ECE 6345

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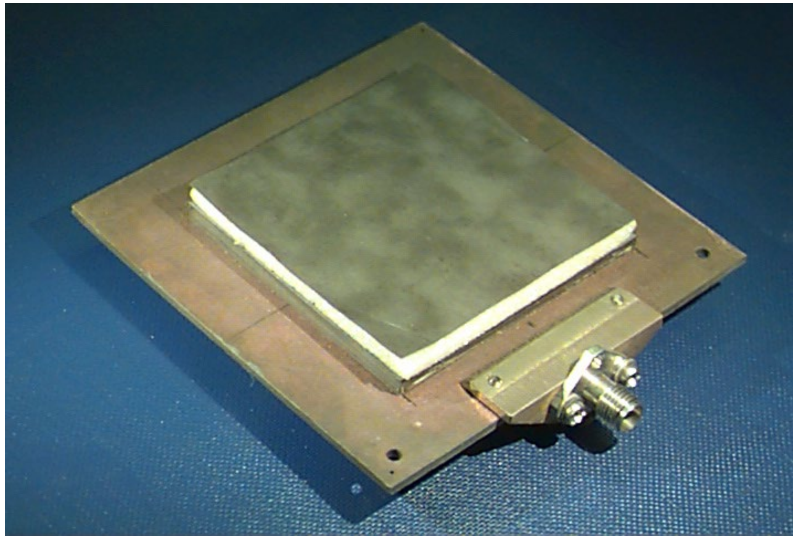


Notes 29

Overview

- In this set of notes we derive the SDI formulation using a more mathematical, but general, approach (we directly Fourier transform Maxwell's equations).

- This allows for all possible types of sources to be treated in one derivation.

General SDI Method

Start with Ampere's law: $$\nabla \times \underline{H} = \underline{J}^i = j\omega\varepsilon \underline{E}$$

$$\nabla = \nabla_t + \underline{\hat{z}} \frac{\partial}{\partial z}$$

where

$$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

Assume a 2D spatial transform: $$\tilde{\nabla}_t = \underline{\hat{x}}\left(-jk_x\right) + \underline{\hat{y}}\left(-jk_y\right)$$

$$= -j\left(\underline{\hat{x}}\, k_x + \underline{\hat{y}}\, k_y\right)$$

$$= -j\underline{k}_t$$

$$= -jk_t \underline{\hat{u}}$$

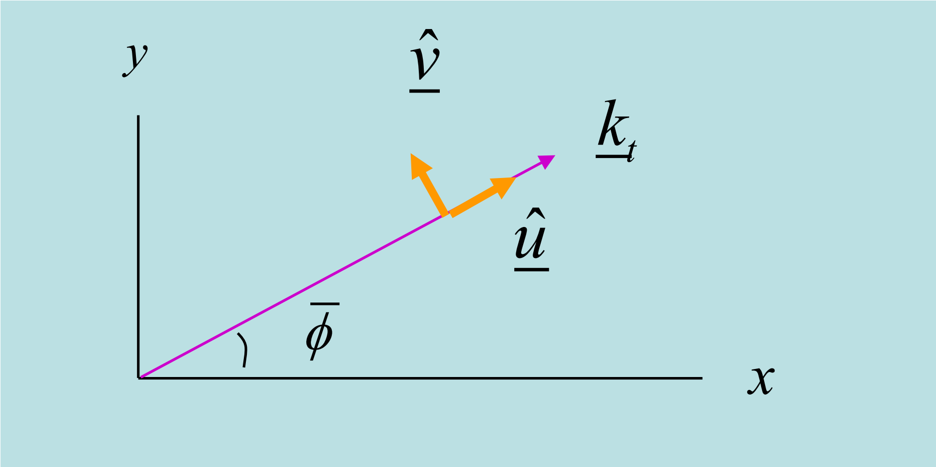
General SDI Method (cont.)

