ECE 6345

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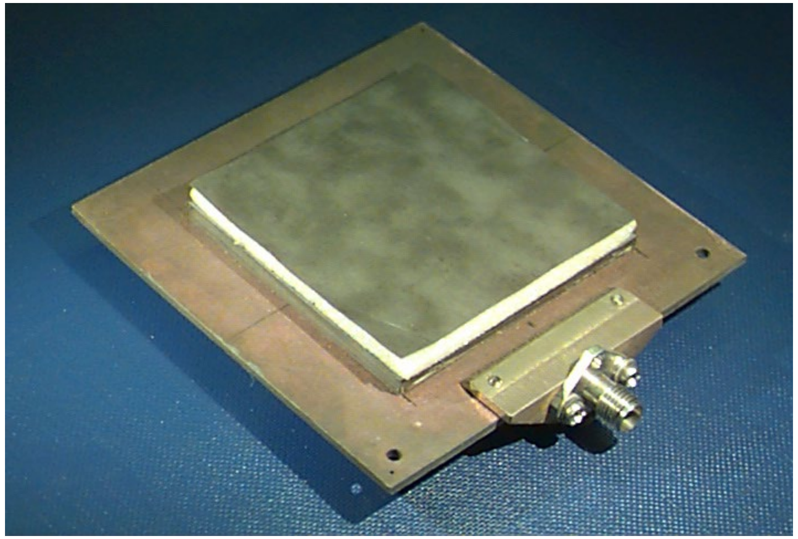
Prof. David R. Jackson
ECE Dept.



Notes 28

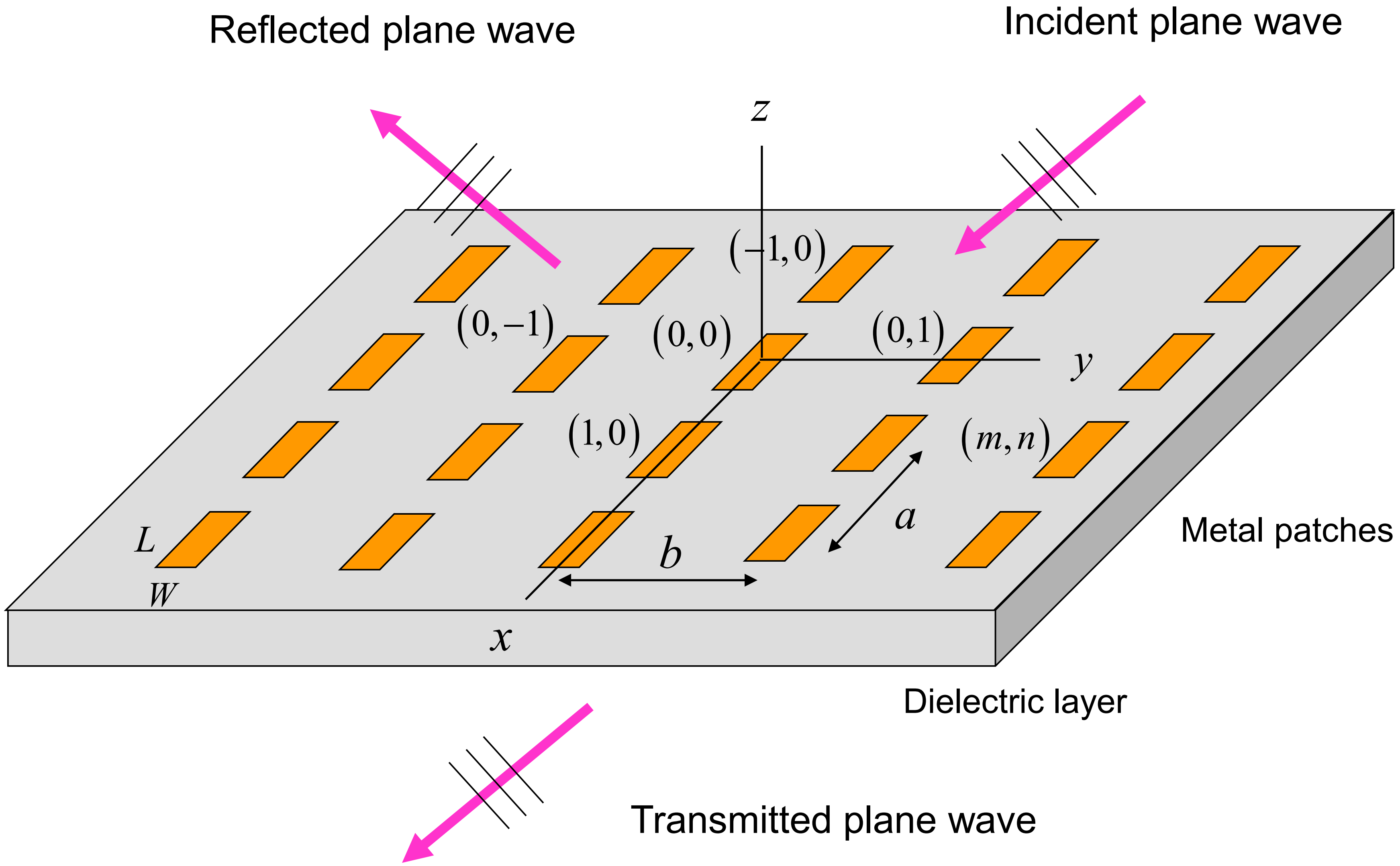
Overview

❖ In this set of notes we extend the spectral-domain method to analyze infinite periodic structures.

Two typical examples of infinite periodic problems:

- Scattering from a frequency selective surface (FSS)
- Input impedance of a microstrip phased array

FSS Geometry



FSS Geometry (cont.)

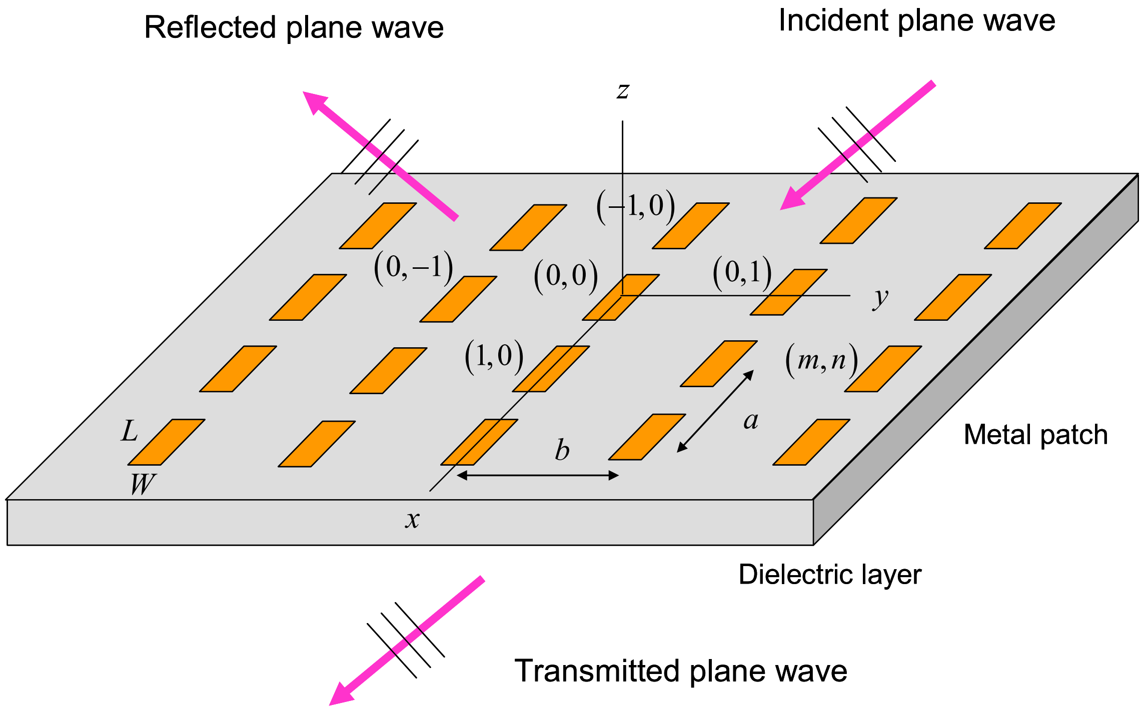
$$\psi^{\text{inc}} = A e^{-j\left(k_{x0}x + k_{y0}y\right)} e^{+jk_{z0}z}$$

$$\left(\theta_0, \phi_0\right) = \text{ arrival angles}$$

$$k_{x0} = -k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = -k_0 \sin\theta_0 \sin\phi_0$$

$$k_{z0} = k_0 \cos\theta_0$$

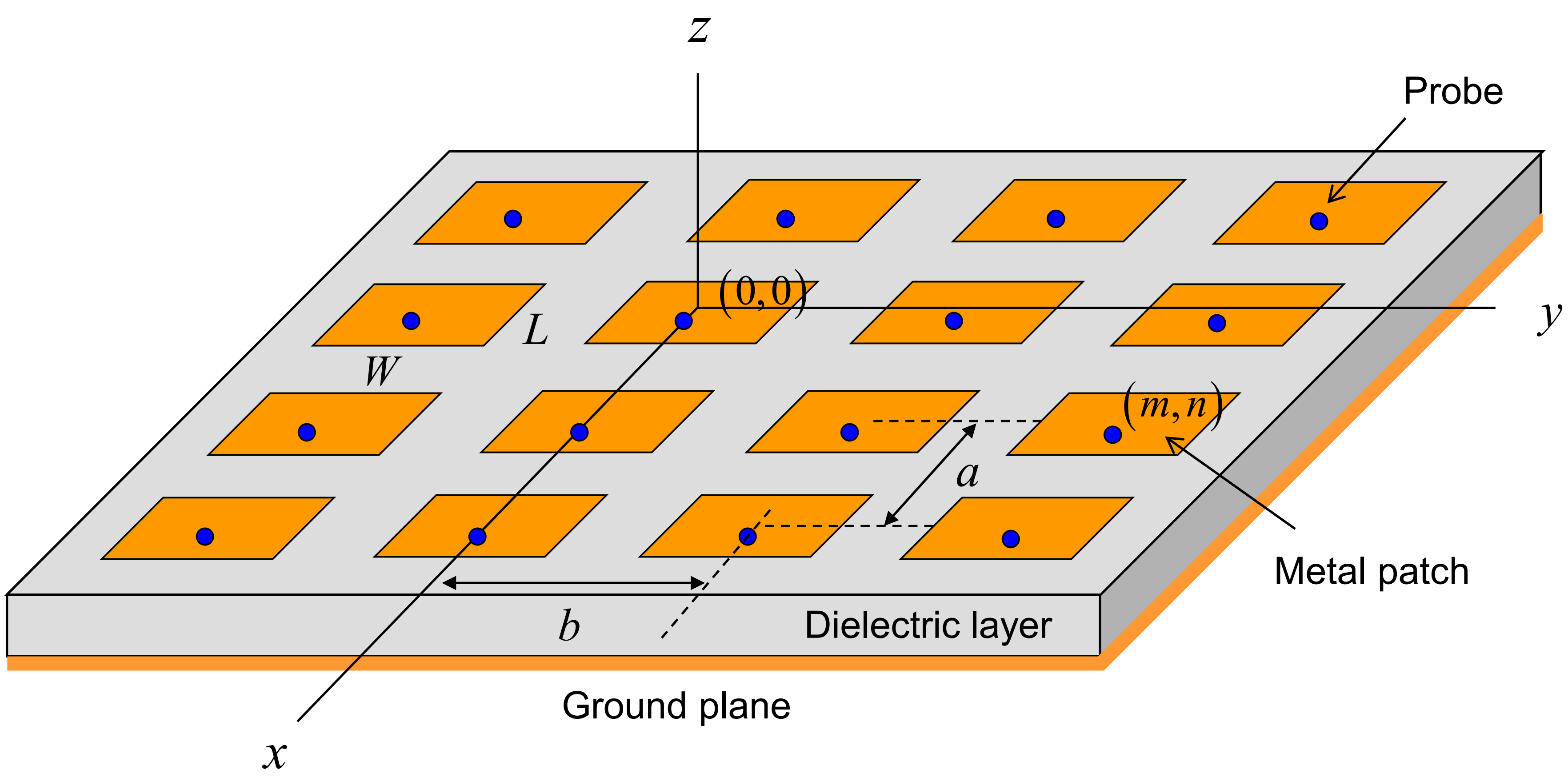


Note:
ψ denotes any field component of interest.

