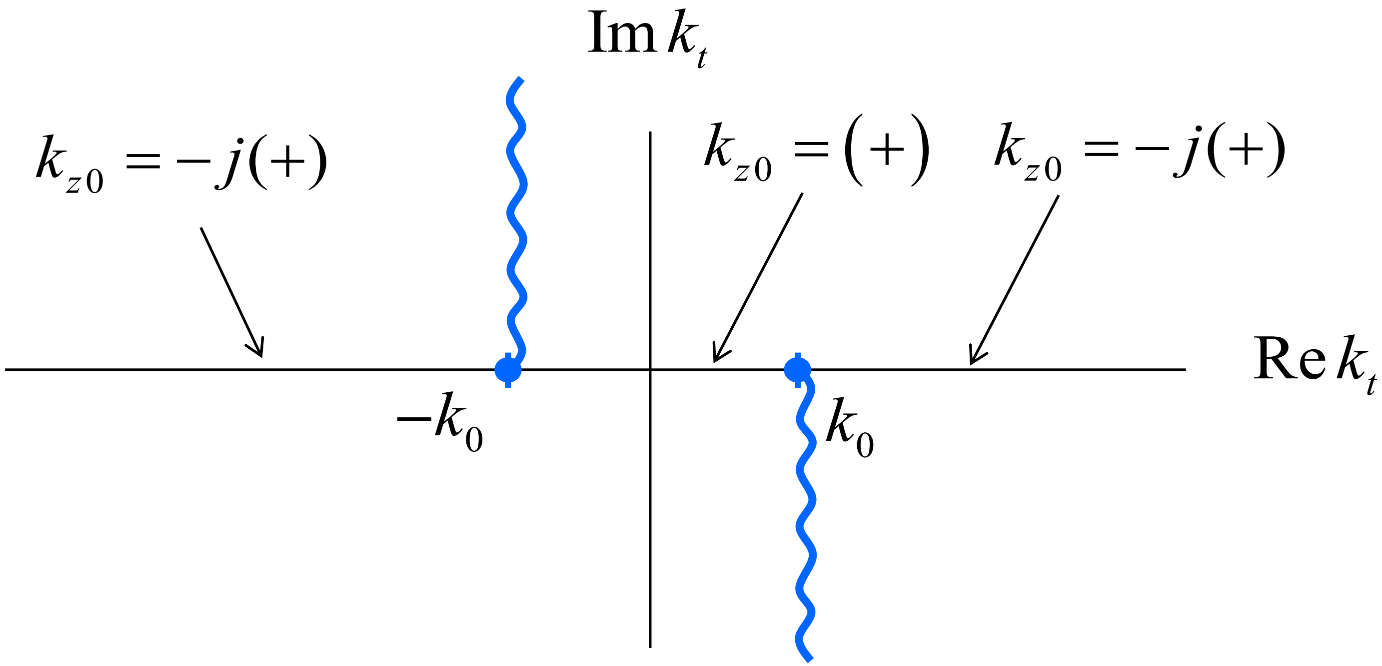
$$k_{z0} = -j\sqrt{|k_t - k_0|}\,\sqrt{|k_t - (-k_0)|}\;e^{j\phi_1/2}\,e^{j\phi_2/2}$$

We obtain the correct signs for k_{z0} if we choose the following branches:

$$-\pi/2 < \mathrm{Arg}(k_t - k_0) < 3\pi/2$$
$$-3\pi/2 < \mathrm{Arg}(k_t - (-k_0)) < \pi/2$$

The wave is then either decaying or outgoing in the air region when we are on the real axis.

$$\phi_1 = \mathrm{Arg}(k_t - k_0)$$
$$\phi_2 = \mathrm{Arg}(k_t - (-k_0))$$



Branch Points (cont.)

