ECE 6345

Fall 2024

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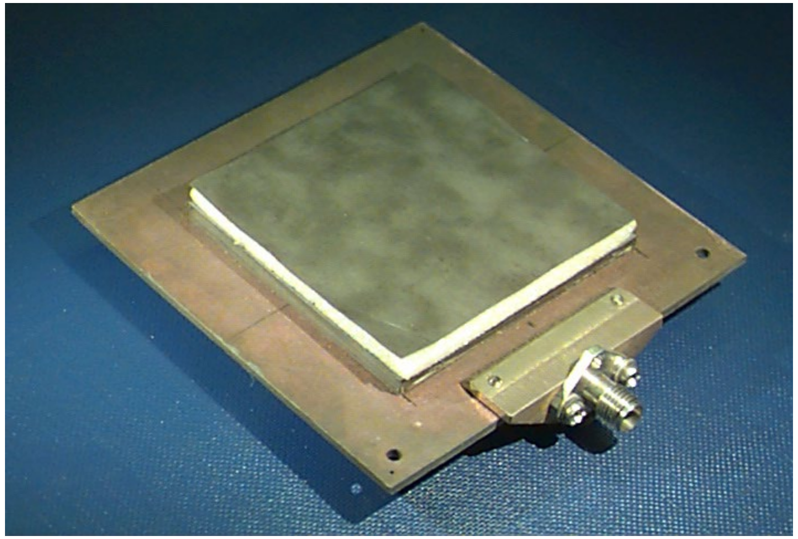


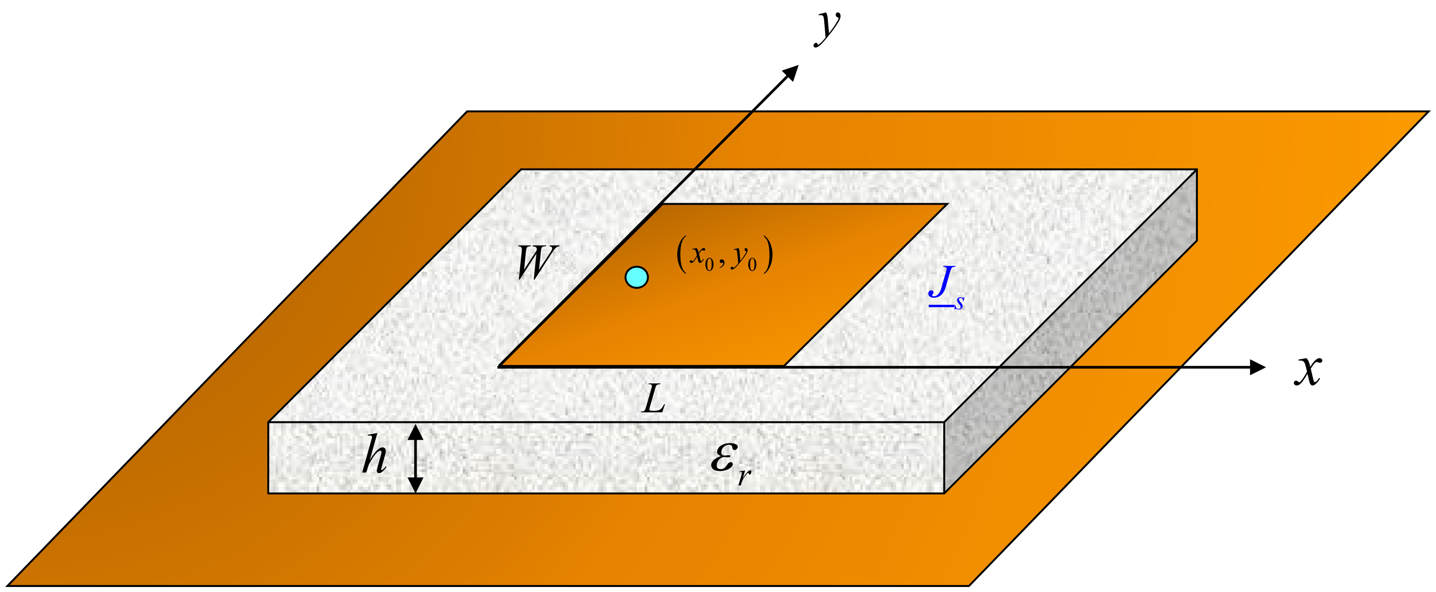
Notes 1

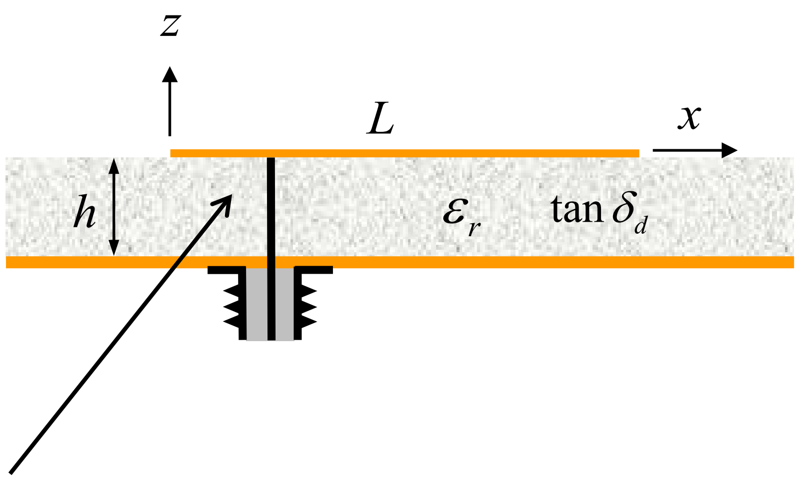
Cavity Model

Cavity Model for Patch Antenna

We use the cavity model and the method of eigenfunction expansion to solve for the input impedance of the rectangular microstrip patch antenna.

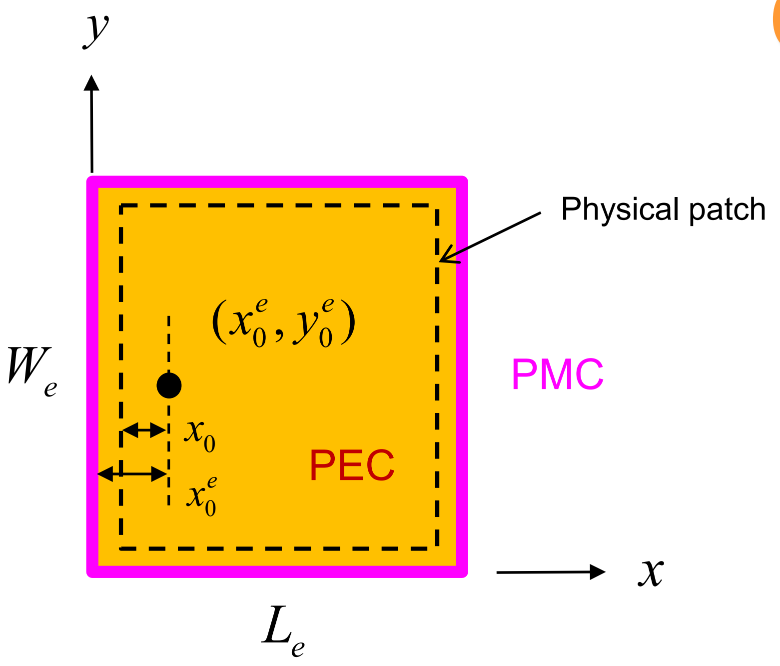
- The cavity model is an efficient and yet accurate method for calculating the input impedance.
- The cavity model justifies the RLC circuit model.