Note: We are following "plane-wave" convention for k_{x0} and k_{y0}, and "transmission-line" convention for k_{z0}.

Microstrip Phased Array Geometry

Probe current mn: $I_{mn} = I_{00}\, e^{-j\left(k_{x0} m a + k_{y0} n b\right)}$



The wavenumbers k_{x0} and k_{y0} are impressed by the feed network.

Microstrip Phased Array Geometry (cont.)

$$I_{mn} = I_{00}\, e^{-j\left(k_{x0} m a + k_{y0} n b\right)}$$

$(\theta_0, \phi_0) =$ radiation angles

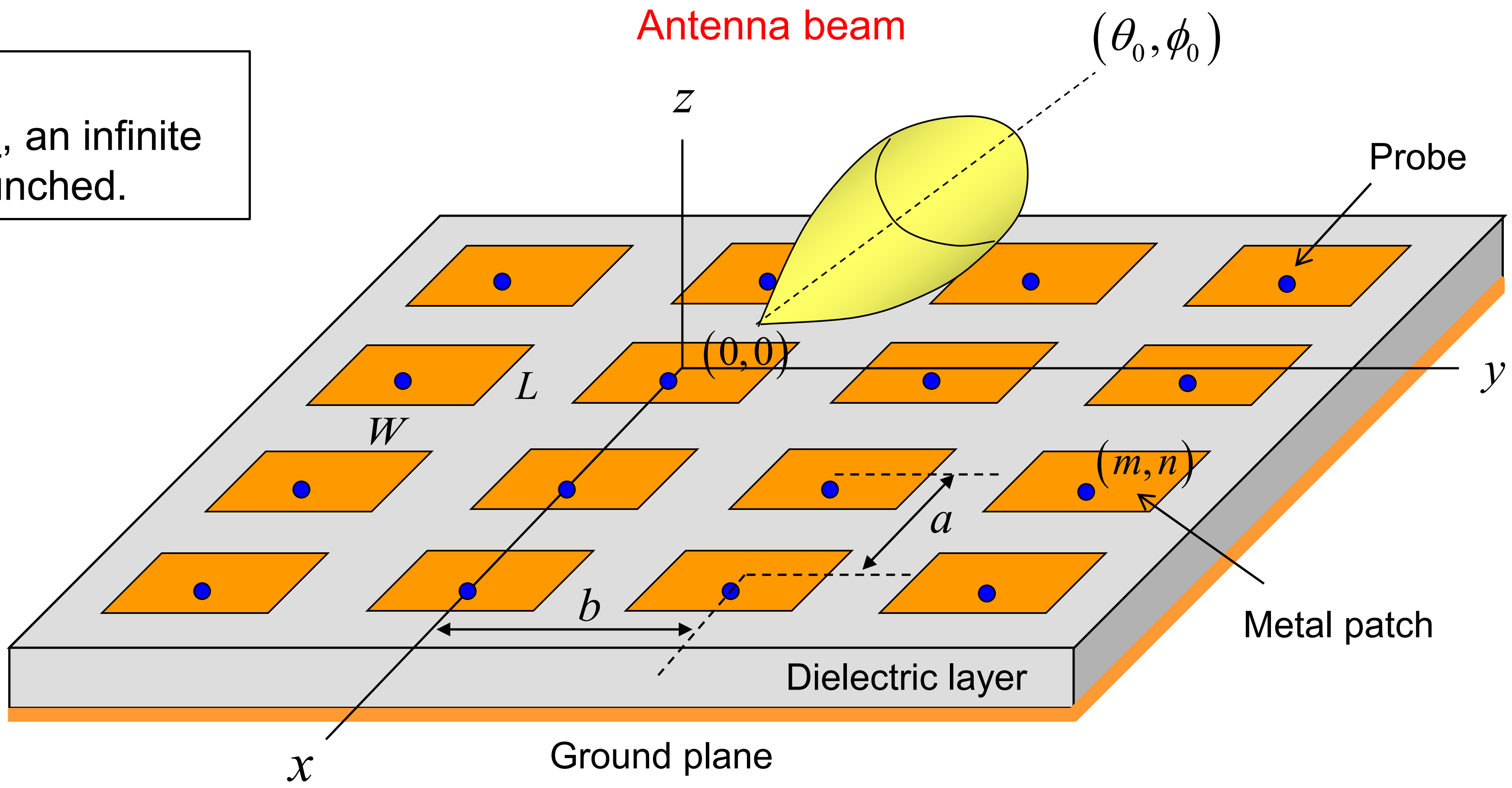
$$k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$

Note:
If the structure is infinite, an infinite plane wave gets launched.

Center of patch (m,n):

$$x = ma,\; y = nb$$

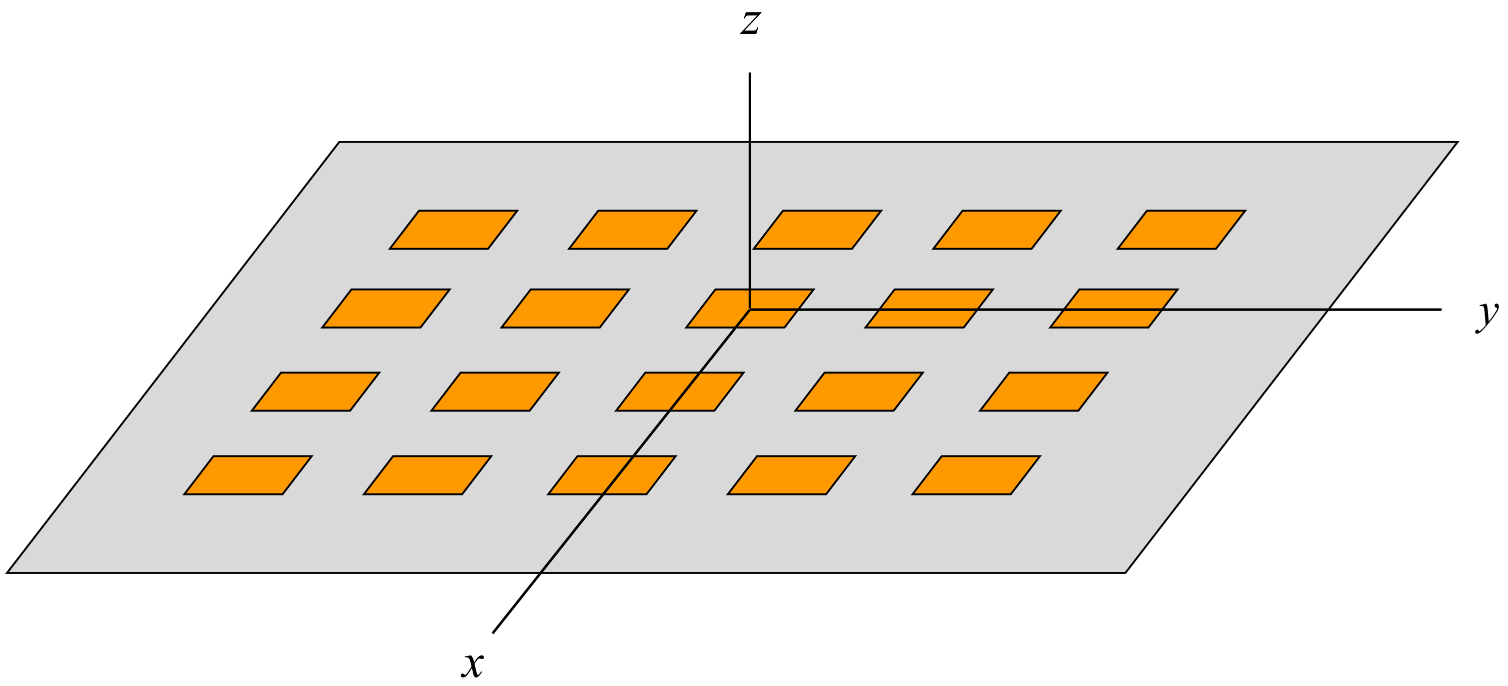


Floquet's Theorem

Fundamental observation:

If the structure is infinite and periodic, and the excitation is periodic except for a phase shift, then all the currents and radiated fields will also be periodic except for a phase shift.

This is sometimes referred to as "*Floquet's theorem*."



Floquet's Theorem (cont.)

From Floquet's theorem:

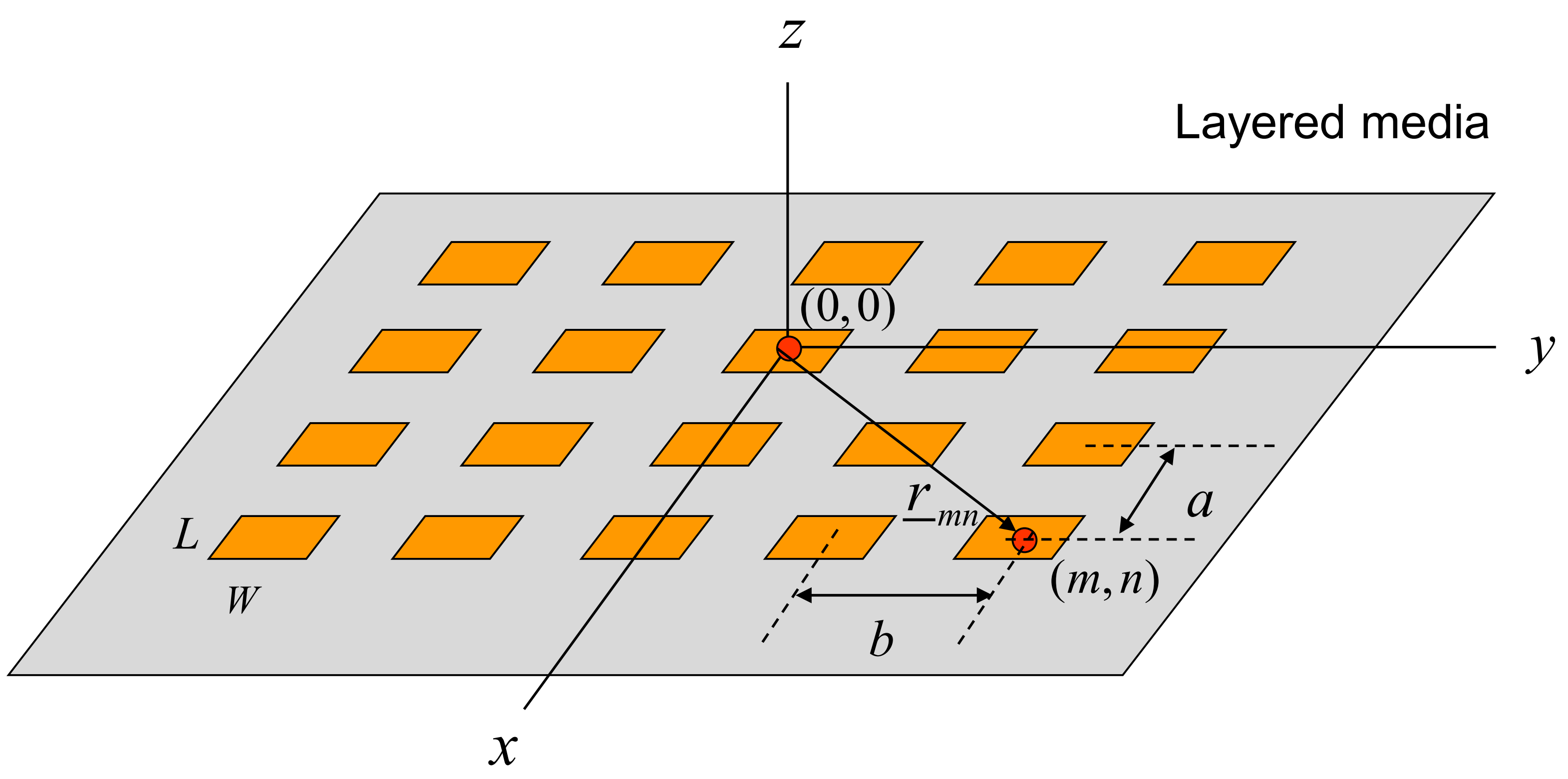
$$\underline{J}_s^{mn}\left(\underline{r}\right)=\underline{J}_s^{00}\left(\underline{r}-\underline{r}_{mn}\right)e^{-j\underline{k}_{t00}\cdot\underline{r}_{mn}}$$

$$\underline{k}_{t00}=\underline{\hat{x}}\,k_{x0}+\underline{\hat{y}}\,k_0$$

$$\underline{r}_{mn}=\underline{\hat{x}}\left(ma\right)+\underline{\hat{y}}\left(nb\right)$$

