ECE 6345

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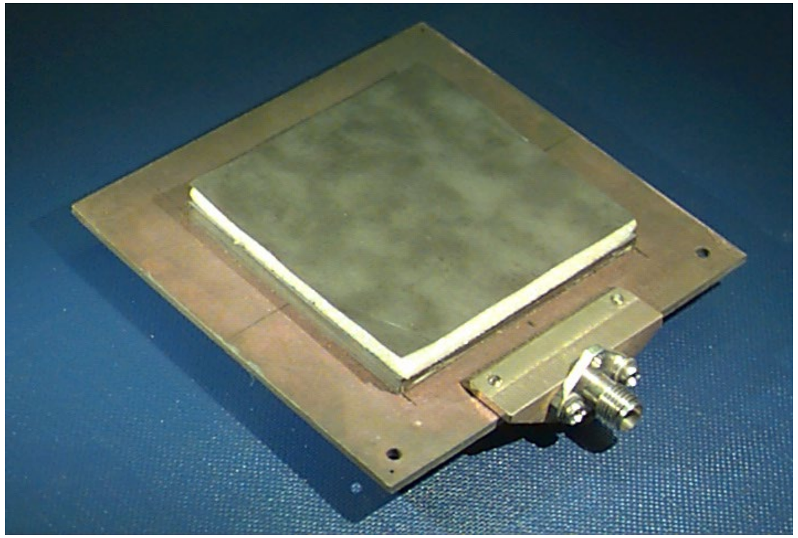
Prof. David R. Jackson
ECE Dept.



Notes 4

Overview

In this set of notes we develop the mode matching method for obtaining the input impedance in the cavity-model problem.

This is an alternative to the eigenfunction expansion method.

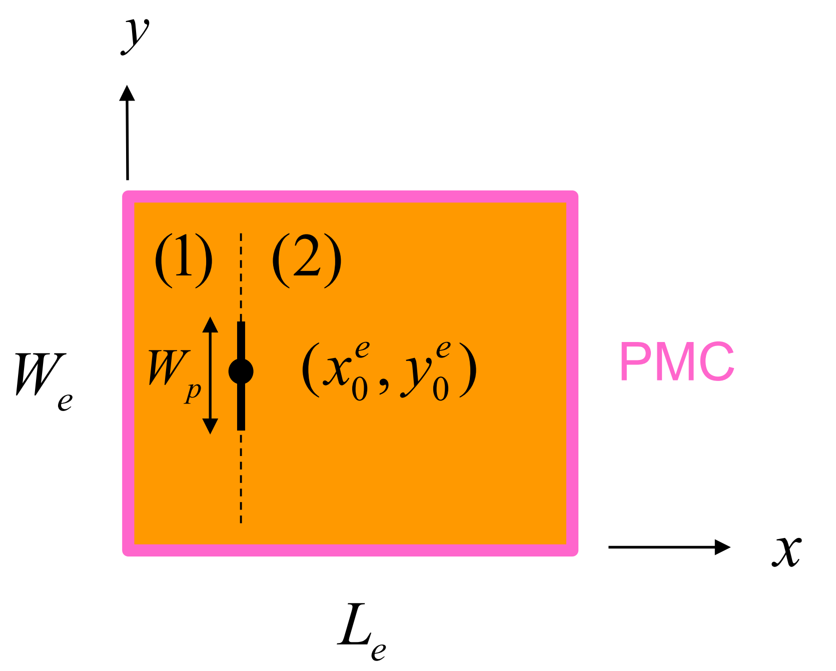
It is numerically convenient, requiring only a single sum instead of a double sum (as in the eigenfunction expansion method).