I_0 current feed at (x_0, y_0)

Cavity Model



Accounting for fringing:

$$L_e = L + 2\Delta L$$

$$W_e = W + 2\Delta W$$

$$x_0^e = x_0 + \Delta L$$

$$y_0^e = y_0 + \Delta W$$

Note: The coordinates (x_0, y_0) are measured from the corner of the physical patch.

$$\Delta L / h = 0.412 \left[\frac{\left(\varepsilon_r^{eff} + 0.3\right)\left(\frac{W}{h} + 0.264\right)}{\left(\varepsilon_r^{eff} - 0.258\right)\left(\frac{W}{h} + 0.8\right)} \right]$$

(Hammerstad formula)

$$\varepsilon_r^{eff} = \frac{\varepsilon_r + 1}{2} + \left(\frac{\varepsilon_r - 1}{2}\right)\left[1 + 12\left(\frac{h}{W}\right)\right]^{-1/2}$$

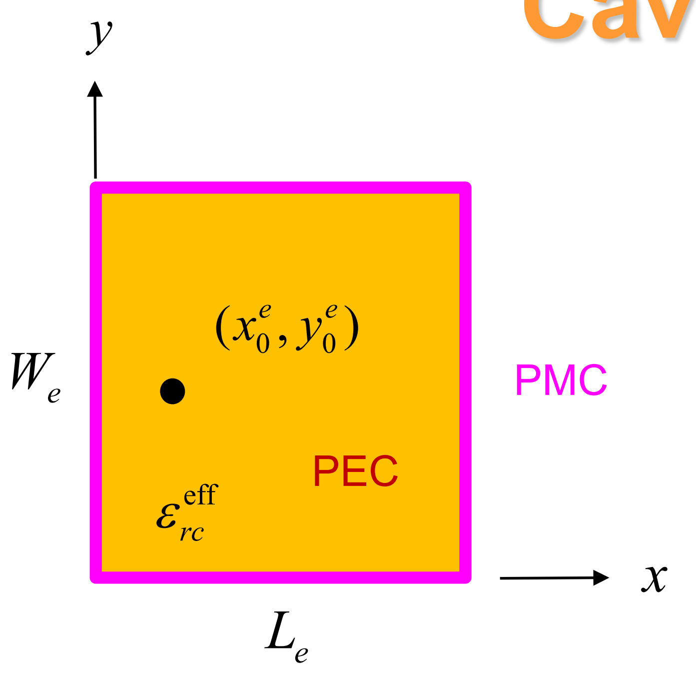
$$\Delta W / h = \frac{\ln 4}{\pi}$$

(Wheeler formula)

Note:

ΔL is often chosen from Hammerstad's formula.
ΔW is often chosen from Wheeler's formula.

Cavity Model (cont.)



Accounting for loss and radiation:

$$k_e = k_0 \sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r \left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan\delta_{\text{eff}} = \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\text{sp}}} + \frac{1}{Q_{\text{sw}}}$$

Note : $\tan\delta_d = \dfrac{1}{Q_d}$

Assume no z variation (the probe current is constant in the z direction.)

Cavity Model (cont.)

$$Q \equiv 2\pi \left(\frac{U_S}{U_D^T} \right)$$

$U_s =$ energy stored inside resonator

$U_D^T =$ energy dissipated per cycle (period T)

$$Q \equiv \frac{2\pi}{T} \left(\frac{U_S}{U_D^T / T} \right)$$

$T = 1/f =$ period

or

$$Q \equiv \omega_0 \left(\frac{U_S}{P_D^{\text{ave}}} \right)$$

$P_D^{\text{ave}} =$ average power "dissipated"

(This includes radiation loss)

Cavity Model (cont.)

CAD Formulas for Q Factors (derived later)

$$Q_d = \frac{1}{\tan \delta_d}$$

$$Q_c = \left(\frac{\eta_0}{2}\right)\left[\frac{(k_0 h)}{R_s^{\text{ave}}}\right] \qquad R_s^{\text{ave}} = \left(R_s^{\text{patch}} + R_s^{\text{ground}}\right)/2 \qquad R_s = \frac{1}{\sigma\delta}$$

$$Q_{\text{sp}} \approx \frac{3}{16}\left(\frac{\varepsilon_r}{p c_1}\right)\left(\frac{L_e}{W_e}\right)\left(\frac{1}{h/\lambda_0}\right) \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$Q_{\text{sw}} = Q_{\text{sp}}\left(\frac{e_r^{\text{hed}}}{1-e_r^{\text{hed}}}\right)$$

Cavity Model (cont.)

CAD Formulas for Q Factors (cont.)

$$e_r^{\text{hed}} = \frac{1}{1+\frac{3}{4}\pi\left(k_0 h\right)\left(\frac{1}{c_1}\right)\left(1-\frac{1}{\varepsilon_r}\right)^3}$$

$$c_1 = 1 - \frac{1}{\varepsilon_r} + \frac{2/5}{\varepsilon_r^2}$$

$$c_2 = -0.0914153$$

$$a_2 = -0.16605$$

$$a_4 = 0.00761$$

$$p = 1+\frac{a_2}{10}\left(k_0 W\right)^2 + \left(a_2^2 + 2a_4\right)\left(\frac{3}{560}\right)\left(k_0 W\right)^4 + c_2\left(\frac{1}{5}\right)\left(k_0 L\right)^2 + a_2 c_2\left(\frac{1}{70}\right)\left(k_0 W\right)^2\left(k_0 L\right)^2$$

Helmholtz Equation for E_z

We first derive the Helmholtz equation for E_z.

$$\nabla \times \underline{H} = \underline{J}^i + j\omega\varepsilon_c^{\text{eff}}\,\underline{E}$$

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \qquad (\mu = \mu_0)$$

(nonmagnetic substrate)

Substituting Faraday's law for $\underline{H}$ into Ampere's law, we have:

$$-\frac{1}{j\omega\mu}\nabla \times \left(\nabla \times \underline{E}\right) = \underline{J}^i + j\omega\varepsilon_c^{\text{eff}}\,\underline{E}$$

$$\Rightarrow \nabla \times \left(\nabla \times \underline{E}\right) = -j\omega\mu\underline{J}^i + k_e^2\underline{E}$$

$$\Rightarrow \nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2\underline{E} = -j\omega\mu\underline{J}^i + k_e^2\underline{E}$$

$$\Rightarrow \nabla^2\underline{E} + k_e^2\underline{E} = j\omega\mu\underline{J}^i$$

Helmholtz Equation for E_z (cont.)

Hence

$$\nabla^2 E_z + k_e^2 E_z = j\omega\mu J_z^i$$

$$J_z^i = J_z^i(x,y) = I_0\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

Feed (impressed) current

Denote

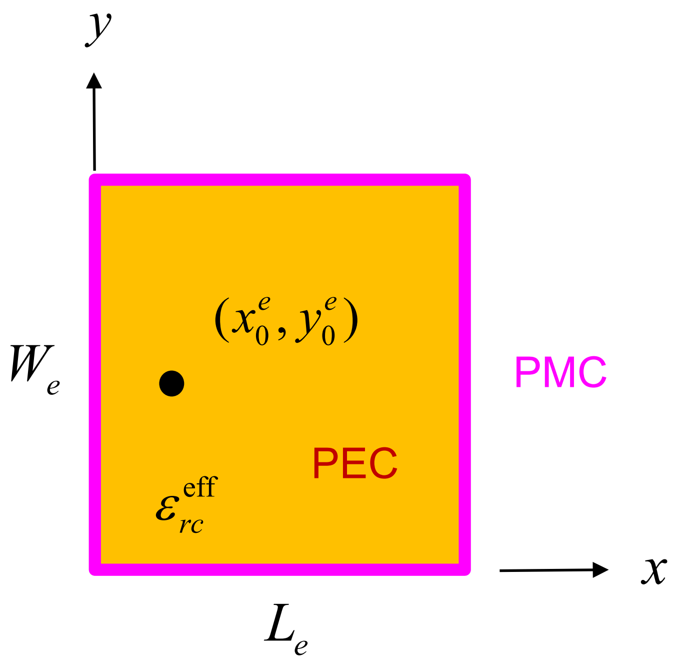
$$\psi(x,y) = E_z(x,y)$$

$$f(x,y) = j\omega\mu J_z^i(x,y)$$

We take it here to be a filamentary source (zero radius).