(vector that points to the center of patch (m,n))

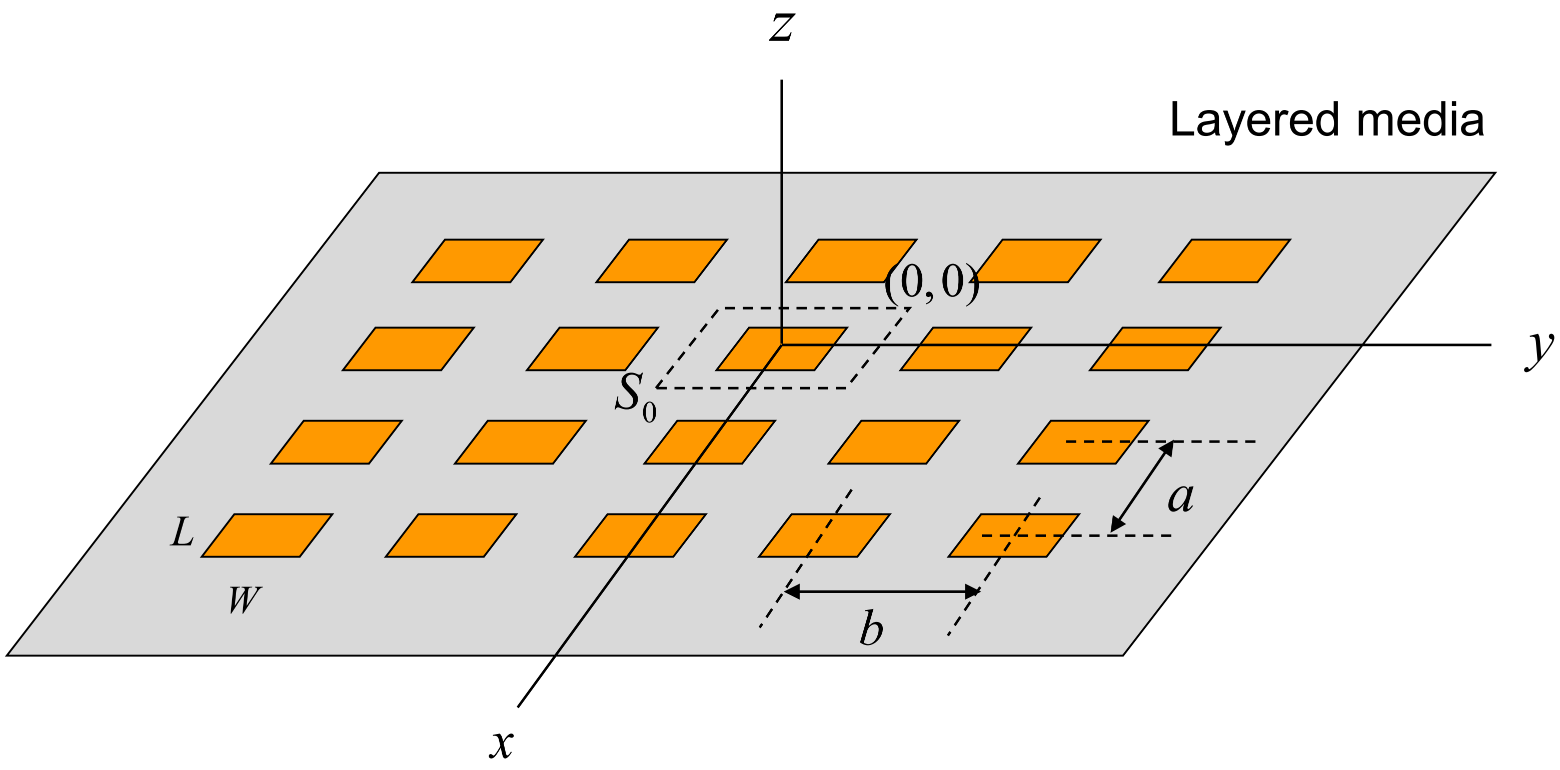
$$e^{-j\underline{k}_{t00}\cdot\underline{r}_{mn}}=e^{-j\left(k_{x0}ma+k_{y0}nb\right)}$$



Floquet's Theorem (cont.)

If we know the current of field at any point within the $(0,0)$ unit cell, we know the current and field everywhere.

$A = ab =$ area of unit cell S_0



Floquet Waves

Let ψ denote any component of the surface current or the field (at a fixed value of z).

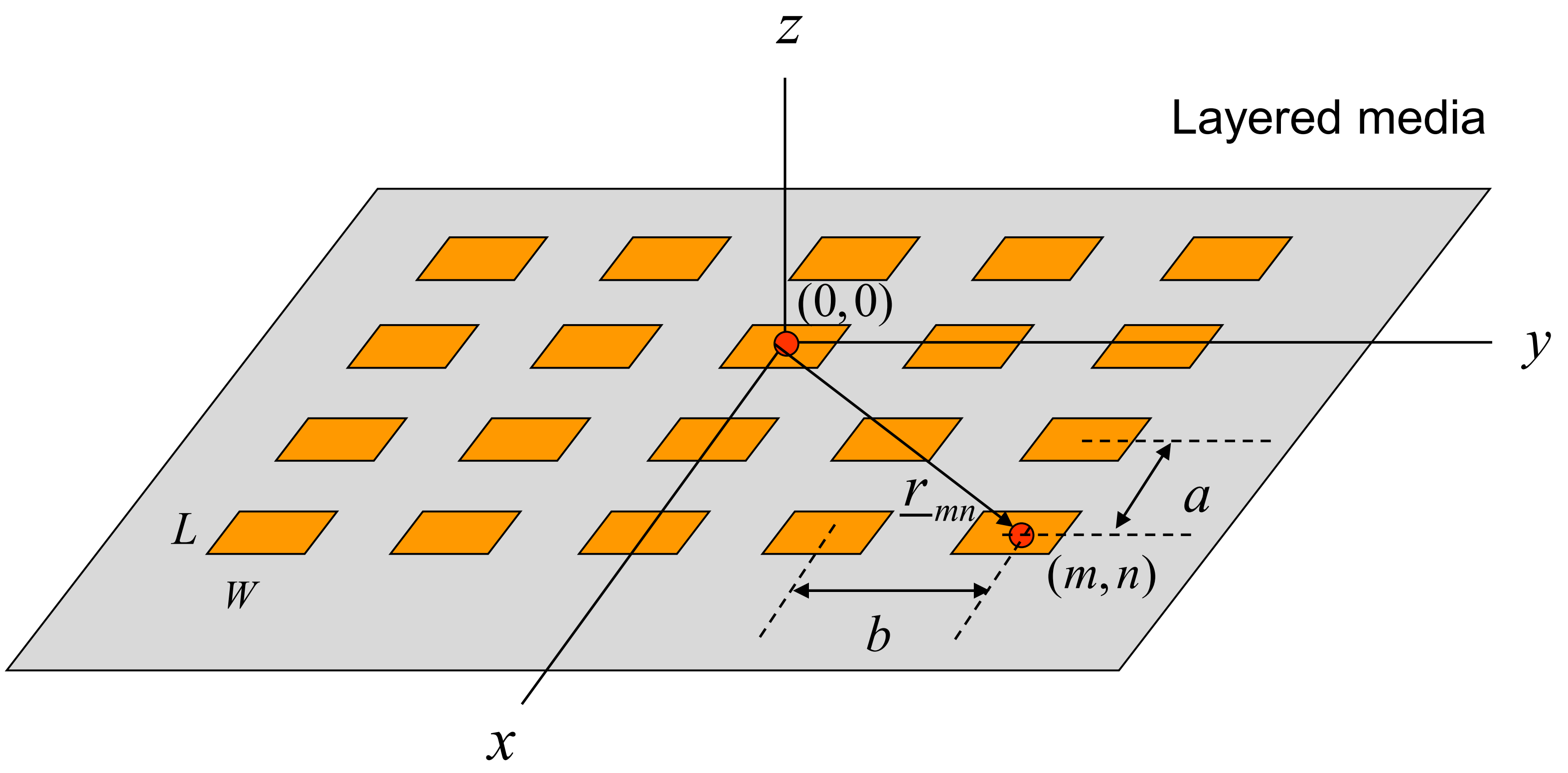
$$\psi(x+a,y)=\psi(x,y)e^{-jk_{x0}a}$$
$$\psi(x,y+b)=\psi(x,y)e^{-jk_{y0}b}$$

$$\psi(x,y)=e^{-j\left(k_{x0}x+k_{y0}y\right)}P(x,y)$$

where

$$P(x+a,y)=P(x,y)$$
$$P(x,y+b)=P(x,y)$$

(a 2D periodic function)



Floquet Waves (cont.)

$$\psi(x,y) = e^{-j\left(k_{x0}x + k_{y0}y\right)} P(x,y)$$

From Fourier-series theory, we know that the 2D periodic function P can be represented as:

$$P(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(\frac{2\pi p}{a}x + \frac{2\pi q}{b}y\right)}$$

Hence, we have:

$$\psi(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq}\, e^{-j\left(\left(k_{x0} + \frac{2\pi p}{a}\right)x + \left(k_{y0} + \frac{2\pi q}{b}\right)y\right)}$$

Floquet Waves (cont.)

Hence, any surface current or field component can be expanded in a set of Floquet waves:

$$\psi(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}A_{pq}\,e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

$$k_{xp}\equiv k_{x0}+\frac{2\pi p}{a}$$

$$k_{yq}\equiv k_{y0}+\frac{2\pi q}{b}$$

$\left(k_{xp},k_{yq}\right)=$ wavenumbers of (p,q) Floquet wave

$$\underline{k}_{tpq}=\underline{\hat{x}}\,\underline{k}_{xp}+\underline{\hat{y}}\,\underline{k}_{yq}$$

$$\underline{k}_{tpq}=\left(\underline{\hat{x}}k_{x0}+\underline{\hat{y}}k_{y0}\right)+\left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}}+\left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right]$$

Incident part — Periodic part

Note:

$$\underline{k}_{tpq}\cdot\underline{r}=k_{xp}x+k_{yq}y$$

where $\underline{r}=\underline{\hat{x}}x+\underline{\hat{y}}y$

Floquet Waves (cont.)

Note: Each Floquet wave repeats from one unit cell to the next, except for a phase shift that corresponds to that of the *incident wave*.

$$\psi_{pq}(x+a,y) = e^{-j\left(k_{xp}(x+a)+k_{yq}y\right)}$$
$$= e^{-j\left(k_{xp}a\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)}$$
$$= e^{-j\left(k_{x0}+\frac{2\pi p}{a}\right)a} e^{-j\left(k_{xp}x+k_{yq}y\right)}$$
$$= e^{-j\left(k_{x0}a\right)} e^{-j\left(\frac{2\pi p}{a}a\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)}$$
$$= e^{-j\left(k_{x0}a\right)} e^{-j\left(2\pi p\right)} e^{-j\left(k_{xp}x+k_{yq}y\right)}$$
$$= e^{-j\left(k_{x0}a\right)} \psi_{pq}(x,y)$$

Hence, we have:

$$\psi_{pq}(x+a,y) = e^{-j\left(k_{x0}a\right)} \psi_{pq}(x,y)$$

Similarly,

$$\psi_{pq}(x,y+b) = e^{-j\left(k_{y0}b\right)} \psi_{pq}(x,y)$$

Periodic SDI

The surface current on the periodic structure is next represented in terms of Floquet waves:

$$\underline{J}_s(x,y) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\underline{k}_{tpq}\cdot\underline{r}}$$

$$\underline{k}_{tpq} = \underline{\hat{x}}k_{xp} + \underline{\hat{y}}k_{yq}$$

$$\underline{k}_{tpq} = \left(\underline{\hat{x}}k_{x0} + \underline{\hat{y}}k_{y0}\right) + \left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}} + \left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right]$$

To solve for the unknown coefficients, multiply both sides by $e^{j\underline{k}_{tp'q'}\cdot\underline{r}}$ and integrate over the $(0,0)$ unit cell S_0:

$$\int_{S_0} \underline{J}_s(x,y)\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS = \int_{S_0} \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\underline{k}_{tpq}\cdot\underline{r}}\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS$$

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \int_{S_0} \underline{a}_{pq}\, e^{-j\left(\frac{2\pi}{a}(p-p')x + \frac{2\pi}{b}(q-q')y\right)} dS$$

Use orthogonality: $\displaystyle\int_{S_0} \underline{J}_s(x,y)\, e^{j\underline{k}_{tp'q'}\cdot\underline{r}}\, dS = \underline{a}_{p'q'}\, A$ $\qquad A = ab =$ area of unit cell S_0

Periodic SDI (cont.)