Mode Matching Method

a_p = probe radius

$$W_p = a_p\, e^{3/2} \doteq 4.482 a_p$$

(uniform strip current model)



$$k_e = k_0 \sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r \left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan \delta_{\text{eff}} = \frac{1}{Q}$$

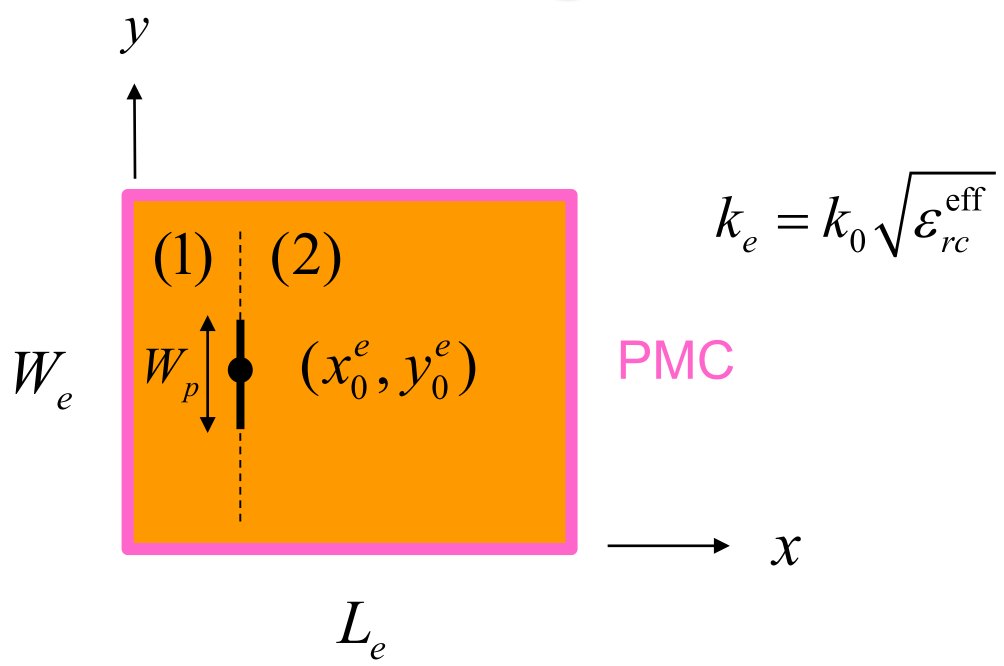
We view the cavity as a finite section of *waveguide*, of length L_e, having PMC side walls.

m^{th} waveguide mode (TM_{m0} mode):

$$E_z(x,y) = \cos\left(\frac{m\pi y}{W_e}\right) e^{-jk_{xm}x} \qquad k_{xm} = \left(k_e^2 - \left(\frac{m\pi}{W_e}\right)^2\right)^{1/2}$$

(propagation wavenumber)

Mode Matching Method



Applying B.C.s at the left and right PMC walls, we have the following field representations:

(1) $$E_{z1} = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right)\cos\left(k_{xm}x\right)$$

(2) $$E_{z2} = \sum_{m=0}^{\infty} B_m \cos\left(\frac{m\pi y}{W_e}\right)\cos\left(k_{xm}\left[x - L_e\right]\right)$$

Mode Matching (cont.)

Boundary conditions at the interface:

at $x = x_0^e$:

$$\begin{cases} E_{z1} = E_{z2} & \text{BC \#1} \\ H_{y2} - H_{y1} = J_{sz}^i = \dfrac{I_0}{W_p}, \quad \left| y - y_0^e \right| < \dfrac{W_p}{2} & \text{BC \#2} \end{cases}$$

To calculate H_y in BC #2 we use (from Faraday's law):

$$H_y = \frac{1}{j\omega\mu}\left(\frac{\partial E_z}{\partial x} - \cancel{\frac{\partial E_x}{\partial z}}\right)$$

Mode Matching (cont.)

From (BC #1) we have:

$$\sum_{m=0}^{\infty} A_m \cos\left(k_{xm} x_0^e\right) \cos\left(\frac{m\pi y}{W_e}\right) = \sum_{m=0}^{\infty} B_m \cos k_{xm}\left[x_0^e - L_e\right] \cos\left(\frac{m\pi y}{W_e}\right)$$

We then multiply both sides by $\cos\left(\frac{n\pi y}{W_e}\right)$ and integrate in y from 0 to W_e.

