ECE 6345

Spring 2024

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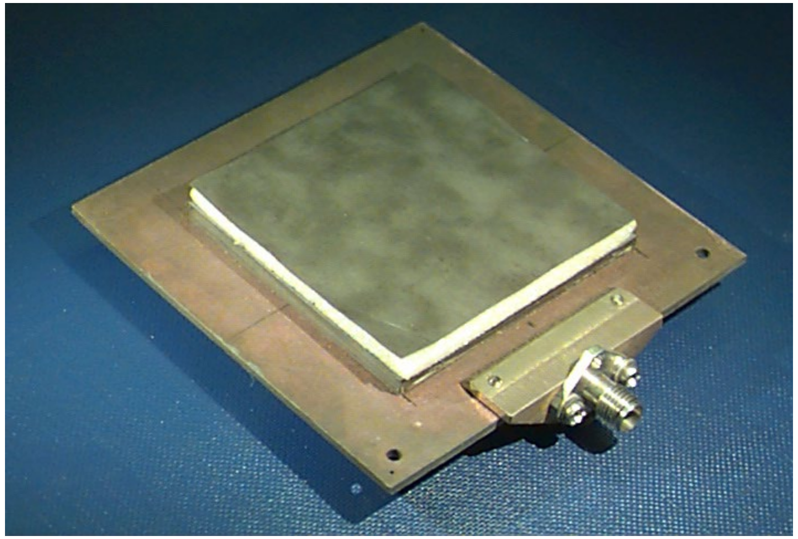
ECE Dept.



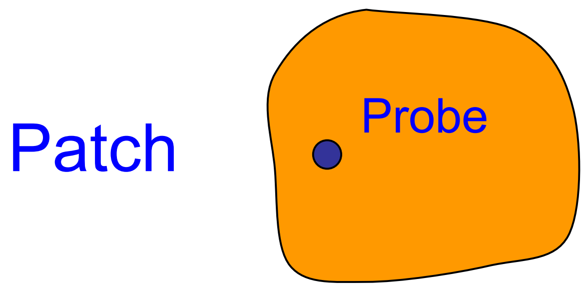
Notes 5

Overview

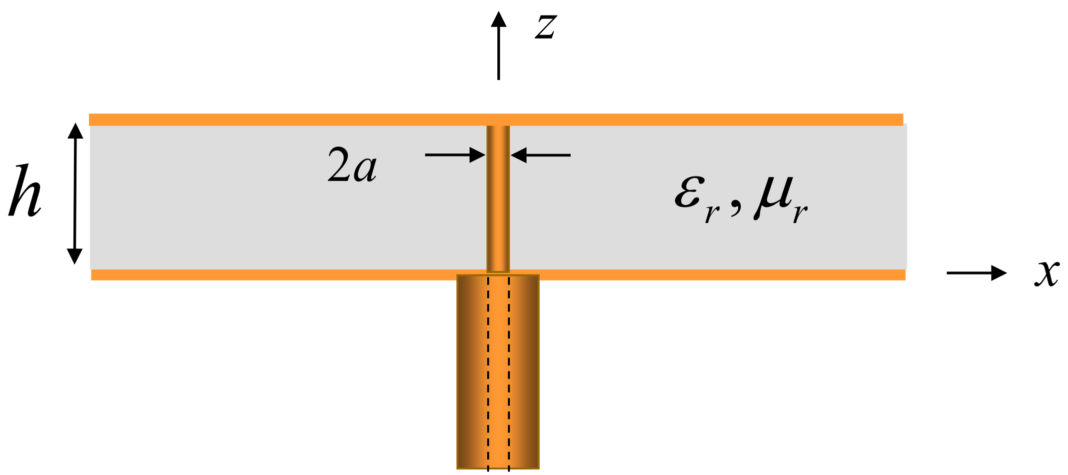
This set of notes discusses the probe inductance of a coax-fed patch.