Then

$$\nabla^2\psi + k_e^2\psi = f(x,y)$$

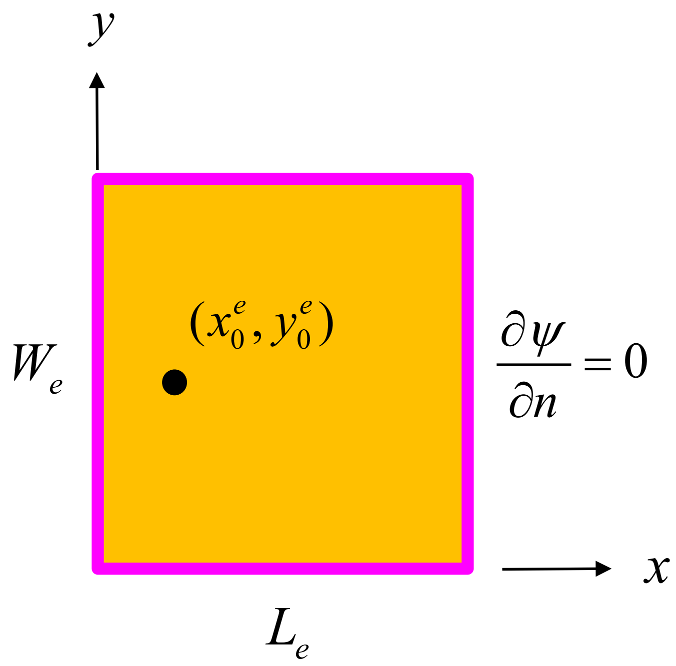


Mathematical Problem

$$\nabla^2\psi + k_e^2\psi = f(x,y)$$

$$\psi(x,y) = E_z(x,y) \qquad k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}} \qquad f(x,y) = (j\omega\mu I_0)\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

The function ψ is really a 2-D Green's function, if the feed current is filamentary.



Eigenvalue Problem

$$\nabla^2 \psi + k_e^2 \psi = f(x,y)$$

Eigenvalue problem:

$$\nabla^2 \psi + k_e^2 \psi = \lambda \psi$$

$$\Rightarrow \quad \nabla^2 \psi = \left(\lambda - k_e^2\right)\psi$$

$$\Rightarrow \quad \nabla^2 \psi = -\lambda'^2 \psi \qquad \left(-\lambda'^2 = \lambda - k_e^2\right)$$

The original eigenvalue problem is thus reduced to this simpler "reduced" eigenvalue problem.

New notation: $\lambda' = \lambda'_{mn}$

Eigenvalue Problem (cont.)

Introduce eigenfunctions of the 2-D Laplace operator:

$$\psi_{mn}(x,y)$$

$$\nabla^2 \psi_{mn}(x,y) = -\lambda_{mn}'^2 \psi_{mn}(x,y)$$

$$\frac{\partial \psi_{mn}}{\partial n} = 0\Big|_C \qquad -\lambda_{mn}'^2 = \text{eigenvalue}$$

For a rectangular patch we have, from separation of variables (or guessing):

$$\psi_{mn}(x,y) = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}'^2 = \left[\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2\right]$$

Note:
The eigenvalues are real and the eigenfunctions are orthogonal.

Eigenfunction Expansion

Assume an “eigenfunction expansion”:

$$\psi(x,y)=\sum_{m,n}A_{mn}\psi_{mn}(x,y) \qquad (m,n)=0,1,2\ldots$$

This must satisfy $\nabla^2\psi + k_e^2\psi = f(x,y)$

Hence
$$\sum_{m,n}A_{mn}\nabla^2\psi_{mn}+k_e^2\sum_{m,n}A_{mn}\psi_{mn}=f(x,y)$$

$$\nabla^2\psi_{mn}(x,y)=-\lambda_{mn}'^2\psi_{mn}(x,y)$$

Using this property of the eigenfunctions, we have:

$$\sum_{m,n}A_{mn}\left(k_e^2-\lambda_{mn}'^2\right)\psi_{mn}(x,y)=f(x,y)$$

Eigenfunction Expansion (cont.)

Multiply the previous equation by $\psi_{m'n'}^{*}(x,y)$ and integrate.

Note that the eigenfunctions are orthogonal, so that

$$\int_S \psi_{mn}(x,y)\,\psi_{m'n'}^{*}(x,y)\,dS = 0 \qquad (m,n) \neq (m',n')$$

Note: The eigenfunctions are real, so we can drop the conjugate here if we want.

Define $$\langle u,v \rangle \equiv \int_S u(x,y)\,v^{*}(x,y)\,dS$$

$$\Rightarrow \quad \langle \psi_{mn},\psi_{m'n'} \rangle = \int_S \psi_{mn}(x,y)\,\psi^{*}_{m'n'}(x,y)\,dS = 0, \quad (m,n) \neq (m',n')$$

We then have:

$$A_{m'n'}\left(k_e^2 - \lambda'^2_{m'n'}\right)\langle \psi_{m'n'},\psi_{m'n'} \rangle = \langle f,\psi_{m'n'} \rangle$$

Eigenfunction Expansion (cont.)

Hence, we have (removing the primes in the notation):

$$A_{mn} = \frac{< f, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >} \left(\frac{1}{k_e^2 - \lambda_{mn}'^2} \right)$$

Recall: $f(x, y) = j\omega\mu J_z^i(x, y)$

Therefore:

$$A_{mn} = j\omega\mu \left(\frac{< J_z^i, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >} \right) \left(\frac{1}{k_e^2 - \lambda_{mn}'^2} \right)$$

The field inside the patch cavity is then given by

$$E_z(x, y) = \psi(x, y) = \sum_{m,n} A_{mn} \psi_{mn}(x, y)$$

Eigenfunction Expansion (cont.)

For the rectangular patch:

$$\psi_{mn} = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}'^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

where

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1 - j l_{\text{eff}}\right)$$

We need:

$$\langle \psi_{mn}, \psi_{mn} \rangle = \int_0^{L_e} \cos^2\left(\frac{m\pi x}{L_e}\right)dx \int_0^{W_e} \cos^2\left(\frac{n\pi y}{W_e}\right)dy$$

Eigenfunction Expansion (cont.)

The result is:

$$\langle \psi_{mn}, \psi_{mn} \rangle = \left(\frac{W_e}{2} \right) \left(\frac{L_e}{2} \right) (1 + \delta_{m0})(1 + \delta_{n0})$$

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Eigenfunction Expansion (cont.)

For a filamentary feed current we have:

$$\left\langle J_z^i, \psi_{mn} \right\rangle = \int_{-W_e/2}^{W_e/2} \int_{-L_e/2}^{L_e/2} I_0 \delta\left(x - x_0^e\right) \delta\left(y - y_0^e\right) \psi_{mn}^*\left(x, y\right) dx dy$$

$$= I_0 \psi_{mn}^*\left(x_0^e, y_0^e\right)$$

Hence, we have

$$\left\langle J_z^i, \psi_{mn} \right\rangle = I_0 \cos\left(\frac{m\pi x_0^e}{L_e}\right) \cos\left(\frac{n\pi y_0^e}{W_e}\right)$$

Eigenfunction Expansion (cont.)