From orthogonality of the cosine functions, we then have (letting n be relabeled as m):

$$A_m \cos\left(k_{xm} x_0^e\right) = B_m \cos k_{xm}\left[x_0^e - L_e\right]$$

Mode Matching (cont.)

From (BC #2) we have:

$$H_{y2} - H_{y1} = J_{sz}^{i} = \frac{I_0}{W_p}, \qquad \left|y - y_0^e\right| < \frac{W_p}{2}$$

This gives us:

$$\sum_{m=0}^{\infty} \frac{1}{j\omega\mu}\left[B_m k_{xm}\left(-\sin k_{xm}\left(x_0^e - L_e\right)\right) - A_m k_{xm}\left(-\sin k_{xm} x_0^e\right)\right]\cos\left(\frac{m\pi y}{W_e}\right) = \frac{I_0}{W_p}$$

$$\left|y - y_0^e\right| < \frac{W_p}{2}$$

We then multiply both sides by $\cos\left(\frac{n\pi y}{W_e}\right)$ and integrate in y from 0 to W_e.

Mode Matching (cont.)

Using orthogonality of the cosine functions, we have (letting n be relabeled as m):

$$-\frac{1}{j\omega\mu}\left(k_{xm}\right)\left[B_m \sin k_{xm}\left(x_0^e - L_e\right) - A_m \sin k_{xm} x_0^e\right]\left(\frac{W_e}{2}\left[1+\delta_{m0}\right]\right)$$

$$= \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{m\pi y}{W_e}\right)\frac{I_0}{W_p}\,dy$$

$$= I_0 \cos\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Mode Matching (cont.)

Hence, we have the following two equations:

$$B_m = A_m \left[\frac{\cos\left(k_{xm} x_0^e\right)}{\cos\left[k_{xm}\left(x_0^e - L_e\right)\right]} \right]$$

(from BC #1)

$$B_m \sin k_{xm}\left(x_0^e - L_e\right) - A_m \sin k_{xm} x_0^e$$
$$= I_0 \cos\left(\frac{m\pi y_0^e}{W_e}\right) \operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)\left(\frac{2}{W_e}\right)\left(\frac{1}{1+\delta_{m0}}\right)\left(\frac{-j\omega\mu}{k_{xm}}\right)$$

(from BC #2)

Mode Matching (cont.)

For the second equation, we have (substituting in B_m from the first equation):

$$\text{LHS} = B_m \sin k_{xm}\left(x_0^e - L_e\right) - A_m \sin k_{xm} x_0^e$$

$$= A_m \left\{ \left(\frac{\cos k_{xm} x_0^e}{\cos k_{xm}\left(x_0^e - L_e\right)} \right) \sin k_{xm}\left(x_0^e - L_e\right) - \sin k_{xm}\, x_0^e \right\}$$

$$= A_m \sec k_{xm}\left(x_0^e - L_e\right)\left\{ \cos k_{xm} x_0^e \sin k_{xm}\left(x_0^e - L_e\right) - \sin k_{xm}\, x_0^e \cos k_{xm}\left(x_0^e - L_e\right) \right\}$$

$$= A_m \sec k_{xm}\left(x_0^e - L_e\right)(-1)\sin k_{xm}\left[x_0^e - \left(x_0^e - L_e\right)\right]$$

$$= -A_m \sec k_{xm}\left(x_0^e - L_e\right)\sin k_{xm} L_e$$

Also, we have:

$$\text{RHS} = I_0 \cos\left(\frac{m\pi y_0^e}{W_e}\right)\text{sinc}\left(\frac{m\pi W_p}{2W_e}\right)\left(\frac{2}{W_e}\right)\left(\frac{1}{1+\delta_{m0}}\right)\left(\frac{-j\omega\mu}{k_{xm}}\right)$$

Mode Matching (cont.)