Hence, we have: $$\left(-jk_t\underline{\hat{u}}+\underline{\hat{z}}\frac{\partial}{\partial z}\right)\times\underline{\tilde{H}}=\underline{\tilde{J}}^i+j\omega\varepsilon\underline{\tilde{E}}$$

Next, represent the field as $$\underline{\tilde{H}}=\underline{\hat{u}}\,\tilde{H}_u+\underline{\hat{v}}\,\tilde{H}_v+\underline{\hat{z}}\tilde{H}_z$$
$$=\underline{\hat{u}}\left(\underline{\tilde{H}}\cdot\underline{\hat{u}}\right)+\underline{\hat{v}}\left(\underline{\tilde{H}}\cdot\underline{\hat{v}}\right)++\underline{\hat{z}}\left(\underline{\tilde{H}}\cdot\underline{\hat{z}}\right)$$

Note that

$$\underline{\hat{u}}\times\underline{\hat{v}}=\underline{\hat{z}}$$
$$\underline{\hat{z}}\times\underline{\hat{u}}=\underline{\hat{v}}$$
$$\underline{\hat{z}}\times\underline{\hat{v}}=-\underline{\hat{u}}$$



Then take the $\underline{\hat{z}},\underline{\hat{u}},\underline{\hat{v}}$ components of the transformed Ampere's equation.

General SDI Method (cont.)

$$\underline{\hat{z}}) \; -jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z$$

$$\underline{\hat{u}}) \quad -\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u$$

$$\underline{\hat{v}}) \quad jk_t\tilde{H}_z + \frac{\partial\tilde{H}_u}{\partial z} = \tilde{J}_v^i + j\omega\varepsilon\tilde{E}_v$$

Examine TM_z field: $\left(\tilde{E}_u, \tilde{H}_v, \tilde{E}_z\right)$ Ignore the $\underline{\hat{v}}$ equation.

$$-jk_t\tilde{H}_v = \tilde{J}_z^i + j\omega\varepsilon\tilde{E}_z \quad (1)$$

$$-\frac{\partial\tilde{H}_v}{\partial z} = \tilde{J}_u^i + j\omega\varepsilon\tilde{E}_u \quad (2)$$

TM_z Fields

We wish to eliminate $\tilde{E}_z$. To do this, use Faraday's law:

$$\nabla \times \underline{E} = -\underline{M}^i - j\omega\mu\underline{H}$$

$$\left(-jk_t\underline{\hat{u}} + \underline{\hat{z}}\frac{\partial}{\partial z}\right) \times \underline{\tilde{E}} = -\underline{\tilde{M}}^i - j\omega\mu\underline{\tilde{H}}$$

Take the $\underline{\hat{v}}$ component of the transformed Faraday's Law:

$$jk_t\tilde{E}_z + \frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v \qquad (3)$$

TM$_z$ Fields (cont.)

Substitute $\tilde{E}_z$ from (1) into (3) to obtain

$$jk_t\left[\frac{1}{j\omega\varepsilon}\left(-\tilde{J}_z^i - jk_t\tilde{H}_v\right)\right] + \frac{\partial\tilde{E}_u}{\partial z} = -\tilde{M}_v^i - j\omega\mu\tilde{H}_v$$

Putting all the sources on the RHS:

$$\frac{\partial\tilde{E}_u}{\partial z} + \frac{k_t^2}{j\omega\varepsilon}\tilde{H}_v + j\omega\mu\tilde{H}_v = -\tilde{M}_v^i + \frac{k_t}{\omega\varepsilon}\tilde{J}_z^i$$

Note that

$$\begin{aligned}\frac{k_t^2}{j\omega\varepsilon} + j\omega\mu &= \frac{1}{j\omega\varepsilon}\left(k_t^2 - \omega^2\mu\varepsilon\right)\\ &= \frac{1}{j\omega\varepsilon}\left(k_t^2 - k^2\right)\\ &= \frac{-1}{j\omega\varepsilon}k_z^2\end{aligned}$$

TM_z Fields (cont.)

Hence, we have:

$$\frac{\partial \tilde{E}_u}{\partial z} - \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i \qquad (4)$$

TM_z Fields (cont.)

Equations (2) and (4) are rewritten as

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM$_z$ Fields (cont.)