The final form for the field inside the patch cavity is then given by:

$$E_z(x,y) = \psi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\psi_{mn}(x,y)$$

$$\psi_{mn} = \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$\lambda_{mn}'^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$J_z^i = J_z^i(x,y) = I_0\,\delta\left(x - x_0^e\right)\delta\left(y - y_0^e\right)$$

$$A_{mn} = j\omega\mu\left(\frac{< J_z^i, \psi_{mn} >}{< \psi_{mn}, \psi_{mn} >}\right)\left(\frac{1}{k_e^2 - \lambda_{mn}'^2}\right)$$

$$\left\langle J_z^i, \psi_{mn}\right\rangle = I_0\cos\left(\frac{m\pi x_0^e}{L_e}\right)\cos\left(\frac{n\pi y_0^e}{W_e}\right)$$

$$\left\langle \psi_{mn}, \psi_{mn}\right\rangle = \left(\frac{W_e}{2}\right)\left(\frac{L_e}{2}\right)\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)$$

$$k_e^2 - \lambda_{mn}'^2 = k_e^2 - \left(\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2\right)$$

Final Field Inside Cavity

Substituting in for all of the terms, we have:

$$E_z(x,y) = j\omega\mu I_0 \left(\frac{4}{W_e L_e}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(1+\delta_{m0})(1+\delta_{n0})} \left(\frac{\cos\left(\dfrac{m\pi x_0^e}{L_e}\right)\cos\left(\dfrac{n\pi y_0^e}{W_e}\right)}{k_e^2 - \left(\dfrac{m\pi}{L_e}\right)^2 - \left(\dfrac{n\pi}{W_e}\right)^2} \right) \cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

$$k_e = k_0 \sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r \left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan\delta_{\text{eff}} = \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c} + \frac{1}{Q_{\text{sp}}} + \frac{1}{Q_{\text{sw}}}$$

Note:
It is usually the (1,0) mode that is resonant.

Note: It is not obvious, but the field goes to infinity when $(x,y) \to \left(x_0^e, y_0^e\right)$

Green's Function

Using a Green's function notation, we have (setting $I_0 = 1$):

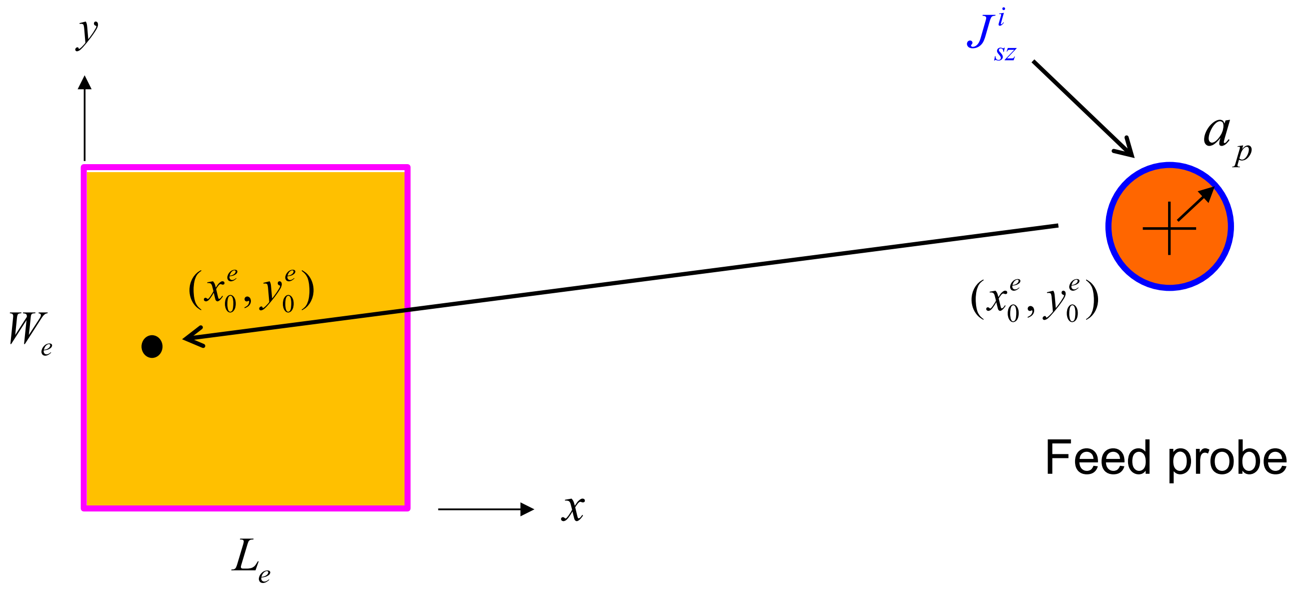
$$G(x,y;x',y') = j\omega\mu\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x'}{L_e}\right)\cos\left(\frac{n\pi y'}{W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

For an arbitrary impressed current excitation inside the cavity, we then have:

$$E_z(x,y) = \int_0^{W_e}\int_0^{L_e} J_z^i(x',y')\,G(x,y;x',y')\,dx'dy'$$

Input Impedance

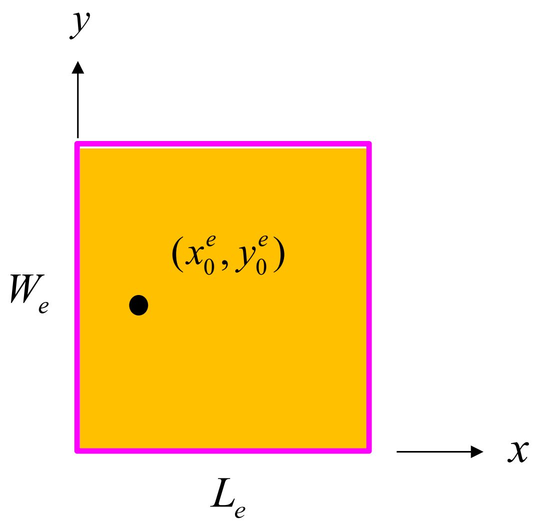
To calculate the input impedance, we need to consider a nonzero radius of the feed probe.

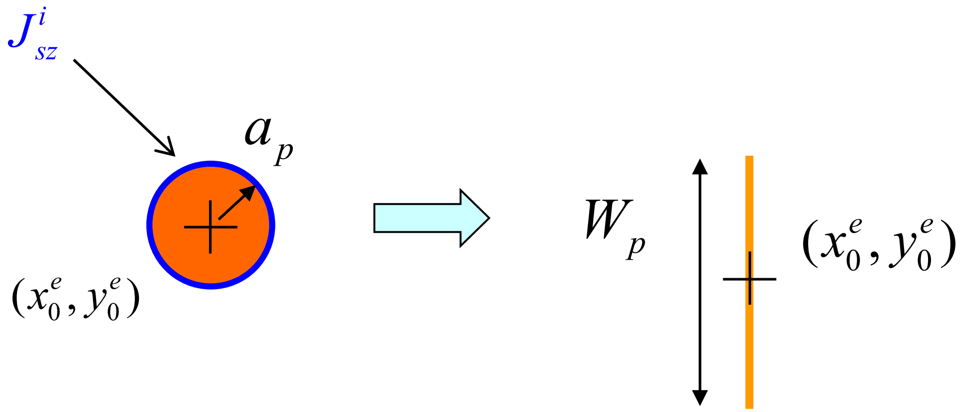


Note: Because the probe is made of PEC, there is a surface current on it.

Input Impedance (cont.)

- We first calculate the electric field E_z inside the patch cavity due to the probe.
- It is convenient to use a strip model of the probe.



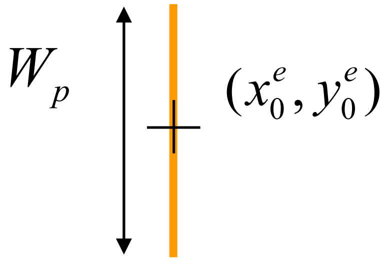


Input Impedance (cont.)

For a “Maxwell” strip current assumption, we have:

$$J_{sz}^{i}\left(y'\right)=\frac{I_0}{\pi\sqrt{\left(\frac{W_p}{2}\right)^2-\left(y'-y_0^e\right)^2}},\qquad y'\in\left(y_0^e-\frac{W_p}{2},\,y_0^e+\frac{W_p}{2}\right)$$

$$W_p=4a_p$$



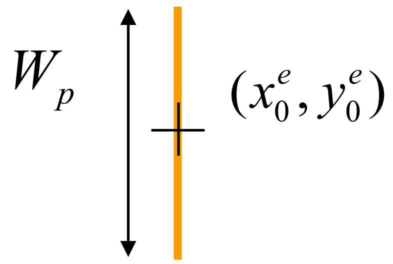
Note: The total probe current is I_0 amps.

Input Impedance (cont.)