- Introduce probe model for a parallel-plate waveguide
- Use this model to calculate the probe inductance

Probe Inductance



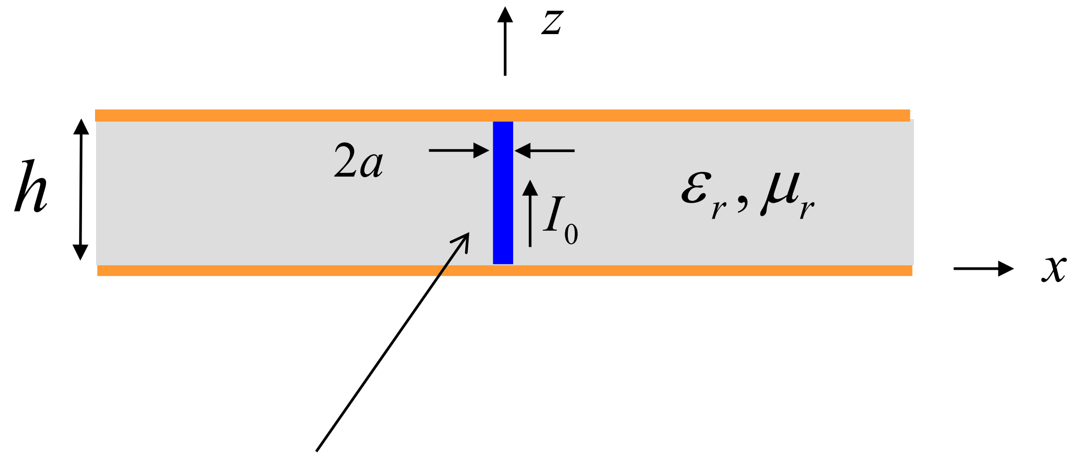
Parallel-Plate Waveguide Model



Probe Inductance

Assume that $\frac{\partial}{\partial z} = 0$

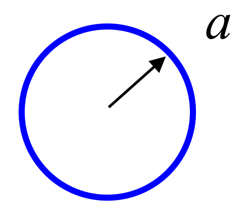
The probe current is assumed to be uniform in the z direction, and the metal is removed by the equivalence principle. Radiation from the coax “frill” is neglected.



$$k = k_0 \sqrt{\varepsilon_r \mu_r}$$

$$\eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Hollow tube of uniform surface current



$$J_{sz} = \frac{I_0}{2\pi a}$$

Probe Inductance (cont.)

Assume $\underline{E} = \hat{\underline{z}}\, E_z(\rho)$ since $\dfrac{\partial}{\partial \phi} = 0 \quad \dfrac{\partial}{\partial z} = 0$

$$\nabla^2 E_z + k^2 E_z = 0$$

General solution:

$$E_z = \begin{pmatrix} \cos(n\phi) \\ \sin(n\phi) \end{pmatrix} \begin{pmatrix} J_n(k_\rho \rho) \\ Y_n(k_\rho \rho) \end{pmatrix} \begin{pmatrix} \cos(k_z z) \\ \sin(k_z z) \end{pmatrix}$$

$$k_\rho = \left(k^2 - k_z^2\right)^{1/2}$$

$$k_z = \frac{m\pi}{h}$$

Choose $m = 0,\ n = 0$

Probe Inductance (cont.)