Hence, we have:

$$\underline{a}_{pq} = \frac{1}{A} \int_{S_0} \underline{J}_s(x,y)\, e^{j\underline{k}_{tpq} \cdot \underline{r}}\, dS$$

Therefore, we have:

$$\begin{aligned}
\underline{a}_{pq} &= \frac{1}{A} \int_{S_0} \underline{J}_s(x,y)\, e^{j\left(k_{xp}x + k_{yq}y\right)}\, dS \\
&= \frac{1}{A} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{J}_s^{00}(x,y)\, e^{j\left(k_{xp}x + k_{yq}y\right)}\, dS \\
&= \frac{1}{A} \underline{\tilde{J}}_s^{00}\left(k_{xp}, k_{yq}\right)
\end{aligned}$$

The current $\underline{J}_s^{00}$ is the current on the $(0,0)$ patch.

We then have:

$$\underline{a}_{pq} = \frac{1}{A} \underline{\tilde{J}}_s^{00}\left(k_{xp}, k_{yq}\right)$$

Periodic SDI (cont.)

Hence the current on the 2D periodic structure can be represented as

$$\underline{J}_s(x,y)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_s^{00}\left(k_{xp},k_{yq}\right)e^{-j\underline{k}_{tpq}\cdot\underline{r}}$$

We now calculate the Fourier transform of the 2D periodic current $\underline{J}_s(x, y)$ (this is what we need in the SDI method):

$$F\left[e^{-j\underline{k}_{tpq}\cdot\underline{r}}\right]=F\left[e^{-jk_{xp}x}e^{-jk_{yq}y}\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{-jk_{xp}x}\,e^{-jk_{yq}y}\right]e^{+j\left(k_x x+k_y y\right)}\,dxdy$$

$$=\int_{-\infty}^{\infty}e^{-jk_{xp}x}\,e^{+jk_x x}\,dx\int_{-\infty}^{\infty}e^{-jk_{yq}y}\,e^{+jk_y y}\,dy$$

$$=2\pi\delta\left(k_x-k_{xp}\right)2\pi\delta\left(k_y-k_{yq}\right)$$

Periodic SDI (cont.)

Hence, we have:

$$\underline{\tilde{J}}_s(k_x,k_y)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_s^{00}(k_{xp},k_{yq})\;2\pi\delta(k_x-k_{xp})2\pi\delta(k_y-k_{yq})$$

Next, we calculate the field produced by the periodic patch currents:

$$\underline{\tilde{E}}(k_x,k_y,z)=\underline{\underline{\tilde{G}}}(k_x,k_y;z,z')\cdot\underline{\tilde{J}}_s(k_x,k_y)$$

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_x,k_y;z,z')\cdot\underline{\tilde{J}}_s(k_x,k_y)e^{-j(k_xx+k_yy)}dk_xdk_y$$

Periodic SDI (cont.)

Hence, we have:

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}\left(k_x,k_y;z,z'\right)\cdot$$

$$\left[\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_s^{00}\left(k_{xp},k_{yq}\right)2\pi\delta\left(k_x-k_{xp}\right)2\pi\delta\left(k_y-k_{yq}\right)\right]$$

$$e^{-j\left(k_x x+k_y y\right)}\,dk_x dk_y$$

Periodic SDI (cont.)

Therefore, integrating over the delta functions, we have:

$$\underline{E}(x,y,z)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\underline{\tilde{G}}}\left(k_{xp},k_{yq};z,z'\right)\cdot\underline{\tilde{J}}_{s}^{00}\left(k_{xp},k_{yq}\right)e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

The field is thus in the form of a double summation of Floquet waves.

Periodic SDI (cont.)

Compare:

Single element (non-periodic):

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_x,k_y;z,z')\cdot\underline{\tilde{J}}_s(k_x,k_y)\,e^{-j(k_x x+k_y y)}dk_x dk_y$$

Infinite periodic array of phased elements:

$$\underline{E}(x,y,z)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_{xp},k_{yq};z,z')\cdot\underline{\tilde{J}}_s^{00}(k_{xp},k_{yq})\,e^{-j(k_{xp}x+k_{yq}y)}$$

Note: $\underline{\tilde{J}}_s^{00}(k_x,k_y)_{\text{phased array}}=\underline{\tilde{J}}_s(k_x,k_y)_{\text{single patch}}$

Periodic SDI (cont.)

Conclusion:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\left(k_x,k_y\right)\,dk_x dk_y \to \frac{\left(2\pi\right)^2}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F\left(k_{xp},k_{yq}\right)$$

where

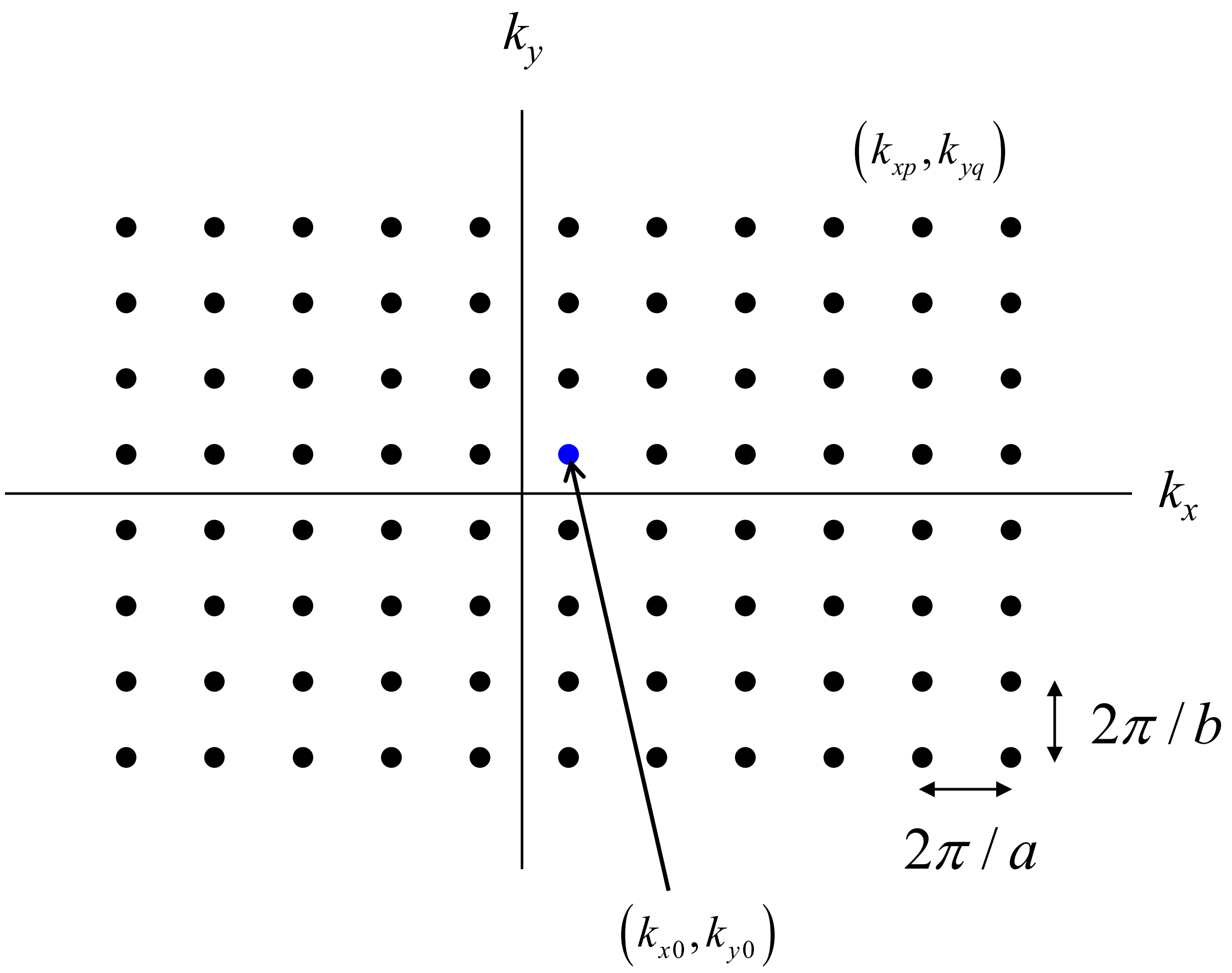
$$k_{xp} \equiv k_{x0} + \frac{2\pi p}{a}$$

$$k_{yq} \equiv k_{y0} + \frac{2\pi q}{b}$$

The double integral is replaced by a double sum, and a factor $\left(2\pi\right)^2 / A$ is introduced.

Periodic SDI (cont.)

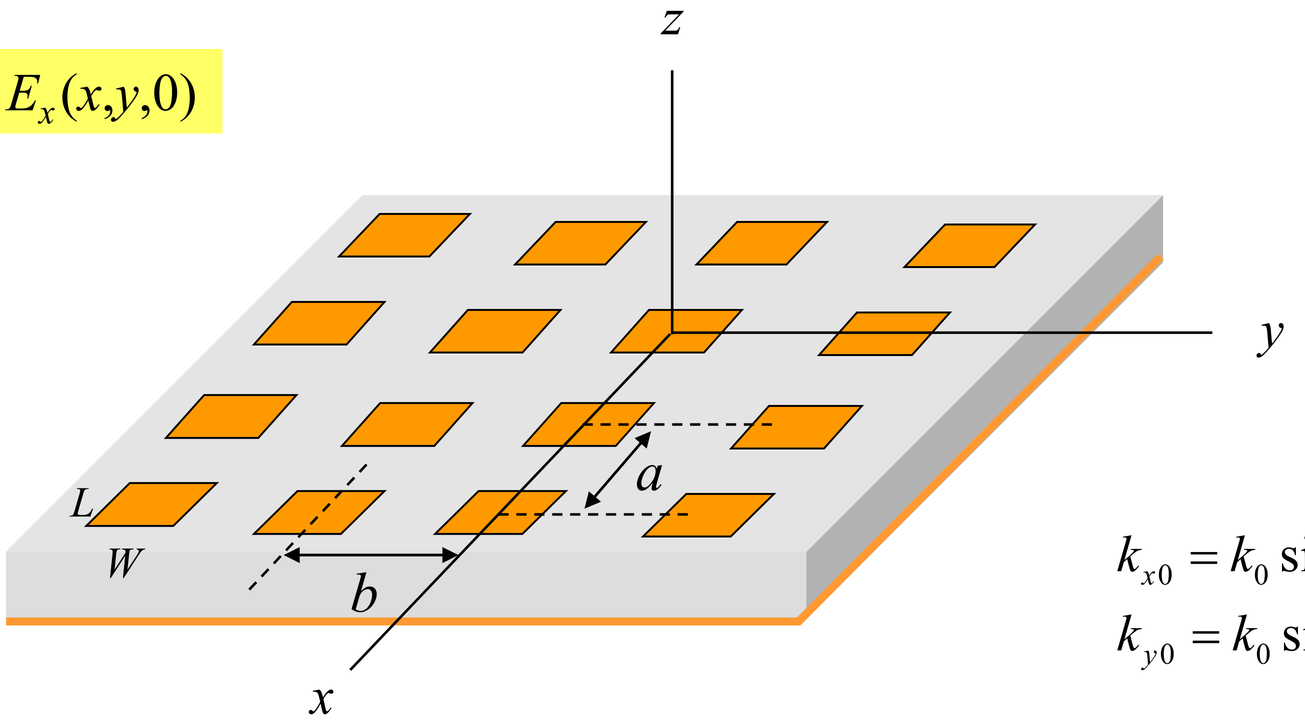
Sample points in the (k_x, k_y) plane



Microstrip Patch Phased Array

Example

Find $E_x(x,y,0)$



$$k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$

$$k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$

Microstrip Patch Phased Array

Phased Array (cont.)

Single patch:

$$E_x(x,y,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}-\frac{1}{k_t^2}\left[\frac{k_x^2}{D_m(k_t)}+\frac{k_y^2}{D_e(k_t)}\right]\tilde{J}_{sx}\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}\;dk_x dk_y$$

$$J_{sx}(x,y)=\cos\left(\frac{\pi x}{L}\right)$$

$$\tilde{J}_{sx}\left(k_x,k_y\right)=\left(\frac{\pi}{2}LW\right)\operatorname{sinc}\left(k_y\frac{W}{2}\right)\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$

$$D^{\mathrm{TM}}(k_t)=Y_0^{\mathrm{TM}}-jY_1^{\mathrm{TM}}\cot\left(k_{z1}h\right)$$

$$D^{\mathrm{TE}}(k_t)=Y_0^{\mathrm{TE}}-jY_1^{\mathrm{TE}}\cot\left(k_{z1}h\right)$$

Phased Array (cont.)