For a uniform strip current assumption, we have:

$$J_{sz}^{i}\left(y'\right)=\frac{I_0}{W_p}, \qquad y' \in \left(y_0^e-\frac{W_p}{2},\, y_0^e+\frac{W_p}{2}\right)$$

$$W_p = a_p\, e^{\frac{3}{2}} \doteq 4.482\, a_p$$



Note: The total probe current is I_0 amps.

(We will use this model.)

Input Impedance (cont.)

Field inside cavity due to probe:

$$E_z(x,y) = \int_0^{W_e}\int_0^{L_e} J_z^i(x',y')\,G(x,y;x',y')\,dx'dy'$$ (arbitrary impressed volumetric current in cavity)

$$E_z(x,y) = \int_C J_{sz}^i(x',y')\,G(x,y;x',y')\,dS'$$ (arbitrary impressed surface current on a contour)

$$\Rightarrow \quad E_z(x,y) = \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} J_{sz}^i(y')\,G\left(x,y;x_0^e,y'\right)dy'$$ (strip current)

$$J_{sz}^i(y') = \frac{I_0}{W_p}$$ (uniform strip current model)

$$G(x,y;x_0^e,y') = j\omega\mu\left(\frac{4}{W_eL_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)\cos\left(\frac{n\pi y'}{W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

Term that needs to be integrated

Input Impedance (cont.)

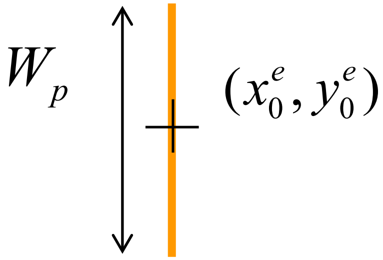
Integration over the strip current:

$$\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} J_{sz}^i\left(y'\right)\cos\left(\frac{n\pi y'}{W_e}\right)dy' = \int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} \frac{I_0}{W_p}\cos\left(\frac{n\pi y'}{W_e}\right)dy$$

$$= \frac{I_0}{W_p}\int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi}{W_e}\left[y_0^e + y''\right]\right)dy'' \qquad \left(y'' \equiv y' - y_0^e\right)$$

Integrates to zero (odd)

$$= \frac{I_0}{W_p}\int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi y_0^e}{W_e}\right)\cos\left(\frac{n\pi y''}{W_e}\right) - \sin\left(\frac{n\pi y_0^e}{W_e}\right)\sin\left(\frac{n\pi y''}{W_e}\right)dy''$$



$$\frac{I_0}{W_p}\cos\left(\frac{n\pi y_0^e}{W_e}\right)\int_{-\frac{W_p}{2}}^{+\frac{W_p}{2}} \cos\left(\frac{n\pi y''}{W_e}\right)dy'$$

$$= \frac{I_0}{W_p}\left[\cos\left(\frac{n\pi y_0^e}{W_e}\right)W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right] \qquad \operatorname{sinc}\left(x\right) \equiv \frac{\sin x}{x}$$

Input Impedance (cont.)

The field inside the cavity due to the strip probe current is then:

$$E_z(x,y) = j\omega\mu\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)\left[\frac{I_0}{W_p}\cos\left(\frac{n\pi y_0^e}{W_e}\right)W_p\,\mathrm{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right]\cos\left(\frac{m\pi x}{L_e}\right)\cos\left(\frac{n\pi y}{W_e}\right)$$

We next use the field inside the cavity to find the input impedance. We first calculate the complex power going into the patch, which is the complex power radiated by the probe current inside the cavity.

$$P_{\mathrm{in}} = -\frac{1}{2}h\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} E_z\left(x_0^e, y\right)J_{sz}^{i*}(y)\,dy$$

$$P_{\mathrm{in}} = \frac{1}{2}Z_{\mathrm{in}}\left|I_0\right|^2$$

$$J_{sz}^{i}(y) = \frac{I_0}{W_p}$$

$$\Rightarrow \quad Z_{\mathrm{in}} = 2\frac{P_{\mathrm{in}}}{\left|I_0\right|^2}$$

Input Impedance (cont.)

$$P_{\text{in}} = -\frac{1}{2} h \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} E_z\left(x_0^e, y\right) J_{sz}^{i*}\left(y\right) \, dy \qquad J_{sz}^{i*}\left(y\right) = \frac{I_0^*}{W_p}$$

$$E_z(x,y) = j\omega\mu\left(\frac{4}{W_e L_e}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{\left(1+\delta_{m0}\right)\left(1+\delta_{n0}\right)} \left(\frac{\cos\left(\frac{m\pi x_0^e}{L_e}\right)}{k_e^2 - \left(\frac{m\pi}{L_e}\right)^2 - \left(\frac{n\pi}{W_e}\right)^2} \right) \left[\frac{I_0}{W_p}\cos\left(\frac{n\pi y_0^e}{W_e}\right) W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)\right] \cos\left(\frac{m\pi x}{L_e}\right) \cos\left(\frac{n\pi y}{W_e}\right)$$

We need this integral:

$$\int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{n\pi y}{W_e}\right)\left(\frac{I_0^*}{W_p}\right) dy = \frac{I_0^*}{W_p}\cos\left(\frac{n\pi y_0^e}{W_e}\right) W_p \operatorname{sinc}\left(\frac{n\pi W_p}{2W_e}\right)$$

Input Impedance (cont.)

The final result is:

$$Z_{\text{in}} = -j\omega\mu h\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos^2\left(\dfrac{m\pi x_0^e}{L_e}\right)\cos^2\left(\dfrac{n\pi y_0^e}{W_e}\right)\operatorname{sinc}^2\left(\dfrac{n\pi W_p}{2W_e}\right)}{k_e^2-\left(\dfrac{m\pi}{L_e}\right)^2-\left(\dfrac{n\pi}{W_e}\right)^2}\right)$$

$$W_p = a_p\, e^{\frac{3}{2}} \doteq 4.482\, a_p$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

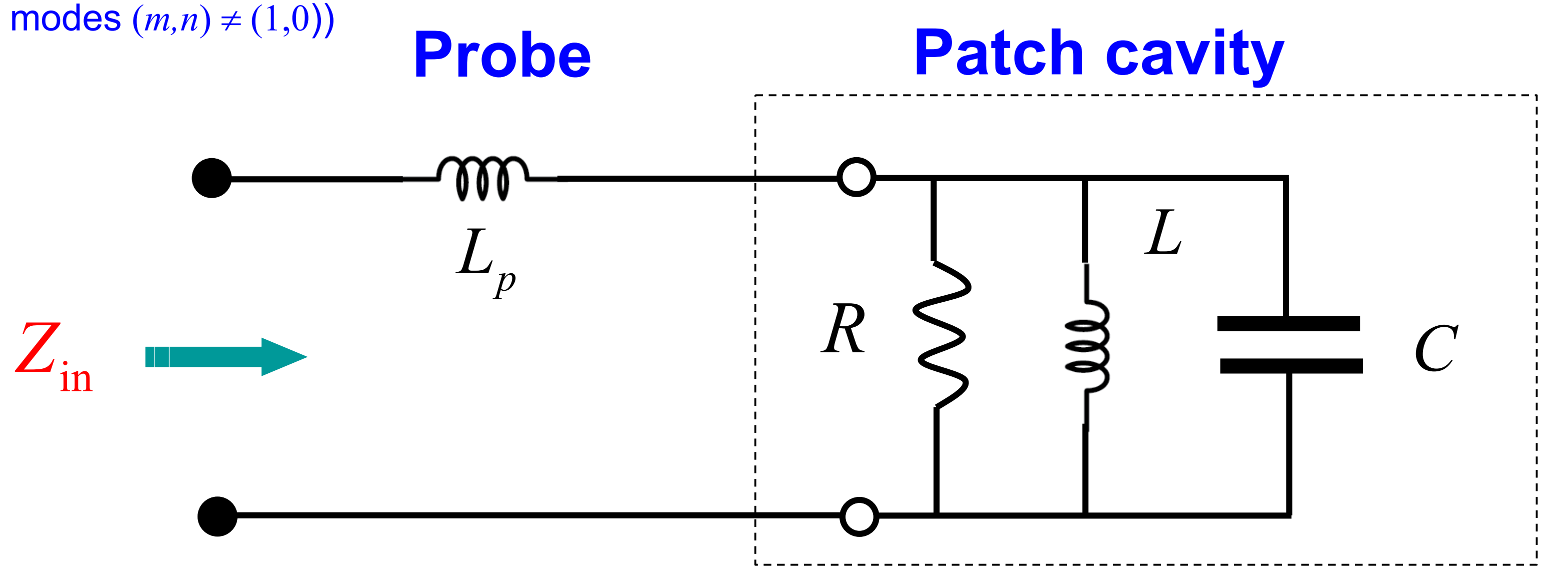
$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1-jl_{\text{eff}}\right)$$

$$l_{\text{eff}} = 1/Q$$

Note:
We cannot assume a probe of zero radius, or else the series will not converge – the input reactance will be infinite.