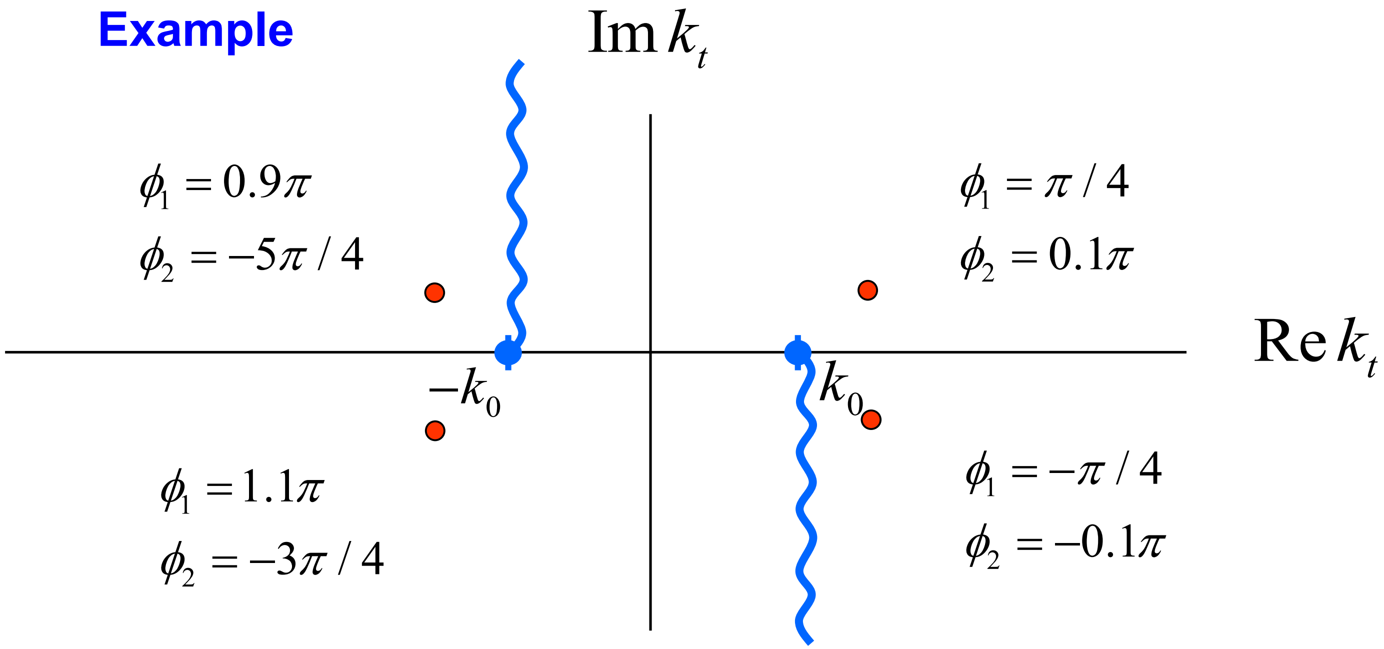
The wavenumber k_{z0} is then uniquely defined everywhere in the complex plane:

$$k_{z0} = -j\sqrt{|k_t - k_0|}\,\sqrt{|k_t - (-k_0)|}\; e^{j\phi_1/2}\, e^{j\phi_2/2}$$

$$-\pi/2 < \phi_1 < 3\pi/2$$
$$-3\pi/2 < \phi_2 < \pi/2$$

$$\phi_1 = \mathrm{Arg}(k_t - k_0)$$
$$\phi_2 = \mathrm{Arg}(k_t - (-k_0))$$

Example



Riemann Surface

- ❖ The Riemann surface is a pair of complex planes, connected by “ramps” (where the branch cuts used to be).
- ❖ The angles (and hence the function) change continuously over the surface.
- ❖ All possible values of the function are found on the surface.

Riemann Surface (cont.)