2D phased array of patches:

$$E_x(x,y,0)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}-\frac{1}{k_{tpq}^2}\left[\frac{k_{xp}^2}{D_m\left(k_{tpq}\right)}+\frac{k_{yq}^2}{D_e\left(k_{tpq}\right)}\right]\tilde{J}_{sx}^{00}\left(k_{xp},k_{yq}\right)e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

where

$$J_{sx}^{00}(x,y)=\cos\left(\frac{\pi x}{L}\right)$$

$$\tilde{J}_{sx}^{00}\left(k_{xp},k_{yq}\right)=\left(\frac{\pi}{2}LW\right)\operatorname{sinc}\left(k_{yq}\frac{W}{2}\right)\left[\frac{\cos\left(k_{xp}\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_{xp}\frac{L}{2}\right)^2}\right]$$

$$k_{tpq}=\sqrt{k_{xp}^2+k_{yq}^2}$$

Phased Array (cont.)

The field is of the following form:

$$E_x(x,y,0)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}A_{pq}\,e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}A_{pq}\,e^{-j\underline{k}_{tpq}\cdot\underline{r}}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}A_{pq}\,\psi_{pq}(x,y)$$

where

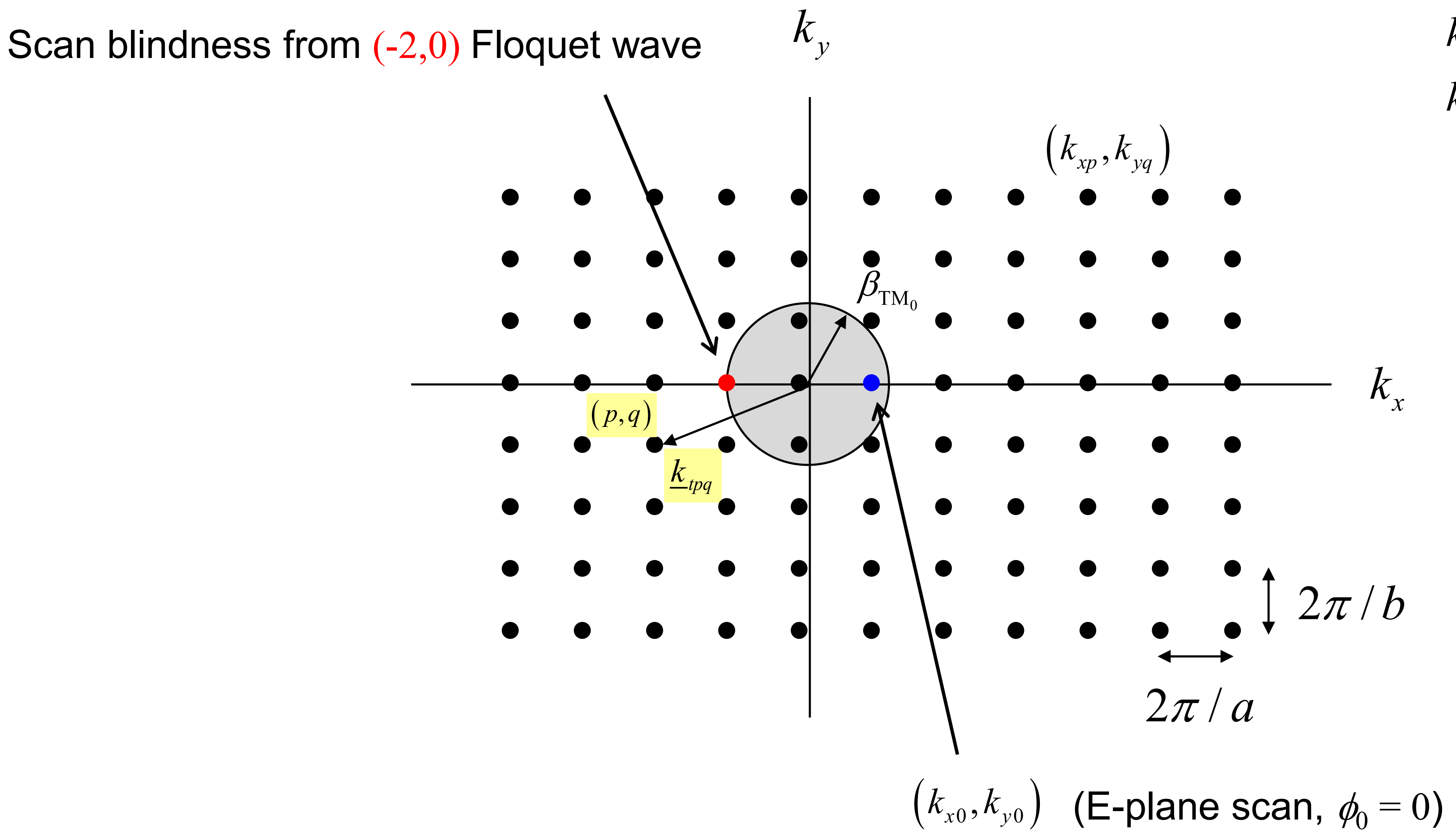
$$\underline{k}_{tpq}=\underline{\hat{x}}k_{xp}+\underline{\hat{y}}k_{yq}=\left(\underline{\hat{x}}k_{x0}+\underline{\hat{y}}k_{y0}\right)+\left[\left(\frac{2\pi p}{a}\right)\underline{\hat{x}}+\left(\frac{2\pi q}{b}\right)\underline{\hat{y}}\right]$$

The field is thus represented as a “sum of Floquet waves.”

Scan Blindness in a Phased Array

This occurs when one of the sample points (p,q) lies on the surface-wave circle (shown for $(-2, 0)$).

$$k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$
$$k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$



Scan Blindness (cont.)

The scan blindness condition is:

$$k_{tpq} = \left|\underline{k}_{tpq}\right| = \beta_{\mathrm{TM}_0} \quad \text{(for some } (p,q)\text{)}$$

$$E_x(x,y,0) = \frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} -\frac{1}{k_{tpq}^2}\left[\frac{k_{xp}^2}{D_m(k_{tpq})} + \frac{k_{yq}^2}{D_e(k_{tpq})}\right]\tilde{J}_{sx}^{00}(k_{xp}, k_{nq})\, e^{-j(k_{xp}x + k_{yq}y)}$$

↑

$$D_m(k_{tpq}) = D_m(\beta_{\mathrm{TM}_0}) = 0$$

The field produced by an *impressed* set of infinite periodic phased surface-current sources will be infinite.

Scan Blindness (cont.)

Physical interpretation: All of the surface-wave fields excited from the patches add up <u>in phase</u> in the direction of the transverse phasing vector:

$$\underline{k}_{tpq} = \underline{\hat{x}} k_{xp} + \underline{\hat{y}} k_{yq} \quad \Rightarrow \quad \cos\phi_{pq} = \left(\frac{k_{xp}}{k_{tpq}}\right)$$

$\cos\phi_{pq}$ = angle of (p,q) Floquet wave

Proof:

Start with the *surface-wave array factor*:

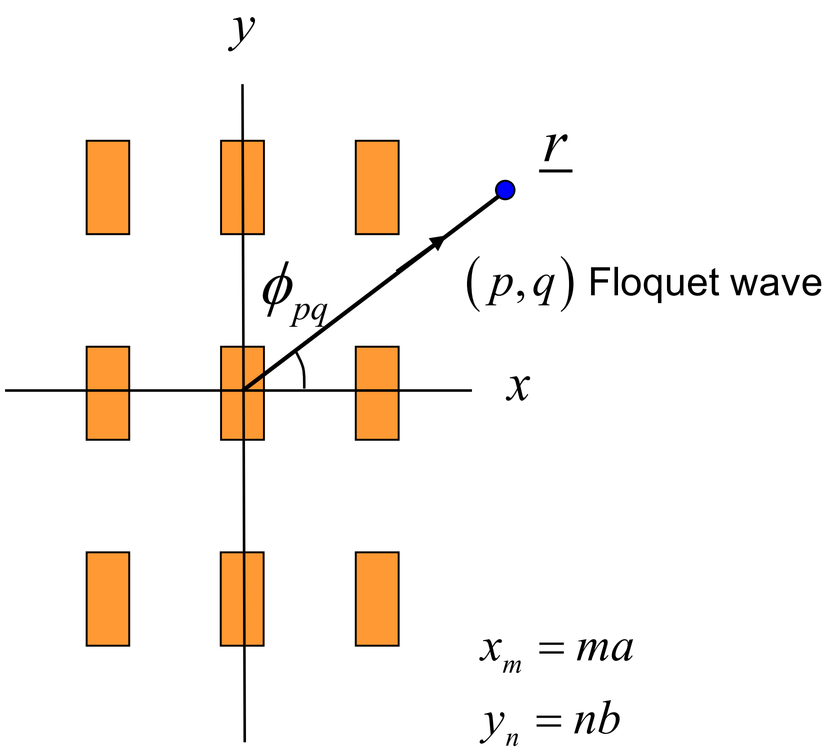
$\phi = \phi_{pq}$

$$\mathrm{AF}_{\mathrm{sw}} = \sum_{m,n} A_{mn}\, e^{+j\left(\beta_{\mathrm{TM}_0} x_m \cos\phi + \beta_{\mathrm{TM}_0} y_n \sin\phi\right)}$$

$$\sum_{m,n} A_{mn}\, e^{+j\left(\beta_{\mathrm{TM}_0} x_m \cos\phi_{pq} + \beta_{\mathrm{TM}_0} y_n \sin\phi_{pq}\right)}$$

$$= \sum_{m,n} A_{mn}\, e^{+j\left(\beta_{\mathrm{TM}_0} x_m \left(\frac{k_{xp}}{k_{tpq}}\right) + \beta_{\mathrm{TM}_0} y_n \left(\frac{k_{yq}}{k_{tpq}}\right)\right)}$$

$$\sum_{m,n} A_{mn}\, e^{+j\left(\beta_{\mathrm{TM}_0} x_m \left(\frac{k_{xp}}{\beta_{\mathrm{TM}_0}}\right) + \beta_{\mathrm{TM}_0} y_n \left(\frac{k_{yq}}{\beta_{\mathrm{TM}_0}}\right)\right)}$$



$x_m = ma$

$y_n = nb$

N elements

Scan Blindness (cont.)