Circuit Model

Probe (feed) inductance
(accounts for all modes $(m,n) \neq (1,0)$)



(1,0 mode resonator)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$Z_{\text{in}} \approx j\omega L_p + \frac{R}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)}$$

Resonance frequency: $\text{Re}\left(k_e^2\right) = \left(\dfrac{\pi}{L_e}\right)^2$

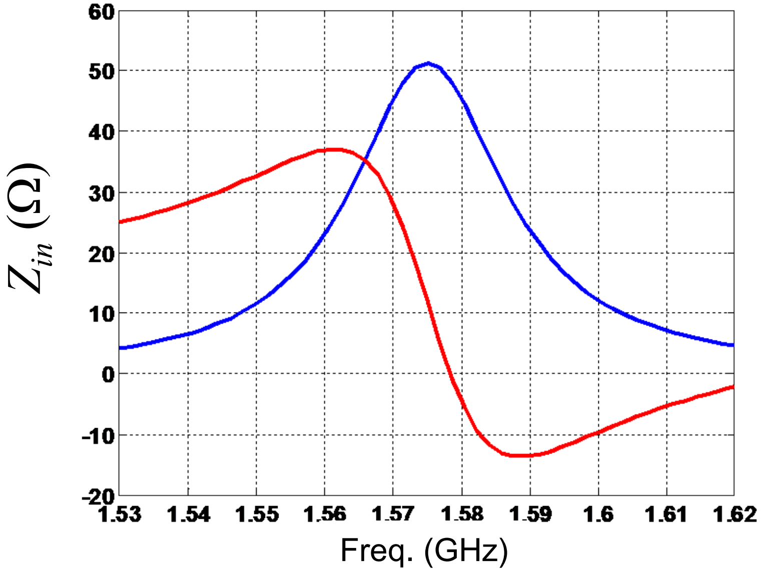
$$\Rightarrow \quad k_0^2 \varepsilon_r' = \left(\frac{\pi}{L_e}\right)^2 \quad \Rightarrow \quad k_0 \sqrt{\varepsilon_r'} = \frac{\pi}{L_e}$$

(Usually ε_r' is just called ε_r.)

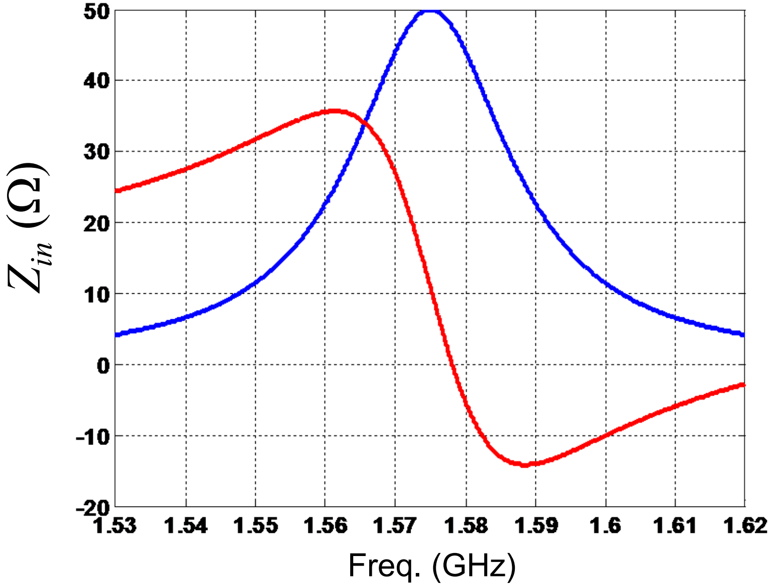
This formula, which describes the above circuit model, comes from approximating the input impedance formula. (This is done in just a bit!)

Results

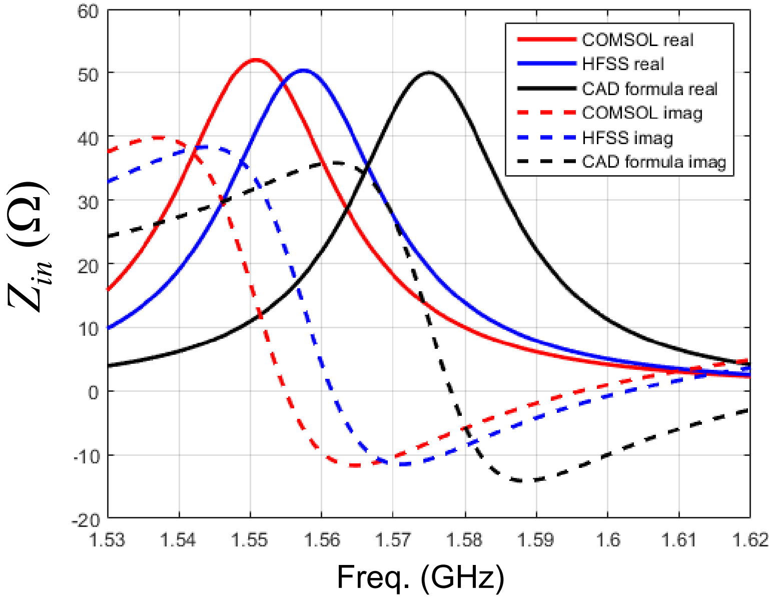
Cavity model (eigenfunction expansion) of patch



CAD Circuit model of patch



Patch	Feed
$\varepsilon_r = 2.2$	$x_0 = 1.85$ cm
$\tan\delta = 0.001$	$y_0 = W/2$
$h = 1.524$ mm	$a = 0.635$ mm
$L = 6.255$ cm	
$W/L = 1.5$	
$\sigma = 3.0\times10^7$ S/m	



Probe Inductance

$$Z_{\text{in}} = -j\omega\mu h\left(\frac{4}{W_e L_e}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos^2\left(\frac{m\pi x_0^e}{L_e}\right)\cos^2\left(\frac{n\pi y_0^e}{W_e}\right)\text{sinc}^2\left(\frac{n\pi W_p}{2W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)$$