Riemann surface for $z^{1/2}$

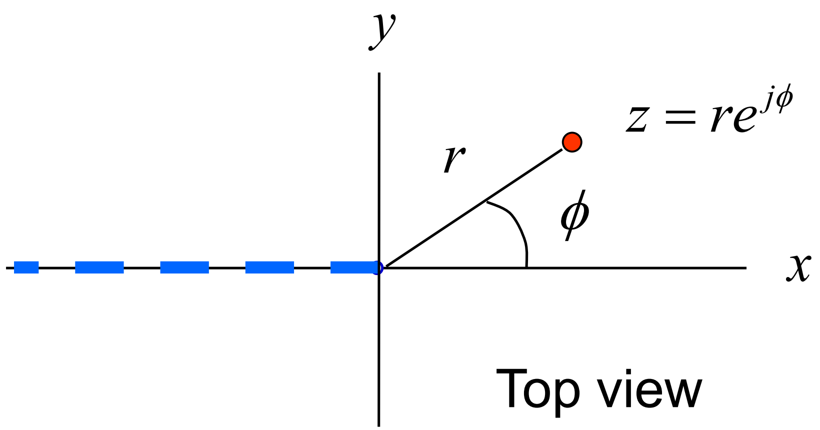
Top sheet

$-\pi < \phi < \pi$

Bottom sheet

$\pi < \phi < 3\pi$

MATLAB : $-\pi < \phi \leq \pi$



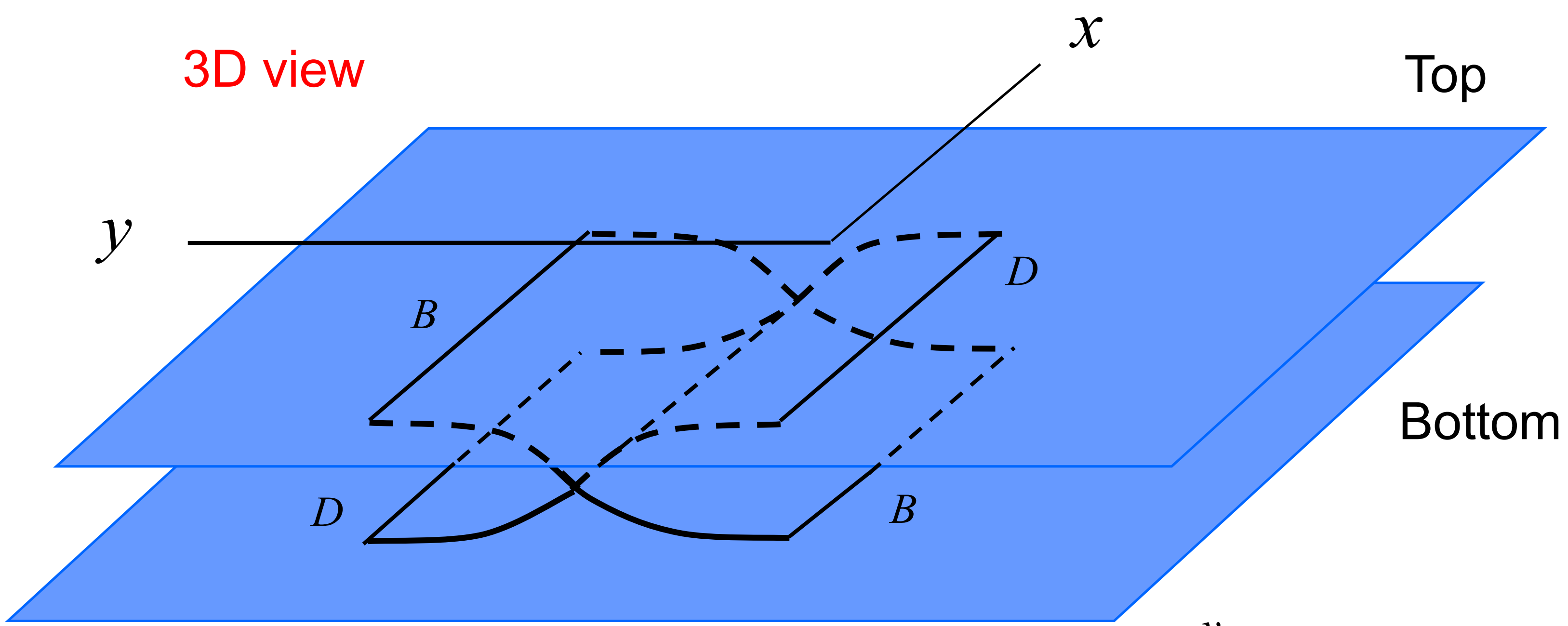
Note: A horizontal branch cut has been arbitrarily chosen.

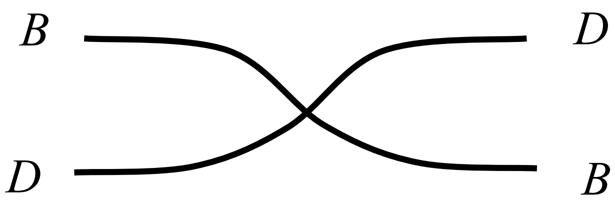
A “ramp” or “escalator” now exists where the branch cut used to be.