Hence, we have in this direction ($\phi = \phi_{pq}$), that

$$\mathrm{AF}_{\mathrm{sw}} = \sum_{m,n} A_{mn}\, e^{+j\left(k_{xp} x_m + k_{yq} y_n\right)}$$

$$= \sum_{m,n} A_{mn}\, e^{+j\left(k_{xp}(x_0+ma) + k_{yq}(y_0+nb)\right)}$$

$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(k_{xp}ma + k_{yq}nb\right)}$$

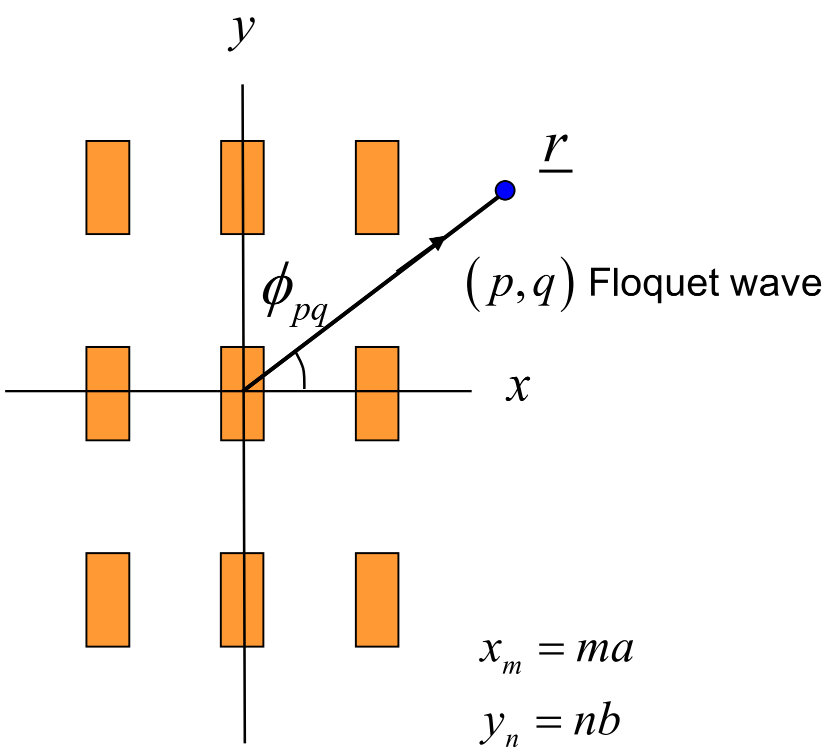
$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(\left(k_{x0}+\frac{2\pi p}{a}\right)ma + \left(k_{y0}+\frac{2\pi q}{b}\right)nb\right)}$$

$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(k_{x0}ma + k_{y0}nb\right)} e^{+j\left(\frac{2\pi p}{a}\right)ma} e^{+j\left(\frac{2\pi q}{b}\right)nb}$$

$$= e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{mn}\, e^{+j\left(k_{x0}ma + k_{y0}nb\right)} e^{+j(2\pi pm)} e^{+j(2\pi qn)}$$

$$e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} \sum_{m,n} A_{00}\, e^{-j\left(k_{x0}ma + k_{y0}nb\right)}\, e^{+j\left(k_{x0}ma + k_{y0}nb\right)}$$

$$e^{+jk_{xp}x_0} e^{+jk_{yq}y_0} A_{00} N$$



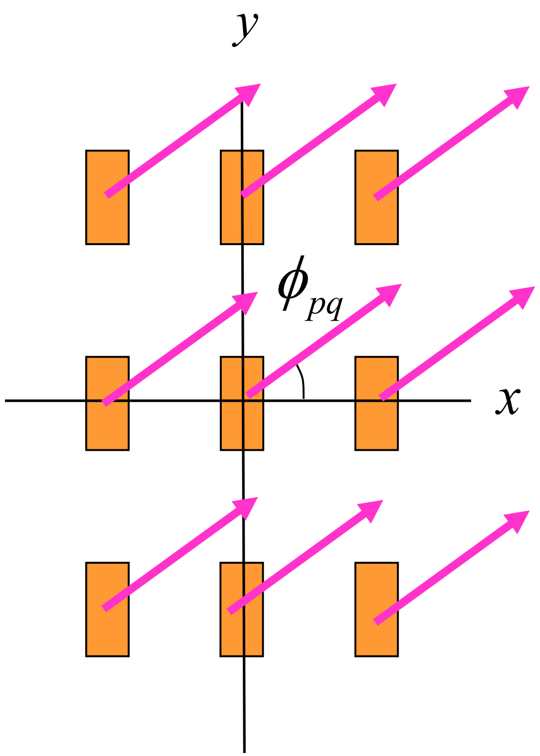
$x_m = ma$

$y_n = nb$

N elements

$$\cos\phi_{pq} = \left(\frac{k_{xp}}{k_{tpq}}\right)$$

Scan Blindness (cont.)



TM_0 surface wave

N elements

In the direction $\phi = \phi_{pq}$, the surface fields from each patch add up in phase.

$$\cos\phi_{pq} = \left(\frac{k_{xp}}{\beta_{TM_0}}\right)$$

$$\Rightarrow \quad AF_{sw} = N\left(A_{00}e^{+jk_{xp}x_0}e^{+jk_{yq}y_0}\right)$$

Note: There is also a surface-wave element pattern as well, with the field decaying as $1/\rho^{1/2}$, but this is ignored here.

Scan Blindness (cont.)