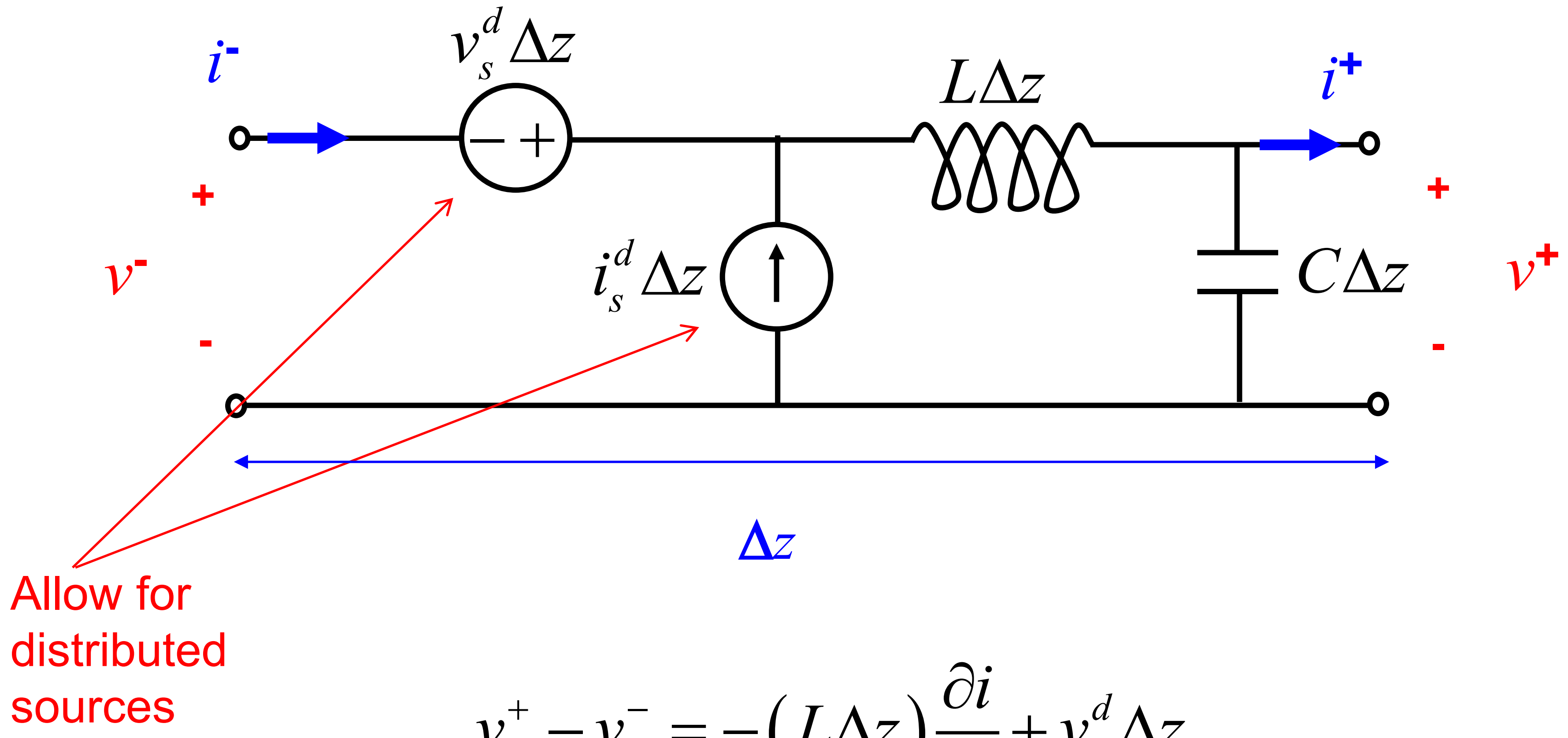
Define:

$$V^{\mathrm{TM}}(z) = \tilde{E}_u(k_x, k_y, z)$$
$$I^{\mathrm{TM}}(z) = \tilde{H}_v(k_x, k_y, z)$$

We then have:

$$\frac{\partial I^{\mathrm{TM}}}{\partial z} = -j\omega\varepsilon V^{\mathrm{TM}} - \tilde{J}_u^i$$
$$\frac{\partial V^{\mathrm{TM}}}{\partial z} = \left(\frac{k_z^2}{j\omega\varepsilon}\right) I^{\mathrm{TM}} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

Telegrapher's Equations



$$v^+ - v^- = -(L\Delta z)\frac{\partial i}{\partial t} + v_s^d \Delta z$$

so $$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} + v_s^d$$

Telegrapher's Equations (cont.)