Riemann Surface (cont.)

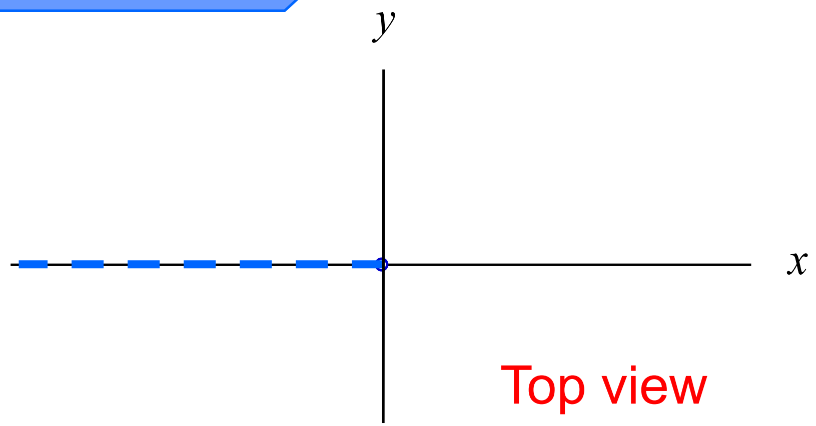
Riemann surface for $z^{1/2}$

3D view





Side view



Top view

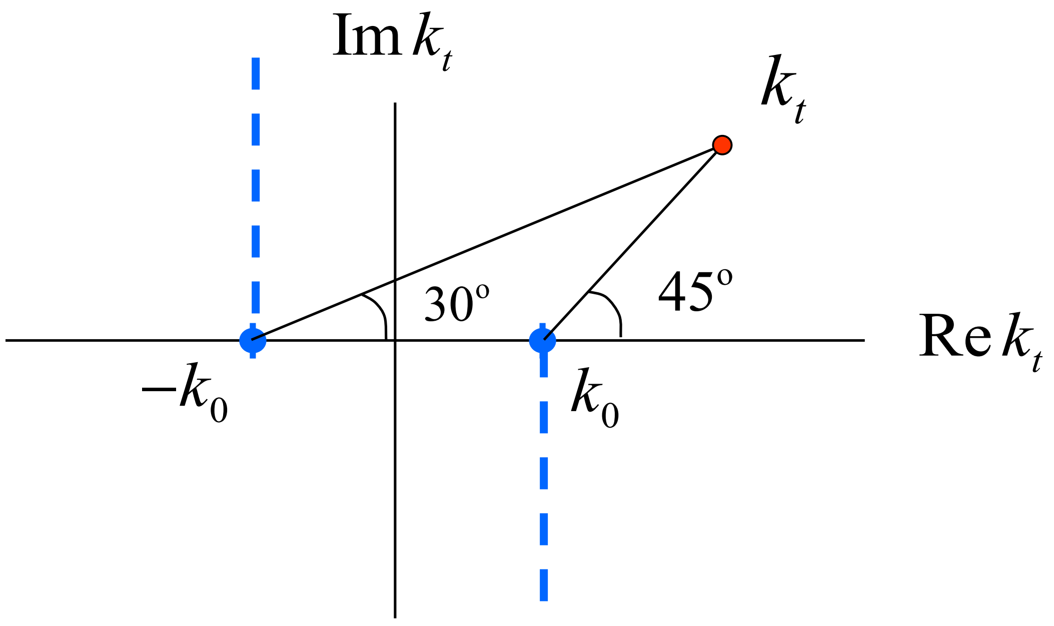
Riemann Surface (cont.)

The Riemann surface can be constructed for the wavenumber function:

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2} = -j\left(k_t - k_0\right)^{1/2}\left(k_t - \left(-k_0\right)\right)^{1/2}$$

Example:

We go counter-clockwise around the branch point at k_0. We start on the top sheet on the real axis and end up back where we started but on the bottom sheet. We track the point shown below (red dot).