Hence

$$E_z = \begin{pmatrix} J_0(k\rho) \\ Y_0(k\rho) \end{pmatrix}$$

$J_0(k\rho)$ ← finite at the origin

$Y_0(k\rho)$ ← infinite at the origin

or

$$E_z = \begin{pmatrix} H_0^{(1)}(k\rho) \\ H_0^{(2)}(k\rho) \end{pmatrix}$$

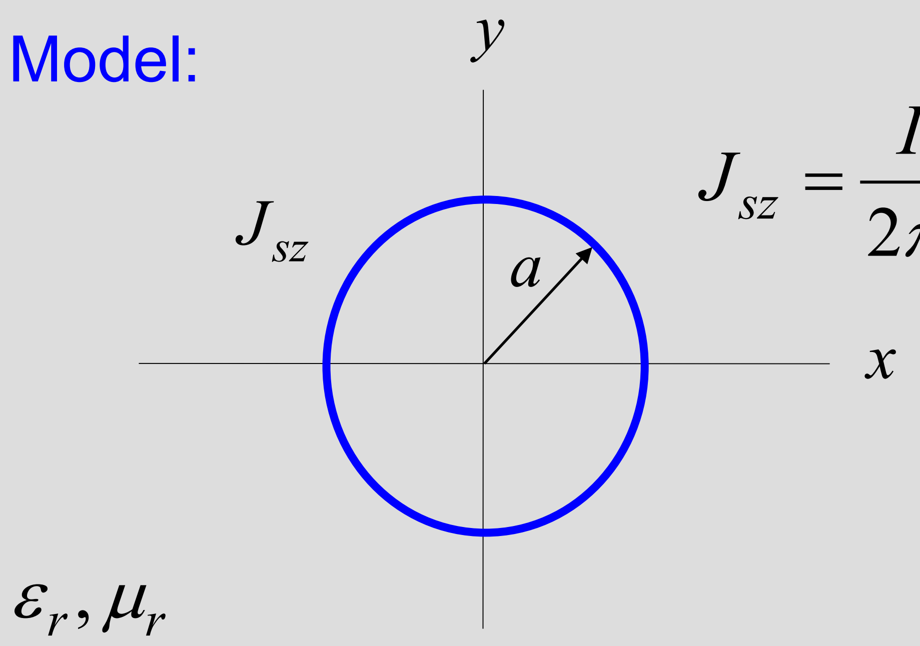
$H_0^{(1)}(k\rho)$ ← incoming wave

$H_0^{(2)}(k\rho)$ ← outgoing wave

$$H_n^{(1)}(z) \equiv J_n(z) + jY_n(z)$$
$$H_n^{(2)}(z) \equiv J_n(z) - jY_n(z)$$

Probe Inductance (cont.)

Model:



$$J_{sz} = \frac{I_0}{2\pi a} \quad [\mathrm{A/m}]$$

Hollow tube of current

Note: The tube may be thought of as being infinite in the z direction (image theory).

Probe Inductance (cont.)

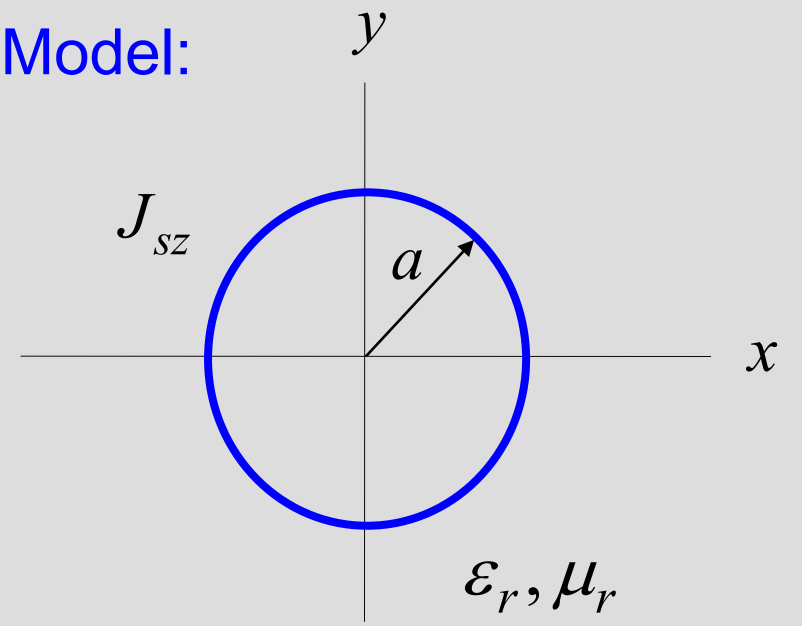
$\rho < a$ $$E_z^- = A^- J_0(k\rho)$$

$\rho > a$ $$E_z^+ = A^+ H_0^{(2)}(k\rho)$$

$\rho = a$ $$E_z^- = E_z^+ \quad \text{(BC \#1)}$$

$$H_\phi^+ - H_\phi^- = J_{sz} \quad \text{(BC \#2)}$$

Model:



$$J_{sz} = \frac{I_0}{2\pi a} \quad [\mathrm{A/m}]$$

Probe Inductance (cont.)