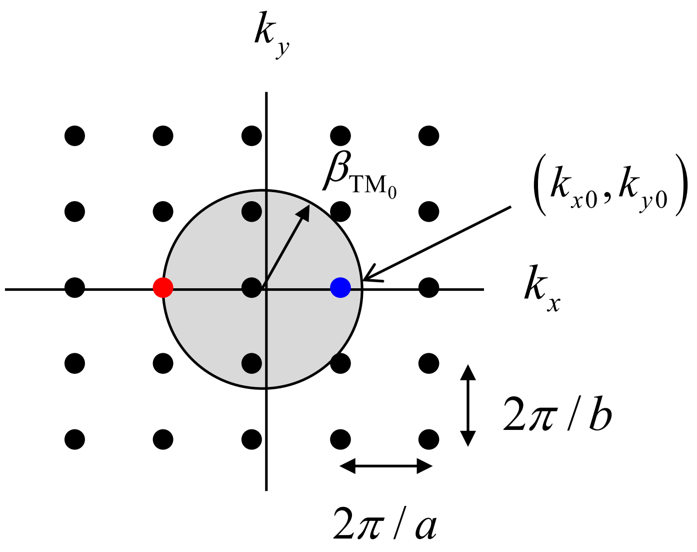
Example

$$p = -2, \quad q = 0$$

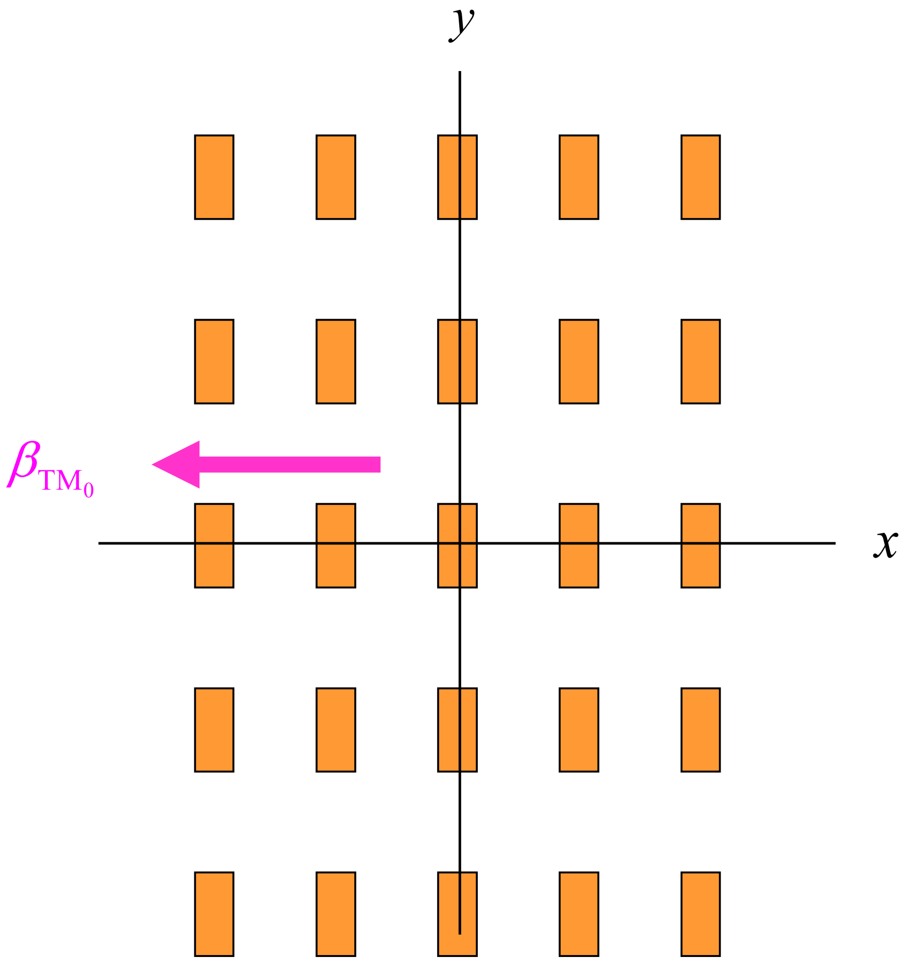
$$k_{xp} = -\beta_{TM_0}$$

$$k_{yq} = 0$$

$$\cos\phi_{pq} = \left(\frac{-\beta_{TM_0}}{\beta_{TM_0}}\right) = -1$$



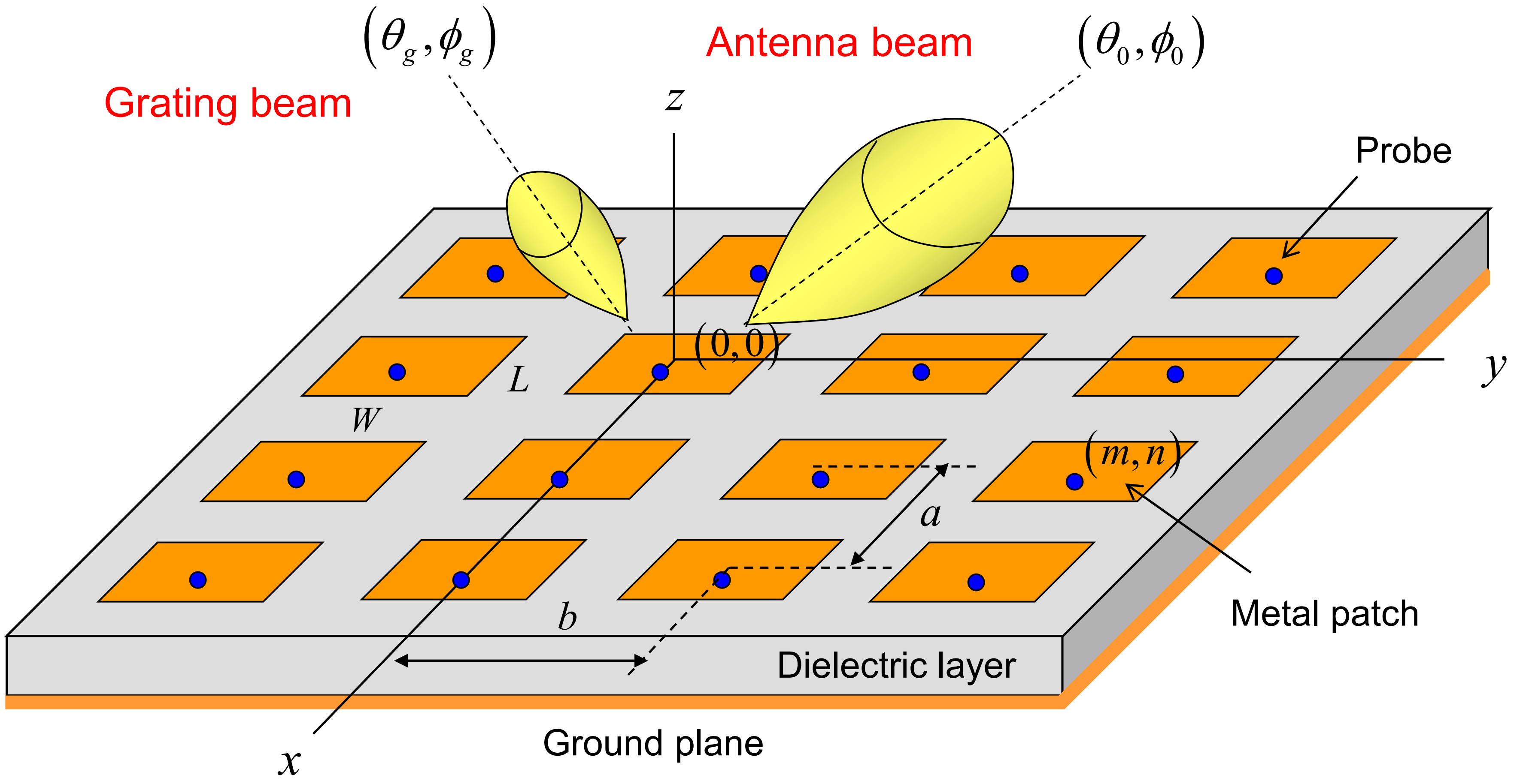
E-plane scan



Grating Lobes

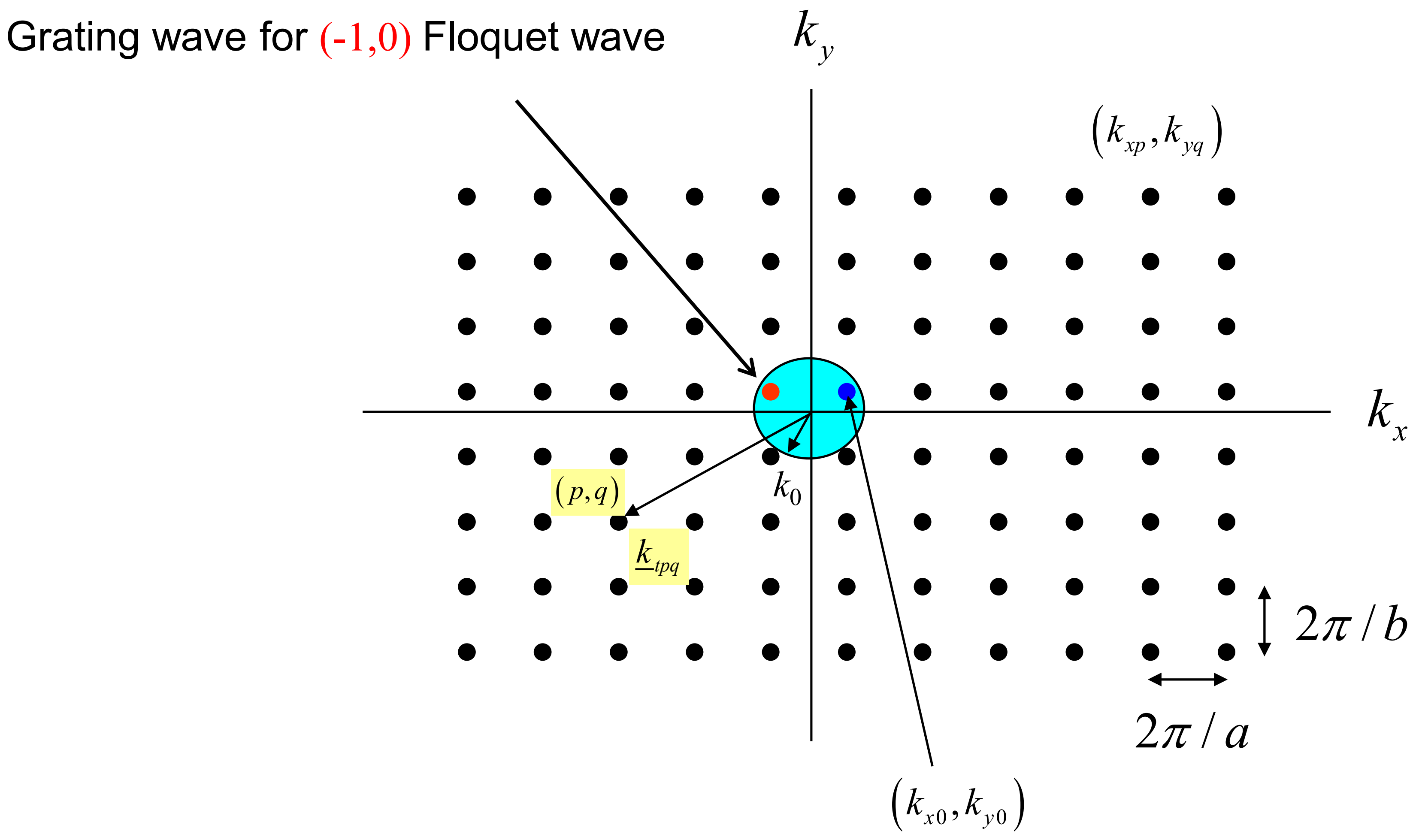
❖ Grating lobes occur when one or more of the higher-order Floquet waves propagates in space.

❖ For a finite-size array, this corresponds to a secondary beam (grating lobe) that gets radiated.



Grating Lobes (cont.)

$$k_{tpq} < k_0 \quad \left(\text{for some } (p,q) \neq (0,0)\right)$$



Pozar Circle Diagram

Define

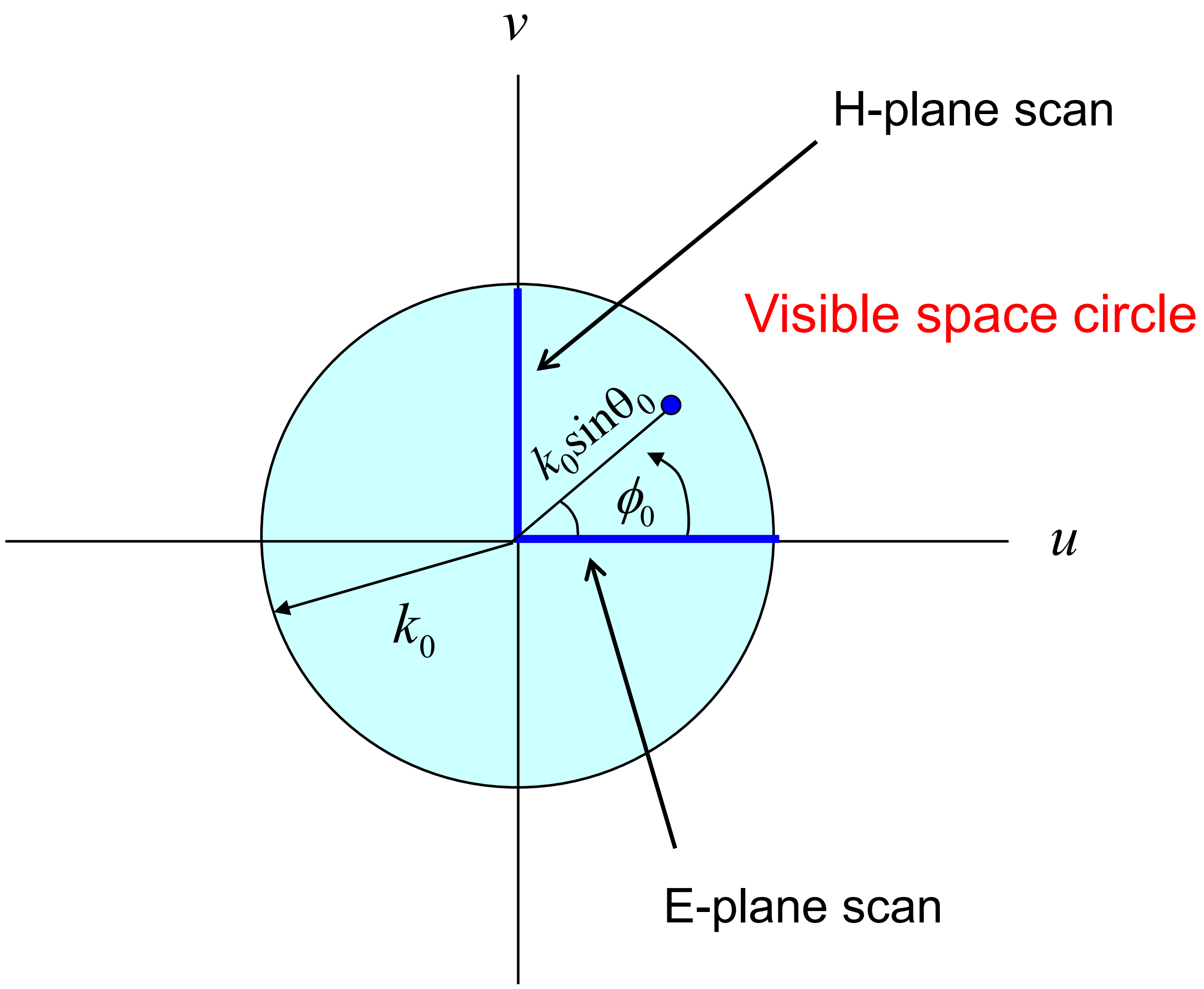
$$u \equiv k_{x0} = k_0 \sin\theta_0 \cos\phi_0$$
$$v \equiv k_{y0} = k_0 \sin\theta_0 \sin\phi_0$$

Radial distance in uv plane:

$$u^2 + v^2 = k_0^2 \sin^2\theta_0$$
$$\Rightarrow \quad \sqrt{u^2 + v^2} = k_0 \sin\theta_0$$

Angle in uv plane:

$$\tan\phi_{uv} = v / u = \tan\phi_0$$
$$\Rightarrow \quad \phi_{uv} = \phi_0$$



Pozar Circle Diagram (cont.)

Grating Lobes

$$k_{tpq} < k_0 \quad \text{for} \quad (u, v) \in \text{visible space circle}$$

So, we require that

$$k_{xp}^2 + k_{yq}^2 < k_0^2 \qquad \text{for} \quad u^2 + v^2 < k_0^2$$

(A grating lobe then appears from the (p,q) Floquet wave.)

The first inequality gives us:

$$\left(k_{x0} + \frac{2\pi p}{a}\right)^2 + \left(k_{y0} + \frac{2\pi q}{b}\right)^2 < k_0^2$$

or

$$\left(k_0 \sin\theta_0 \cos\phi_0 + \frac{2\pi p}{a}\right)^2 + \left(k_0 \sin\theta_0 \sin\phi_0 + \frac{2\pi q}{b}\right)^2 < k_0^2$$

or

$$\left(u + \frac{2\pi p}{a}\right)^2 + \left(v + \frac{2\pi q}{b}\right)^2 < k_0^2$$

Pozar Circle Diagram (cont.)

Therefore, we have:

$$\left(u-\left(-\frac{2\pi p}{a}\right)\right)^2+\left(v-\left(-\frac{2\pi q}{b}\right)\right)^2<k_0^2$$

or

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2<k_0^2$$ (We are inside a shifted (p,q) visible space circle.)

$$u_p\equiv-\frac{2\pi p}{a},\quad v_q\equiv-\frac{2\pi q}{b}$$ (This is the center of the (p,q) circle.)

Summary of grating lobe condition:

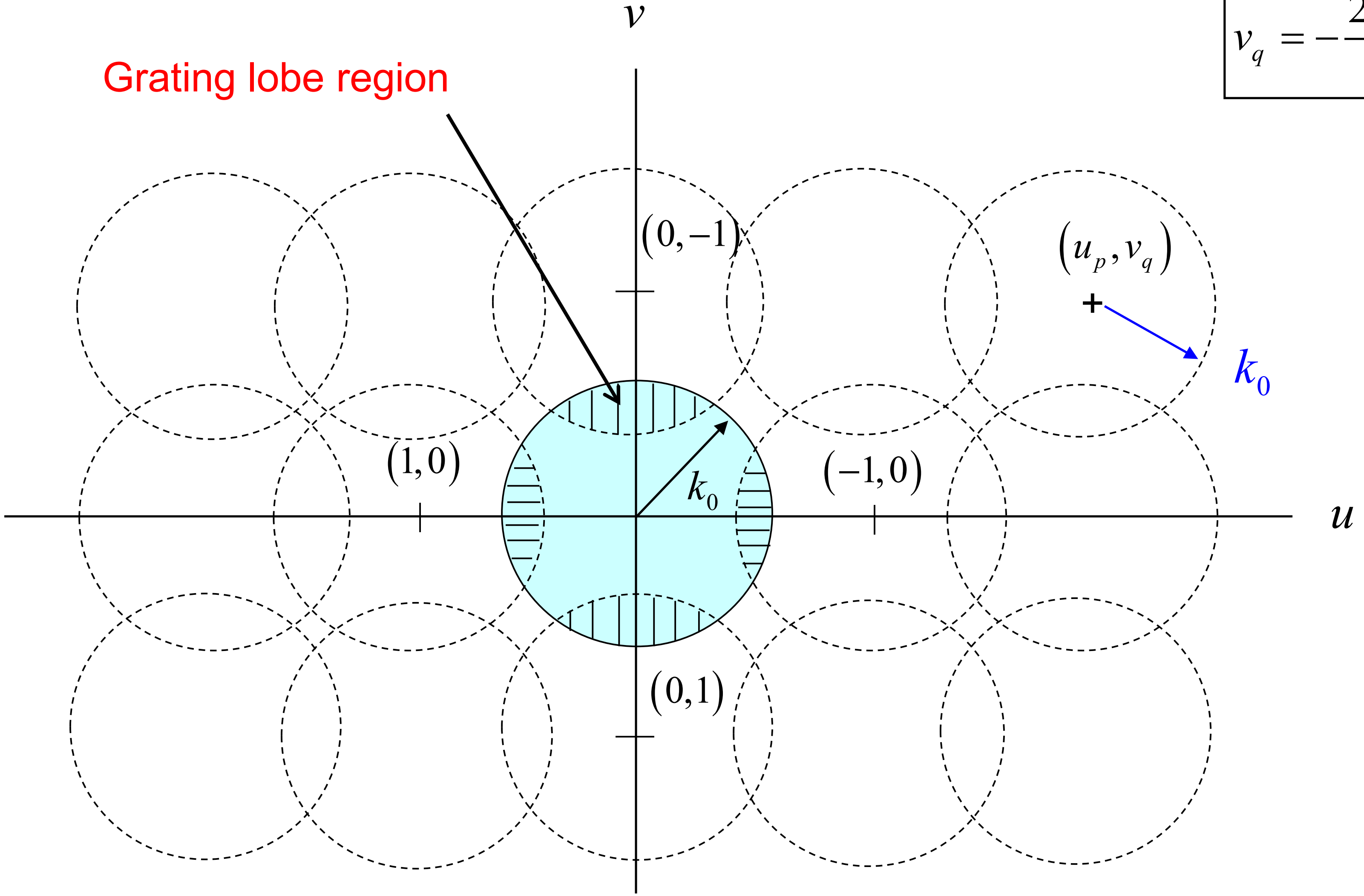
$$\left(u-u_p\right)^2+\left(v-v_q\right)^2<k_0^2$$
$$u^2+v^2<k_0^2$$

Part of the *interior* of the (p,q) circle is also inside the visible space circle.