Top sheet

$\phi_1 = \pi / 4$

$\phi_2 = \pi / 6$

Bottom sheet

$\phi_1 = \pi / 4 + 2\pi$

$\phi_2 = \pi / 6$

Sommerfeld Branch Cuts

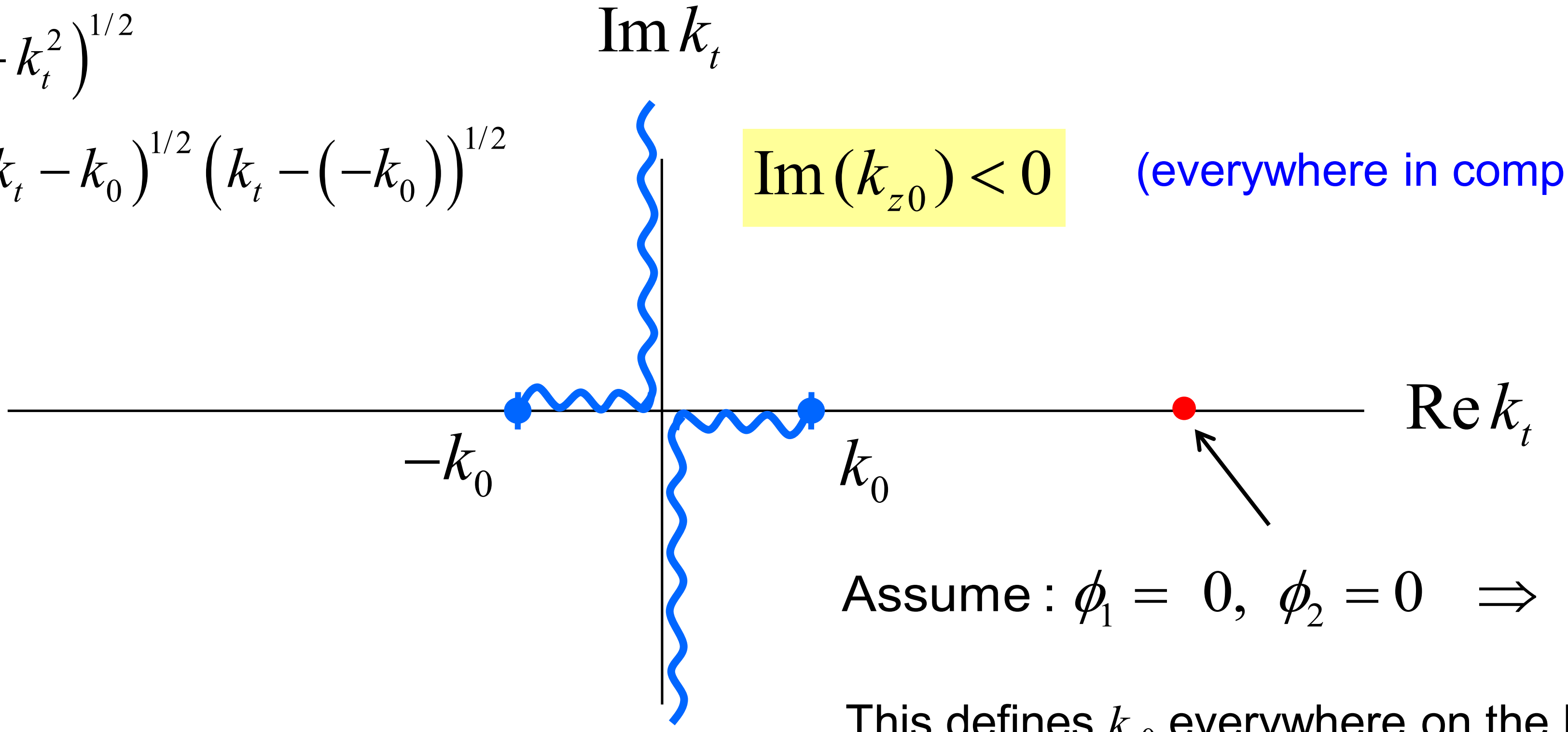
Sommerfeld branch cuts are a convenient choice for theoretical purposes (discussed more in ECE 6341 and ECE 6382):