BC 1: $A^- J_0(ka) = A^+ H_0^{(2)}(ka)$

BC 2: $H_\phi = \dfrac{-1}{j\omega\mu}(\nabla \times \underline{E}) \cdot \underline{\hat{\phi}}$

$$= \frac{-1}{j\omega\mu}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)$$

$$= \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial \rho}$$

so

$$\frac{k}{j\omega\mu}\left[A^+ H_0^{(2)\prime}(ka) - A^- J_0'(ka)\right] = \frac{I_0}{2\pi a}$$

$\rho < a$ $\quad E_z^- = A^- J_0(k\rho)$

$\rho > a$ $\quad E_z^+ = A^+ H_0^{(2)}(k\rho)$

$\rho = a$

$E_z^- = E_z^+$ (BC 1)

$H_\phi^+ - H_\phi^- = J_{sz}$ (BC 2)

Probe Inductance (cont.)

Hence, eliminating A^- using BC 1, we have:

$$A^+ H_0^{(2)\prime}(ka) - \left[A^+ \frac{H_0^{(2)}(ka)}{J_0(ka)} \right] J_0{}'(ka) = \left(\frac{I_0}{2\pi a} \right) \left(\frac{j\omega\mu}{k} \right)$$

or

$$A^+ \left[J_0(ka) H_0^{(2)\prime}(ka) - J_0{}'(ka)\, H_0^{(2)}(ka) \right] = \frac{I_0}{2\pi a} \left(\frac{j\omega\mu}{k} \right) J_0(ka)$$

Probe Inductance (cont.)

Wronskian Identity:

$$J_n(x)\,H_n^{(2)\prime}(x) - J_n{}'(x)\,H_n^{(2)}(x) = -j\left(\frac{2}{\pi x}\right)$$

Hence

$$A^{+}\left(-j\frac{2}{\pi(ka)}\right) = \left(\frac{I_0}{2\pi a}\right)\left(\frac{j\omega\mu}{k}\right)J_0(ka)$$

or

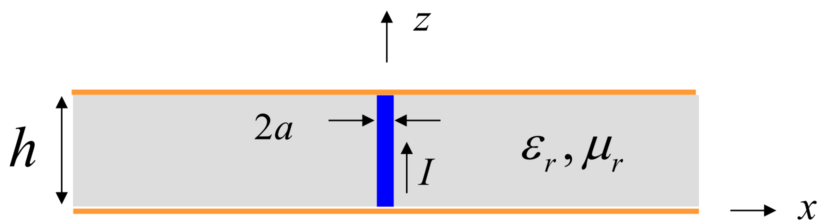
$$A^{+} = -\left(\frac{I_0}{4}\right)(\omega\mu)\,J_0(ka)$$

Next, use $\omega\,\mu = k\eta$ so $A^{+} = -\eta k\left(\frac{I_0}{4}\right)J_0(ka)$

Probe Inductance (cont.)

Hence

$$E_z^+ = -\eta k \left(\frac{I_0}{4} \right) J_0(ka) H_0^{(2)}(k\rho)$$



Next, we use

$$Z_{\text{in}} = \frac{V}{I_0} = \frac{-h \left. E_z \right|_{\rho=a}}{I_0}$$

so

$$Z_{\text{in}} = \eta \frac{(kh)}{4} J_0(ka) H_0^{(2)}(ka)$$

Note: The imaginary part of Z_{in} should be fairly accurate for the probe feed of a patch, but not the real part (the radiation effects are very different).