Hence, in the phasor domain,

$$\frac{\partial V}{\partial z} = -j\omega L I + V_s^d$$

Also, $i^+ - i^- = -\left(C\Delta z\right)\frac{\partial v}{\partial t} + i_s^d \Delta z$

so $\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} + i_s^d$

Hence, in the phasor domain,

$$\frac{\partial I}{\partial z} = -j\omega C V + I_s^d$$

Telegrapher's Equations (cont.)

Compare field equations for TM$_z$ fields with TL equations:

$$\frac{\partial I^{\mathrm{TM}}}{\partial z} = -j\omega(\varepsilon)V^{\mathrm{TM}} + \left(-\tilde{J}_u^i\right)$$

$$\frac{\partial V^{\mathrm{TM}}}{\partial z} = -j\omega\left(\frac{k_z^2}{\omega^2\varepsilon}\right)I^{\mathrm{TM}} + \left[-\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i\right]$$

$$\frac{\partial I}{\partial z} = -j\omega CV + I_s^d$$

$$\frac{\partial V}{\partial z} = -j\omega LI + V_s^d$$

Telegrapher's Equations (cont.)

We then make the following identifications:

$$C = \varepsilon$$

$$L = \frac{k_z^2}{\omega^2 \varepsilon}$$

Hence

$$k_z^{\mathrm{TL}} = \omega\sqrt{LC} = \omega\sqrt{\frac{k_z^2}{\omega^2 \varepsilon}\varepsilon} = k_z$$

$$Z_0^{\mathrm{TL}} = \sqrt{\frac{L}{C}} = \sqrt{\frac{k_z^2}{\omega^2 \varepsilon^2}} = \frac{k_z}{\omega \varepsilon}$$

or

$$k_z^{\mathrm{TL}} = k_z$$

$$Z_0^{\mathrm{TL}} = \frac{k_z}{\omega \varepsilon}$$

Sources: TM$_z$

For the sources we have, for the TM$_z$ case:

$$I_s^{d^{\mathrm{TM}}} = -\tilde{J}_u^i$$

$$V_s^{d^{\mathrm{TM}}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

Sources: TM$_z$ (cont.)

Special case: *planar surface-current sources*

Assume $\underline{J}(x,y,z) = \underline{J}_s(x,y)\,\delta(z)$

$$\underline{M}(x,y,z) = \underline{M}_s(x,y)\,\delta(z)$$

$$I_s^{d\,\mathrm{TM}} = -\tilde{J}_{su}^{i}\,\delta(z)$$

$$V_s^{d\,\mathrm{TM}} = -\tilde{M}_{sv}^{i}\,\delta(z)$$

Then we have $I_s^{\mathrm{TM}} = -\tilde{J}_{su}^{i}$ This is a lumped parallel current generator.

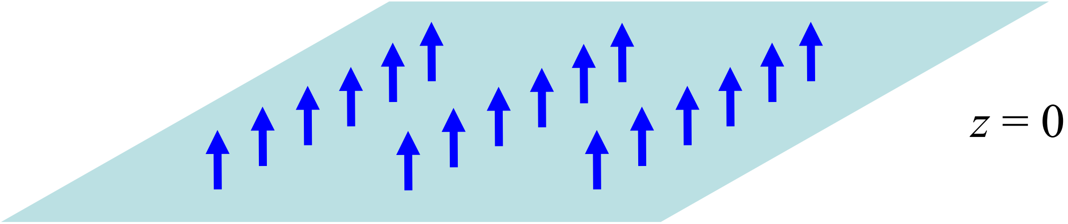
Similarly, we have: $V_s^{\mathrm{TM}} = -\tilde{M}_{sv}^{i}$ This is a lumped series voltage generator.

Sources: TM$_z$ (cont.)