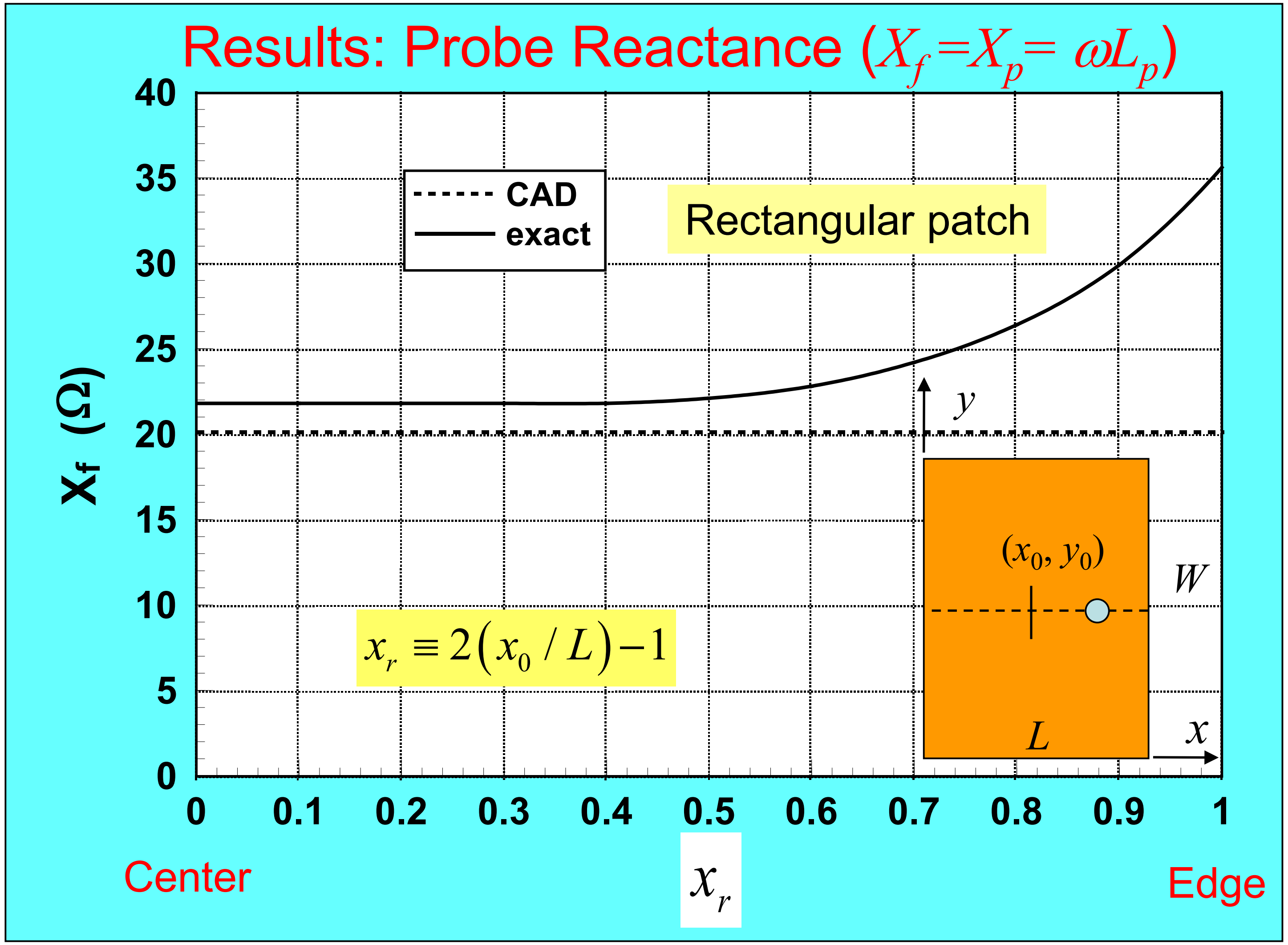
Note that

$(1,0) =$ term that corresponds to the dominant patch mode current (impedance of RLC circuit).

Hence, we have:

$$jX_p = -j\omega\mu h\left(\frac{4}{W_e L_e}\right)\sum_{\substack{(m,n)\\ \neq(1,0)}}\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\left(\frac{\cos^2\left(\frac{m\pi x_0^e}{L_e}\right)\cos^2\left(\frac{n\pi y_0^e}{W_e}\right)\text{sinc}^2\left(\frac{n\pi W_p}{2W_e}\right)}{k_e^2-\left(\frac{m\pi}{L_e}\right)^2-\left(\frac{n\pi}{W_e}\right)^2}\right)$$

Probe Inductance (cont.)



$\varepsilon_r = 2.2$

$W / L = 1.5$

$h = 0.0254\lambda_0$

$a = 0.5\,\text{mm}$

CAD:

$$X_p = \frac{\eta_0}{2\pi}(k_0 h)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\varepsilon_r}(k_0 a)}\right)\right]$$

Exact:

The cavity model for X_p with all infinite modes (excluding the (1,0) term).

The normalized feed location ratio x_r is zero at the center of the patch ($x = L/2$), and is 1.0 at the patch edge ($x = L$).

RLC Model

We can write

$$Z_{\text{in}} = \sum_{m,n} Z_{\text{in}}^{m,n}$$

where

$$Z_{\text{in}}^{m,n} = -j\omega\left(\frac{P_{mn}}{k_e^2 - k_{mn}^2}\right)$$

$$P_{mn} = \mu h\left(\frac{4}{W_e L_e}\right)\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\cos^2\left(\frac{m\pi x_0^e}{L_e}\right)\cos^2\left(\frac{n\pi y_0^e}{W_e}\right)\text{sinc}^2\left(\frac{n\pi W_p}{2W_e}\right)$$

$$k_{mn}^2 = \lambda_{mn}'^2 = \left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

(The P_{mn} coefficients are <u>not</u> a function of frequency.)

RLC Model (cont.)

We can then write

$$Z_{\text{in}}^{m,n} = -j\omega\left(\frac{P_{mn}}{k_e^2 - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{k_1^2\left(1 - jl_{\text{eff}}\right) - k_{mn}^2}\right)$$

$$= -j\omega\left(\frac{P_{mn}}{\left(k_1^2 - k_{mn}^2\right) - jk_1^2 l_{\text{eff}}}\right)$$

$$= \omega\frac{P_{mn}}{k_1^2 l_{\text{eff}} + j\left(k_1^2 - k_{mn}^2\right)}$$

Recall :

$$k_e^2 = k_0^2 \varepsilon_{rc}^{\text{eff}}$$

$$= k_0^2 \varepsilon_r \left(1 - jl_{\text{eff}}\right)$$

$$= k_1^2 \left(1 - jl_{\text{eff}}\right)$$

Note:
k_1 is the wavenumber of a lossless substrate having the (real) relative permittivity ε_r.

RLC Model (cont.)

We can write this as:

$$Z_{\text{in}}^{m,n} = \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\left(\frac{\omega}{\frac{k_1^2}{k_{mn}^2} + j\left(\frac{1}{l_{\text{eff}}}\right)\left(\frac{k_1^2}{k_{mn}^2} - 1\right)}\right)$$

Next, use:

$$\frac{1}{l_{\text{eff}}} = Q$$

$$\frac{k_1^2}{k_{mn}^2} = \frac{\omega^2 \mu_0 \varepsilon_0 \varepsilon_r}{\omega_{mn}^2 \mu_0 \varepsilon_0 \varepsilon_r} = \frac{\omega^2}{\omega_{mn}^2} = \frac{f^2}{f_{mn}^2} = f_{rmn}^2$$

$$f_{rmn} \equiv \left(\frac{f}{f_{mn}}\right)$$

$$\omega_{mn}^2 \equiv k_{mn}^2 / \mu_0 \varepsilon_0 \varepsilon_r$$

Also, define:

$$R_{mn} \equiv \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega_{mn}$$

$$\Rightarrow \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega = \left(\frac{P_{mn}}{k_{mn}^2 l_{\text{eff}}}\right)\omega_{mn}\left(\frac{\omega}{\omega_{mn}}\right) = R_{mn} f_{rmn}$$

f_{mn} = resonance frequency of (m, n) mode in lossless cavity filled with ε_r

RLC Model (cont.)

Then

$$Z_{\text{in}}^{m,n} = R_{mn}\left(\frac{f_{rmn}}{f_{rmn}^2 + jQ\left(f_{rmn}^2 - 1\right)}\right)$$

or

$$Z_{\text{in}}^{m,n} = \left(\frac{R_{mn}}{f_{rmn} + jQ\left(f_{rmn} - \frac{1}{f_{rmn}}\right)}\right)$$

RLC Model (cont.)

Near the resonance of the TM_{10} mode, for $f_{r10}^2 \approx 1$, we have:

$$Z_{\text{in}} \approx \frac{R}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)}$$

$$R \equiv R_{10}$$
$$f_0 \equiv f_{10}$$

(RLC equation)

This justifies the RLC model near resonance of the TM_{10} mode.