Hence, we have:

$$A_m = -\cos k_{xm}\left(x_0^e - L_e\right)\csc\left(k_{xm}L_e\right) \cdot I_0\left(\frac{2}{W_e}\right)\left(\frac{1}{1+\delta_{mo}}\right)\left(\frac{-j\omega\mu}{k_{xm}}\right)\cos\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Mode Matching (cont.)

We now calculate the input impedance, using complex power.

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I_0|^2} = \frac{2}{|I_0|^2}\int_V -\frac{1}{2}E_z\,J_z^{i*}\,dV = \frac{1}{|I_0|^2}\int_S -E_z\,J_{sz}^{i\,*}\,dS$$

$$= -\frac{h}{|I_0|^2}\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} E_z\,J_{sz}^{i\,*}\,dy = -\frac{h}{|I_0|^2}\left(\frac{I_0^*}{W_p}\right)\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}} E_z\left(x_0^e, y\right)dy$$

$\left(\textbf{Note:}\ \frac{I_0^*}{|I_0|^2} = \frac{1}{I_0}\right)$

Next, use (from the region 1 solution):

$$E_z = E_{z1} = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right)\cos\left(k_{xm}x\right)$$

Mode Matching (cont.)

Hence, we have:

$$Z_{\text{in}} = -\frac{h}{W_p I_0} \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi y}{W_e}\right) \cos\left(k_{xm} x_0^e\right) dy$$

$$= -\frac{h}{W_p I_0} \cos\left(k_{xm} x_0^e\right) \sum_{m=0}^{\infty} A_m \int_{y_0^e - \frac{W_p}{2}}^{y_0^e + \frac{W_p}{2}} \cos\left(\frac{m\pi y}{W_e}\right) dy$$

Mode Matching (cont.)

Note that

$$\frac{1}{W_p}\int_{y_0^e-\frac{W_p}{2}}^{y_0^e+\frac{W_p}{2}}\cos\left(\frac{m\pi y}{W_e}\right)dy=\cos\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Hence, we have

$$Z_{\text{in}}=-\frac{h}{I_0}\sum_{m=0}^{\infty}A_m\cos\left(k_{xm}x_0^e\right)\cos\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Summary

$$Z_{\text{in}} = -\frac{h}{I_0}\sum_{m=0}^{\infty} A_m \cos\left(k_{xm} x_0^e\right) \cos\left(\frac{m\pi y_0^e}{W_e}\right) \operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

with

$$A_m = -\cos k_{xm}\left(x_0^e - L_e\right)\csc\left(k_{xm} L_e\right) \cdot I_0\left(\frac{2}{W_e}\right)\left(\frac{1}{1+\delta_{mo}}\right)\left(\frac{-j\omega\mu}{k_{xm}}\right)\cos\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}\left(\frac{m\pi W_p}{2W_e}\right)$$

Final Result

$$Z_{\text{in}} = -j\omega\mu h\left(\frac{2}{W_e}\right)\sum_{m=0}^{\infty}\left(\frac{1}{1+\delta_{mo}}\right)\left(\frac{1}{k_{xm}}\right)$$

$$\cdot\cos\left(k_{xm}x_0^e\right)\cos k_{xm}\left(x_0^e - L_e\right)\csc\left(k_{xm}L_e\right)$$

$$\cdot\cos^2\left(\frac{m\pi y_0^e}{W_e}\right)\operatorname{sinc}^2\left(\frac{m\pi W_p}{2W_e}\right)$$

where

$$k_{xm} = \left(k_e^2 - \left(\frac{m\pi}{W_e}\right)^2\right)^{1/2}$$

$$W_p = a_p\, e^{3/2} \doteq 4.482 a_p$$

$$k_e = k_0\sqrt{\varepsilon_{rc}^{\text{eff}}}$$

$$\varepsilon_{rc}^{\text{eff}} = \varepsilon_r\left(1 - j l_{\text{eff}}\right)$$

$$l_{\text{eff}} = \tan\delta_{\text{eff}} = \frac{1}{Q}$$