Special case: *vertical planar electric current*

If $$J_z^i(x,y,z) = f(x,y)\,\delta(z)$$

Then we have $$V_s^{\mathrm{TM}} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{f}\left(k_x,k_y\right)$$



Example: $f(x,y) = \delta(x)\delta(y)$ (unit-amplitude vertical electric dipole)

$$\tilde{f}\left(k_x,k_y\right) = 1$$

TE_z Fields

Use duality:

$$\underline{E} \to \underline{H}$$

$$\underline{H} \to -\underline{E}$$

$$\underline{J}^i \to \underline{M}^i$$

$$\underline{M}^i \to -\underline{J}^i$$

$$\varepsilon \rightleftarrows \mu$$

$$\frac{\partial \tilde{H}_v}{\partial z} = -\tilde{J}_u^i - j\omega\varepsilon \tilde{E}_u$$

$$\frac{\partial \tilde{E}_u}{\partial z} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i + \left(\frac{k_z^2}{j\omega\varepsilon}\right)\tilde{H}_v$$

TM_z

$$\Downarrow$$

$$-\frac{\partial \tilde{E}_v}{\partial z} = -\tilde{M}_u^i - j\omega\mu \tilde{H}_u$$

$$\frac{\partial \tilde{H}_u}{\partial z} = +\tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i - \left(\frac{k_z^2}{j\omega\mu}\right)\tilde{E}_v$$

TE_z

TE_z (cont.)

Define

$$V^{\mathrm{TE}}(z) = -\tilde{E}_v(k_x, k_y, z)$$

$$I^{\mathrm{TE}}(z) = \tilde{H}_u(k_x, k_y, z)$$

$$\frac{\partial V^{\mathrm{TE}}}{\partial z} = -j\omega(\mu) I^{\mathrm{TE}} + \left[-\tilde{M}_u^i\right]$$

$$\frac{\partial I^{\mathrm{TE}}}{\partial z} = -j\omega\left(\frac{k_z^2}{\omega^2 \mu}\right) V^{\mathrm{TE}} + \tilde{J}_v^i + \left[\left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i\right]$$

We then identify:

$$L = \mu$$

$$C = \frac{k_z^2}{\omega^2 \mu}$$

$\Rightarrow$

$$k_z^{\mathrm{TL}} = k_z$$

$$Z_0^{\mathrm{TE}} = \frac{\omega\mu}{k_z}$$

TE$_z$ (cont.)

For the sources, we have:

$$V_s^{d\,\mathrm{TE}} = -\tilde{M}_u^i$$

$$I_s^{d\,\mathrm{TE}} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of horizontal surface currents:

$$V_s^{\mathrm{TE}} = -\tilde{M}_{su}^i$$

$$I_s^{\mathrm{TE}} = +\tilde{J}_{sv}^i$$

Special case of vertical planar currents:

$$\left(M_z^i = g(x,y)\,\delta(z)\right)$$

$$I_s^{\mathrm{TE}} = \left(\frac{k_t}{\omega\mu}\right)\tilde{g}$$

Summary

$$V^{\mathrm{TM}} = \tilde{E}_u$$
$$I^{\mathrm{TM}} = \tilde{H}_v$$
$$V^{\mathrm{TE}} = -\tilde{E}_v$$
$$I^{\mathrm{TE}} = \tilde{H}_u$$

$$I_s^{d\,\mathrm{TM}} = -\tilde{J}_u^i$$
$$V_s^{d\,\mathrm{TM}} = -\tilde{M}_v^i + \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{J}_z^i$$

$$V_s^{d\,\mathrm{TE}} = -\tilde{M}_u^i$$
$$I_s^{d\,\mathrm{TE}} = \tilde{J}_v^i + \left(\frac{k_t}{\omega\mu}\right)\tilde{M}_z^i$$

Special case of horizontal surface currents:

$$I_s^{\mathrm{TM}} = -\tilde{J}_{su}^i$$
$$V_s^{\mathrm{TM}} = -\tilde{M}_{sv}^i$$

$$I_s^{\mathrm{TE}} = -\tilde{M}_{su}^i$$
$$V_s^{\mathrm{TE}} = +\tilde{J}_{sv}^i$$

Summary (cont.)

Special case of vertical planar currents:

$$V_s^{\mathrm{TM}} = \left(\frac{k_t}{\omega\varepsilon}\right)\tilde{f} \qquad \left(J_z^i = f(x,y)\,\delta(z)\right)$$

$$I_s^{\mathrm{TE}} = \left(\frac{k_t}{\omega\mu}\right)\tilde{g} \qquad \left(M_z^i = g(x,y)\,\delta(z)\right)$$

Example

Calculate G_{zz}

This is the E_z field at (x, y, z) due to a unit-amplitude vertical dipole at (x', y', z').