$\mathrm{Im}(k_{z0}) = 0$ on branch cut

$$k_{z0} = \left(k_0^2 - k_t^2\right)^{1/2}$$

$$k_{z0} = -j\left(k_t - k_0\right)^{1/2}\left(k_t - (-k_0)\right)^{1/2}$$

$\mathrm{Im}(k_{z0}) < 0$ (everywhere in complex plane)



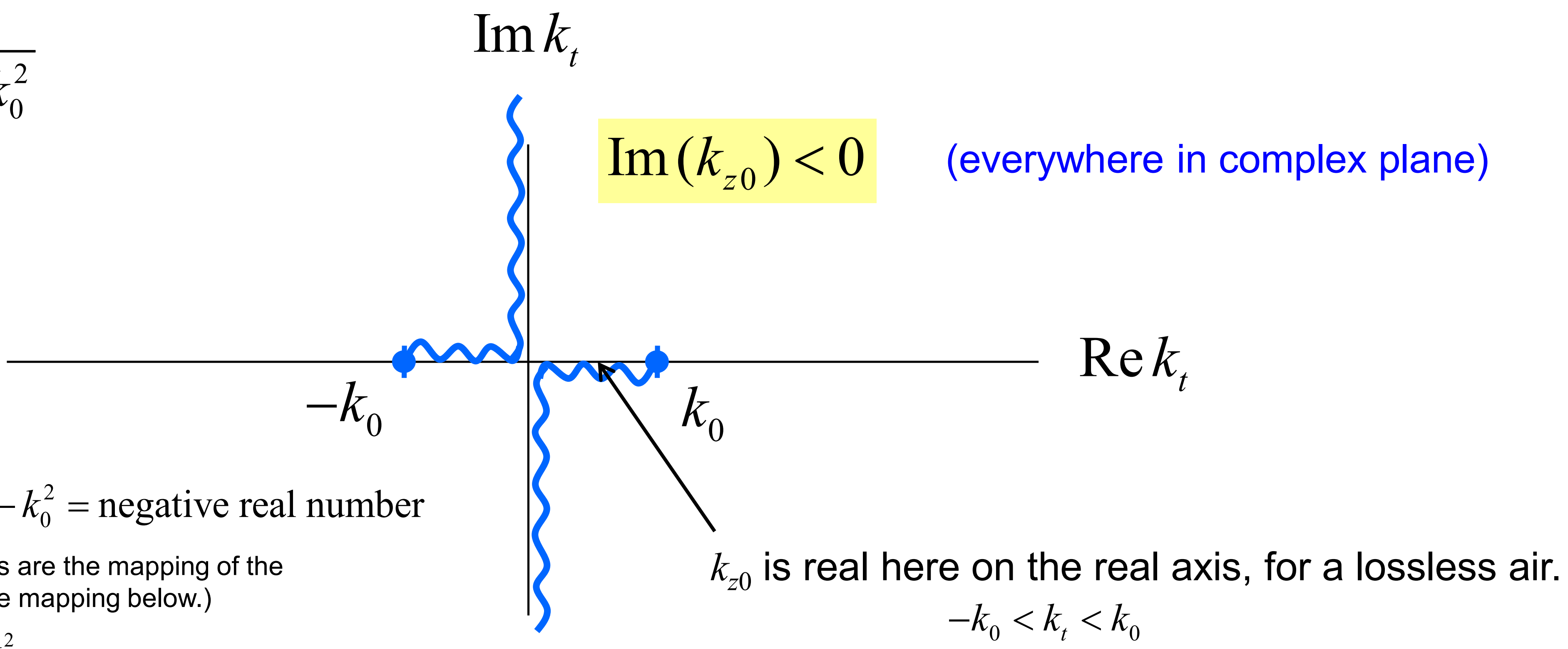
Assume : $\phi_1 = 0,\ \phi_2 = 0 \;\Rightarrow\; k_{z0} = -j(+)$

This defines k_{z0} everywhere on the Riemann surface.

Sommerfeld Branch Cuts (cont.)

$$k_{z0} = -j\sqrt{k_t^2 - k_0^2}$$

Note : $\operatorname{Re}\sqrt{z} \geq 0$



$$k_t \in \text{branch cut} \;\Leftrightarrow\; z = k_t^2 - k_0^2 = \text{negative real number}$$

(The Sommerfeld branch cuts are the mapping of the negative real axis in the mapping below.)