Pozar Circle Diagram (cont.)

$$(u-u_p)^2+(v-v_q)^2<k_0^2$$
$$u^2+v^2<k_0^2$$

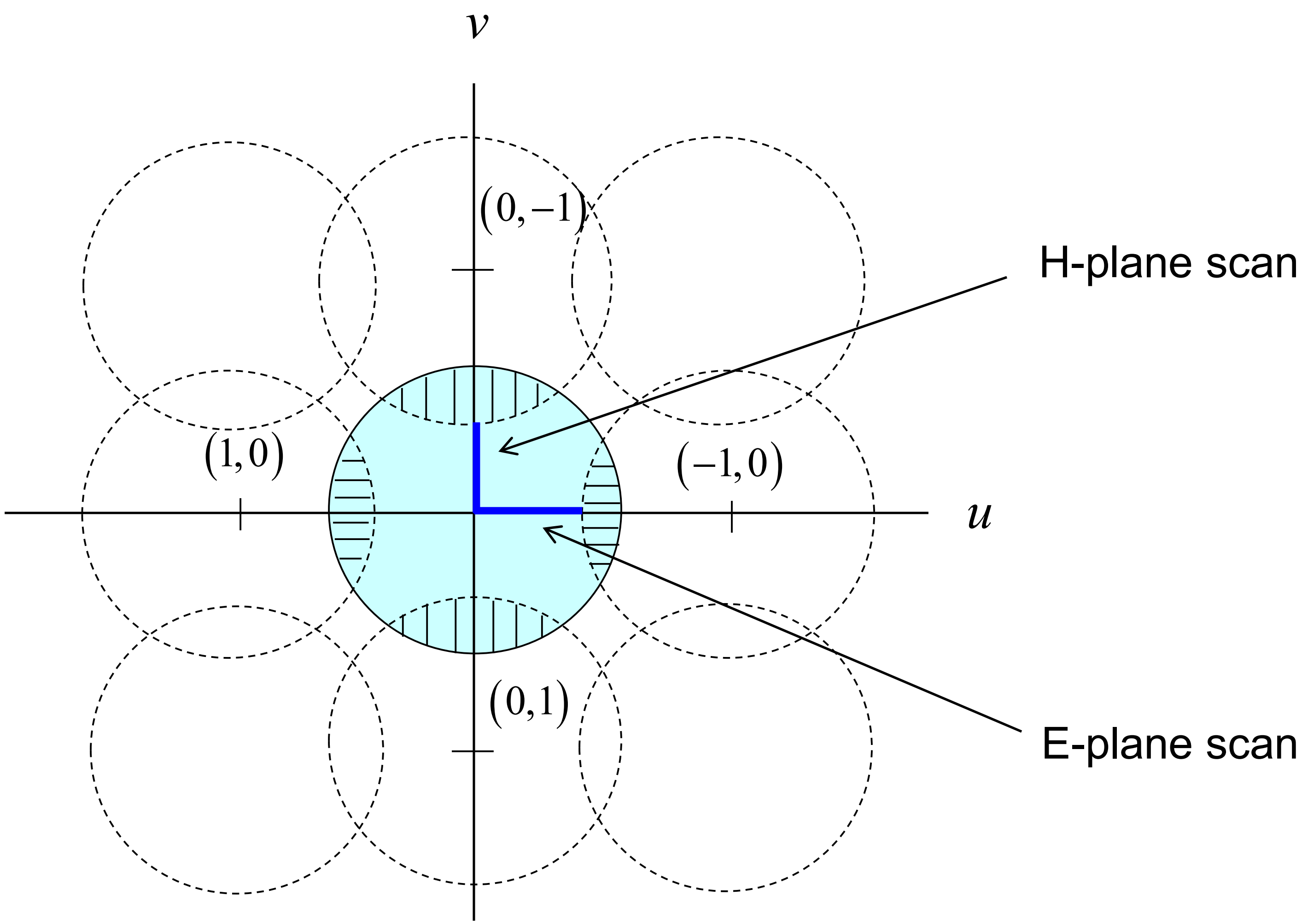
$$u_p=-\frac{2\pi p}{a}$$
$$v_q=-\frac{2\pi q}{b}$$



Pozar Circle Diagram (cont.)

This diagram shows when grating lobes occur in the principal scan planes.

$$u = k_0 \sin\theta_0 \cos\phi_0 \qquad v = k_0 \sin\theta_0 \sin\phi_0$$



Pozar Circle Diagram (cont.)

To avoid grating lobes for all scan angles, we require:

$$u_{-1} > 2k_0$$
$$v_{-1} > 2k_0$$

(The circles do not overlap.)

Therefore, we have:

$$\frac{2\pi}{a} > 2k_0$$
$$\frac{2\pi}{b} > 2k_0$$

or

$$k_0 a < \pi$$
$$k_0 b < \pi$$

Hence, to avoid grating lobes we have:

$$a < \lambda_0 / 2$$
$$b < \lambda_0 / 2$$

Pozar Circle Diagram (cont.)

Scan Blindness

$k_{tpq} = \beta_{\mathrm{TM}_0}$ for $(u, v) \in$ visible space circle

So, we require that

$$k_{xp}^2 + k_{yq}^2 = \beta_{\mathrm{TM}_0}^2 \qquad \text{for } u^2 + v^2 < k_0^2$$

The equation gives us

$$\left(k_{x0} + \frac{2\pi p}{a}\right)^2 + \left(k_{y0} + \frac{2\pi q}{b}\right)^2 = \beta_{\mathrm{TM}_0}^2$$

or

$$\left(k_0 \sin\theta_0 \cos\phi_0 + \frac{2\pi p}{a}\right)^2 + \left(k_0 \sin\theta_0 \sin\phi_0 + \frac{2\pi q}{b}\right)^2 = \beta_{\mathrm{TM}_0}^2$$

or

$$\left(u + \frac{2\pi p}{a}\right)^2 + \left(v + \frac{2\pi q}{b}\right)^2 = \beta_{\mathrm{TM}_0}^2$$

Pozar Circle Diagram (cont.)

We then have:

$$\left(u-\left(-\frac{2\pi p}{a}\right)\right)^2+\left(v-\left(-\frac{2\pi q}{b}\right)\right)^2=\beta_{\mathrm{TM}_0}^2$$

or

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{\mathrm{TM}_0}^2$$

where

$$u_p\equiv-\frac{2\pi p}{a}$$

$$v_q\equiv-\frac{2\pi q}{b}$$

Summary of scan blindness condition:

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{\mathrm{TM}_0}^2$$

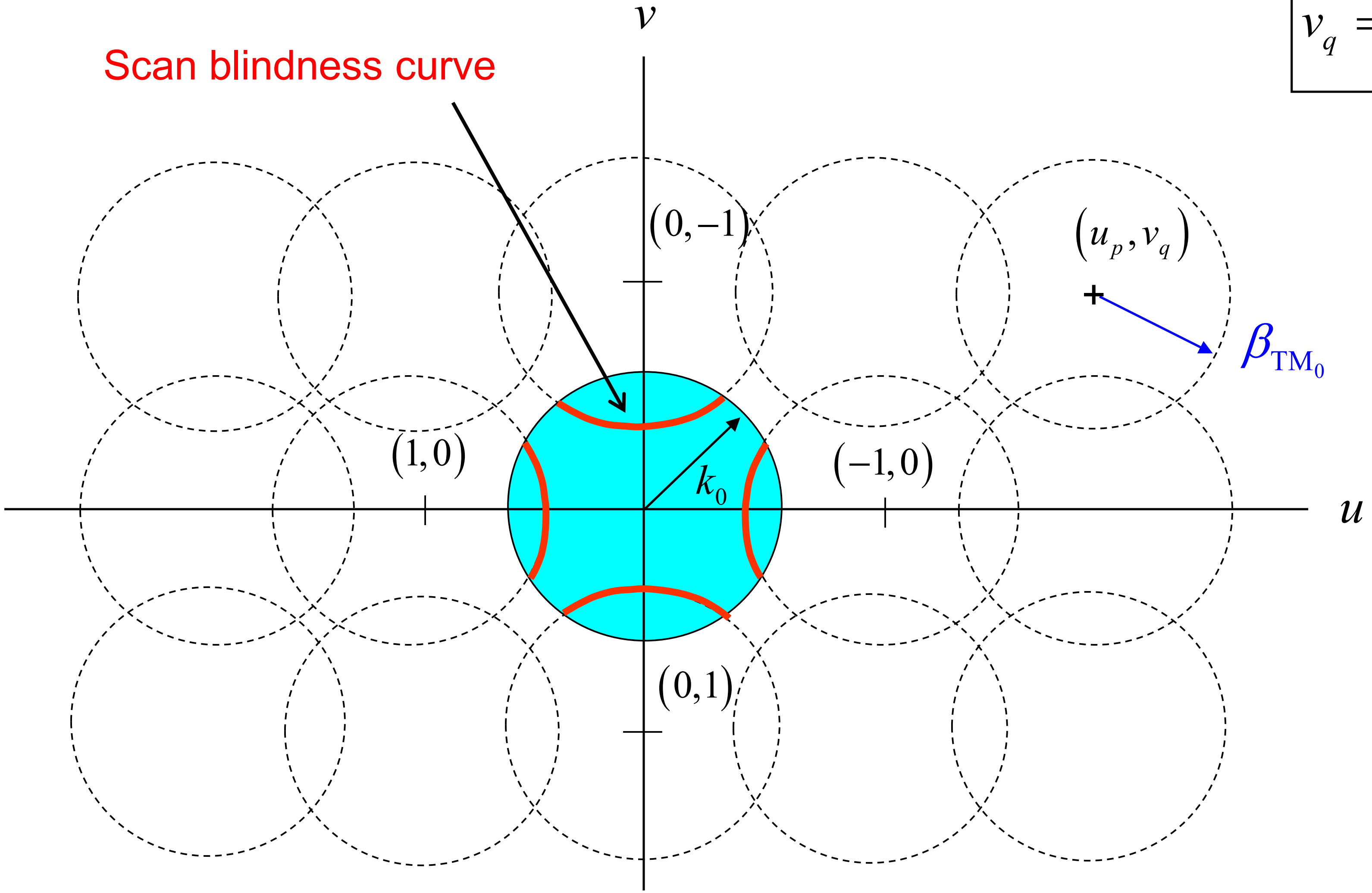
$$u^2+v^2<k_0^2$$

Part of the *boundary* of the (p,q) circle is inside the visible space circle.

Pozar Circle Diagram (cont.)

$$\left(u-u_p\right)^2+\left(v-v_q\right)^2=\beta_{\mathrm{TM}_0}^2$$
$$u^2+v^2<k_0^2$$

$$u_p=-\frac{2\pi p}{a}$$
$$v_q=-\frac{2\pi q}{b}$$



Pozar Circle Diagram (cont.)

To avoid scan blindness for all scan angles, we require:

$$u_{-1} > \beta_{\mathrm{TM}_0} + k_0$$
$$v_{-1} > \beta_{\mathrm{TM}_0} + k_0$$

(The circles do not overlap.)

Therefore, we have:

$$\frac{2\pi}{a} > \beta_{\mathrm{TM}_0} + k_0$$
$$\frac{2\pi}{b} > \beta_{\mathrm{TM}_0} + k_0$$

or

$$\frac{2\pi}{k_0 a} > \beta_{\mathrm{TM}_0} / k_0 + 1$$
$$\frac{2\pi}{k_0 b} > \beta_{\mathrm{TM}_0} / k_0 + 1$$

Hence

$$a / \lambda_0 < \frac{1}{\beta_{\mathrm{TM}_0} / k_0 + 1}$$
$$b / \lambda < \frac{1}{\beta_{\mathrm{TM}_0} / k_0 + 1}$$