To calculate $\tilde{G}_{zz}$ use: $\nabla \times \underline{H} = \underline{J} + j\omega\varepsilon\underline{E}$

We then have

$$E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{1}{j\omega\varepsilon} J_z$$

so that

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_{\text{obs}}}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right) - \frac{1}{j\omega\varepsilon_{\text{obs}}}\tilde{J}_z$$

Example (cont.)

We have:

$$\tilde{H}_x = \tilde{H}_u \cos\overline{\phi} + \tilde{H}_v\left(-\sin\overline{\phi}\right)$$
$$= I^{\mathrm{TE}} \cos\overline{\phi} + I^{\mathrm{TM}}\left(-\sin\overline{\phi}\right)$$

$$\tilde{H}_y = \tilde{H}_u \sin\overline{\phi} + \tilde{H}_v \cos\overline{\phi}$$
$$= I^{\mathrm{TE}} \sin\overline{\phi} + I^{\mathrm{TM}} \cos\overline{\phi}$$

The TE part cancels when we substitute these expressions into the expression for the transform of E_z, so we have:

$$\tilde{E}_z = \frac{-k_t}{\omega\varepsilon_{\mathrm{obs}}}\left(\cos^2\overline{\phi} + \sin^2\overline{\phi}\right) I^{\mathrm{TM}} - \frac{1}{j\omega\varepsilon_{\mathrm{obs}}}\tilde{J}_z$$

Example (cont.)

We then have:

Ignore

$$\tilde{E}_z = \frac{-k_t}{\omega \varepsilon_{\text{obs}}} I^{\text{TM}} - \frac{1}{j\omega\varepsilon_{\text{obs}}} \tilde{J}_z$$

(We assume that we are not inside the source dipole.)

From the strength of the TM voltage generator due to a unit-amplitude vertical dipole at z', we then have:

$$I^{\text{TM}} = \left(\frac{k_t}{\omega \varepsilon_{\text{src}}} \right) I_v^{\text{TM}}$$

Hence, we have:

$$\tilde{E}_z = \frac{-1}{\omega \varepsilon_{\text{obs}}} \left(\frac{k_t^2}{\omega \varepsilon_{\text{src}}} \right) I_v^{\text{TM}}$$

Example (cont.)

In the space domain we have (after taking the 2D inverse Fourier transform):

$$E_z = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-1}{\omega \varepsilon_{\text{obs}}} \left(\frac{k_t^2}{\omega \varepsilon_{\text{src}}} \right) I_v^{\text{TM}} e^{-j\left(k_x x + k_y y\right)} \, dk_x dk_y$$

Example (cont.)

Converting to polar coordinates, we have (for any function F):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\left(k_t\right) e^{-j\left(k_x x + k_y y\right)}\, dk_x dk_y = \int_0^{\infty}\int_0^{2\pi} F\left(k_t\right)\, e^{-j\left(k_x x + k_y y\right)}\, k_t d\overline{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t \rho\right)\left(\cos\overline{\phi}\cos\phi + \sin\overline{\phi}\sin\phi\right)}\, d\overline{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t \rho\right)\cos\left(\overline{\phi} - \phi\right)}\, d\overline{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \int_0^{2\pi} e^{-j\left(k_t \rho\right)\cos\overline{\phi}}\, d\overline{\phi}\, dk_t$$

$$= \int_0^{\infty} F\left(k_t\right) k_t \left(2\pi J_0\left(k_t \rho\right)\right) dk_t$$

Example (cont.)

In the space domain we then have:

$$E_z = \frac{1}{2\pi} \int_0^{\infty} \frac{-1}{\omega\varepsilon_{\text{obs}}} \left(\frac{k_t^3}{\omega\varepsilon_{\text{src}}} \right) I_v^{\text{TM}} J_0\left(k_t \rho\right) dk_t$$

so that

$$G_{zz} = \frac{1}{2\pi} \int_0^{\infty} \frac{-1}{\omega\varepsilon_{\text{obs}}} \left(\frac{k_t^3}{\omega\varepsilon_{\text{src}}} \right) I_v^{\text{TM}} J_0\left(k_t \rho\right) dk_t$$

Note: The integral does not converge when $z = z'$.