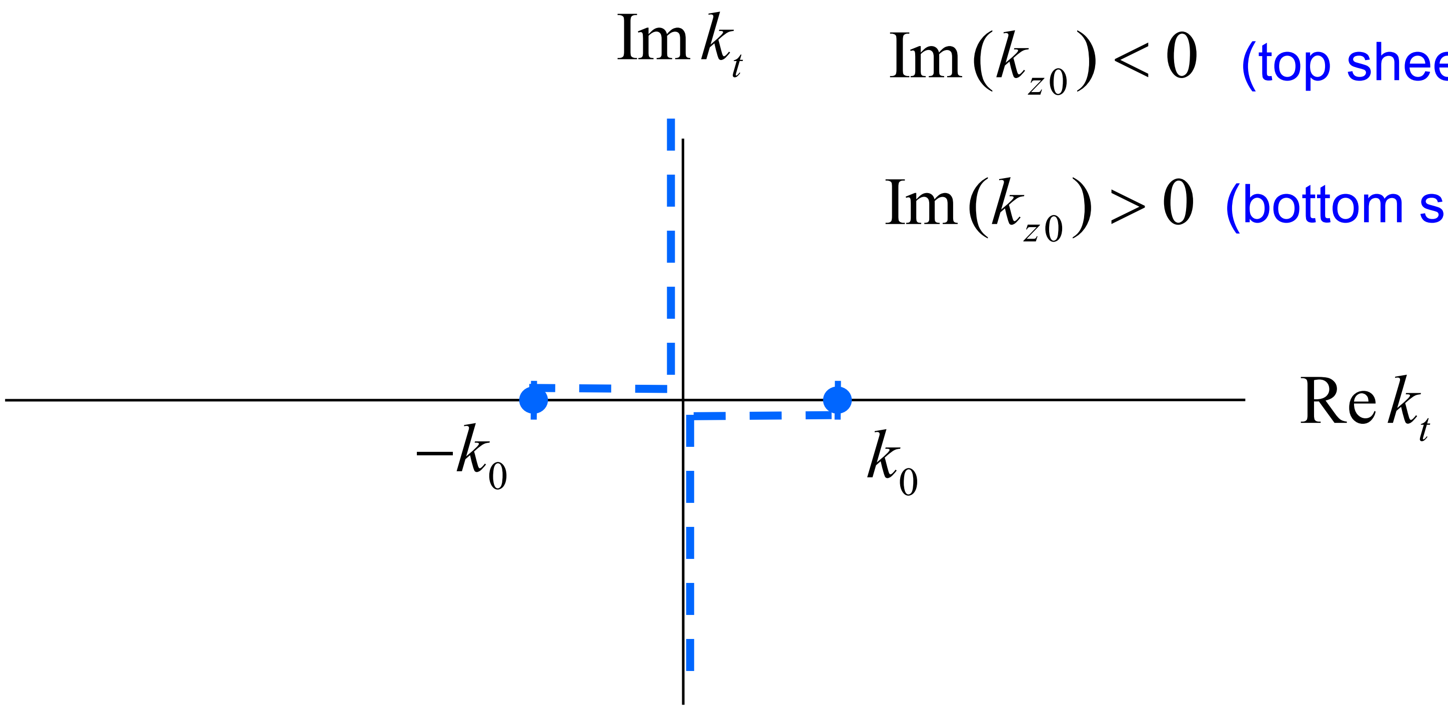
$$z = k_t^2 - k_0^2$$

Practical note:
If we give the air a small amount of loss, we can simply check to make sure that $\operatorname{Im}(k_{z0}) < 0$.

Note: The branch points move off of the axes for a lossy air.

Sommerfeld Branch Cuts (cont.)

The Riemann surface with Sommerfeld branch cuts.



$\text{Im}(k_{z0}) < 0$ (top sheet)

$\text{Im}(k_{z0}) > 0$ (bottom sheet)

Note: Surface wave poles must lie on the top sheet, and leaky-wave poles must lie on the bottom sheet.

Path of Integration

$$E_x = \int_0^{\pi/2} I(\bar{\phi})\, d\bar{\phi}$$

where

$$I(\bar{\phi}) = \int_0^{\infty} F(k_t, \bar{\phi})\, dk_t$$