(0,0) Mode

Note that for the $(0,0)$ mode $\omega_{00} = 0$

Recall : $k_{mn} = \sqrt{\left(\frac{m\pi}{L_e}\right)^2 + \left(\frac{n\pi}{W_e}\right)^2}$

$$Z_{\text{in}}^{m,n} = \omega \frac{P_{mn}}{k_1^2 l_{\text{eff}} + j\left(k_1^2 - k_{mn}^2\right)} \quad \Rightarrow \quad Z_{\text{in}}^{0,0} = \omega \frac{P_{00}}{j\left(k_1^2 - jk_1^2 l_{\text{eff}}\right)}$$

or

$$Z_{\text{in}}^{0,0} \approx \frac{1}{j\omega\left(\dfrac{\mu\varepsilon_0\varepsilon_r}{P_{00}}\right)}$$

(Assume $l_{\text{eff}} = l_{\text{eff}}^{0,0} \ll 1$)

Also, we have:

$$P_{00} = \frac{\mu h}{L_e W_e}$$

Recall: $P_{mn} \equiv \mu h\left(\frac{4}{W_e L_e}\right)\frac{1}{(1+\delta_{m0})(1+\delta_{n0})}\cos^2\left(\frac{m\pi x_0^e}{L_e}\right)\cos^2\left(\frac{n\pi y_0^e}{W_e}\right)\text{sinc}^2\left(\frac{n\pi W_p}{2W_e}\right)$

(0,0) Mode (cont.)

Hence

$$Z_{\text{in}}^{0,0} \approx \frac{1}{j\omega\left(\dfrac{\mu\varepsilon_0\varepsilon_r}{P_{00}}\right)} = \frac{1}{j\omega\left(\dfrac{\mu\varepsilon_0\varepsilon_r}{\left(\dfrac{\mu h}{L_e W_e}\right)}\right)}$$

or

$$Z_{\text{in}}^{0,0} \approx \frac{1}{j\omega\left(\varepsilon_0\varepsilon_r \dfrac{L_e W_e}{h}\right)} = \frac{1}{j\omega C}$$

As expected, the $(0,0)$ mode acts as a parallel-plate capacitor.

Nonresonant Modes

For any other *nonresonant* mode $(m,n) \neq (1,0)$ or $(0,0)$:

$$f_{rmn} < 1$$

Note:
This is not true for the $(0,1)$ mode, but this mode is not excited when feeding on the centerline.

$$Z_{\text{in}}^{m,n} = \left(\frac{R_{mn}}{f_{rmn} + jQ\left(f_{rmn} - \dfrac{1}{f_{rmn}} \right)} \right)$$

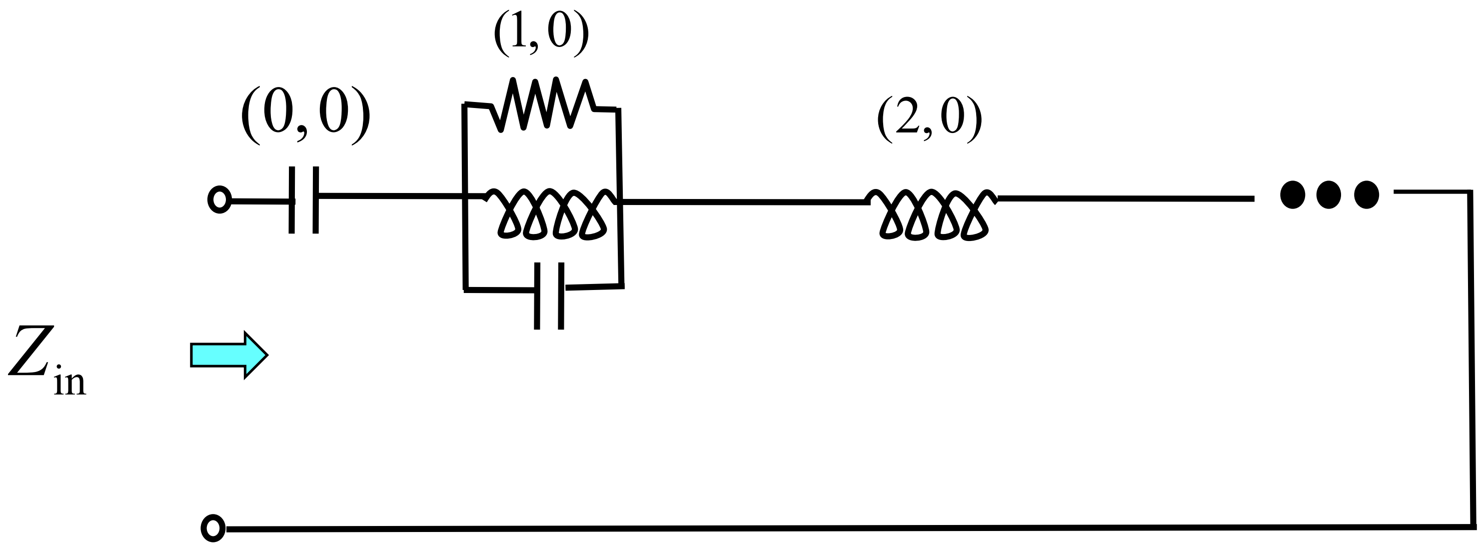
Note: $f_{rmn} - \dfrac{1}{f_{rmn}} < 0$

$Q >> 1$

$$Z_{\text{in}}^{m,n} \approx \left(\frac{R_{mn}}{jQ\left(f_{rmn} - \dfrac{1}{f_{rmn}} \right)} \right) = jX_{mn} \qquad (X_{mn} > 0)$$

RLC Circuit Model

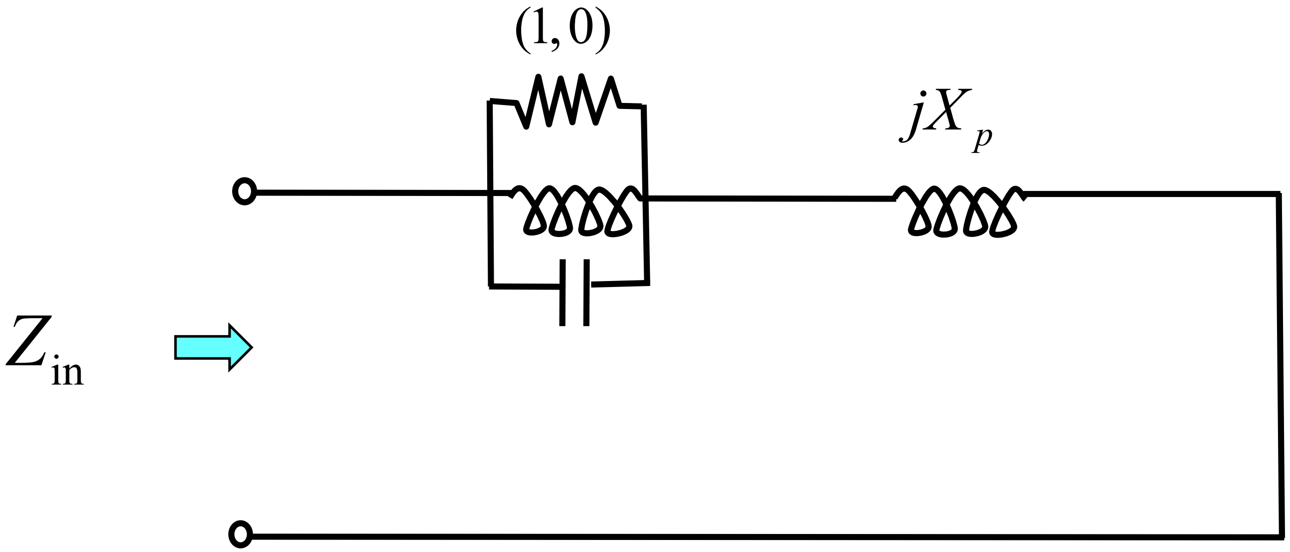
Circuit model:



Note:
This circuit model is accurate as long as we are near the resonance of the (1,0) circuit.

RLC Circuit Model (cont.)

Lumping all of the nonresonant circuits together into a "probe reactance", we have:



This gives us the CAD model for the patch.