Probe Inductance (cont.)

We could also complex power:

$$Z_{\text{in}} = \frac{2P_c}{\left|I(0)\right|^2}$$

$$P_c = \frac{-1}{2} \oint_s \underline{E} \cdot \underline{J}_s^{i*} \, dS$$

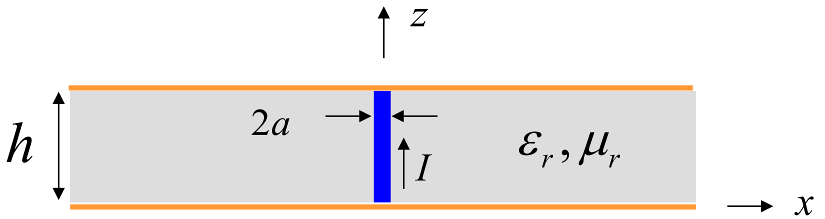
$$= \frac{-1}{2} \int_0^{2\pi} \int_0^{h} E_z(a) J_{zs}^{i*} \, a \, dz \, d\phi$$

$$= -\pi a \int_0^h E_z(a) J_{zs}^{i*} \, dz$$

$$= -\frac{\pi a}{2\pi a} \int_0^h E_z(a) I_0^* dz$$

$$= -\frac{1}{2} \int_0^h \left(-\eta k \left(\frac{I_0}{4} \right) J_0(ka) H_0^{(2)}(ka) \right) \cdot \left(I_0^* \right) dz$$

$$= -\frac{1}{8} \left| I_0^2 \right| h \left(-\eta k \, J_0(ka) H_0^{(2)}(ka) \right)$$



$$Z_{\text{in}} = \eta \frac{(kh)}{4} J_0(ka) H_0^{(2)}(ka)$$

(same result)

Probe Inductance (cont.)

Taking the imaginary part:
$$X_{\text{in}} = -\eta \frac{(kh)}{4} J_0\left(ka\right) Y_0(ka)$$

$$\left(\text{Recall}: H_0^{(2)}(ka) = J_0(ka) - jY_0(ka)\right)$$

For $ka << 1$

$$J_0(ka) \approx 1 \qquad Y_0(ka) \approx \frac{2}{\pi}\left(\gamma + \ln\left(\frac{ka}{2}\right)\right)$$

where $\gamma \doteq 0.57722$ (Euler's constant)

Probe Inductance (cont.)

Approximating the Y_0 Bessel function, we have:

$$X_{\text{in}} = \eta\left(\frac{kh}{2\pi}\right)\left[-\gamma + \ln\left(\frac{2}{ka}\right)\right]$$

or

$$X_{\text{in}} = \frac{\eta_0}{2\pi}\sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\mu_r\varepsilon_r}\left(k_0 h\right)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r\varepsilon_r}\,k_0 a}\right)\right]$$

or

$$X_{\text{in}} = \frac{\eta_0}{2\pi}\mu_r\left(k_0 h\right)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r\varepsilon_r}\,k_0 a}\right)\right]$$

Probe Inductance (cont.)

We then have (with $X_{\text{in}} = X_p$):

$$X_p = \frac{\eta_0}{2\pi}\mu_r\left(k_0 h\right)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r \varepsilon_r}\left(k_0 a\right)}\right)\right] \;[\Omega]$$

Alternative form:

$$X_p = \eta_0\,\mu_r\left(\frac{h}{\lambda_0}\right)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r \varepsilon_r}\left(k_0 a\right)}\right)\right] \;[\Omega]$$

$\gamma \doteq 0.57722$ (Euler's constant) $\qquad \eta_0 \doteq 376.7303\;[\Omega]$

Probe Inductance (cont.)

We can solve for the probe inductance using $L_p = \dfrac{X_p}{\omega} = \dfrac{X_p}{k_0}\sqrt{\mu_0 \varepsilon_0} = \dfrac{X_p}{k_0 c}$

$$L_p = \frac{\eta_0}{2\pi c}\mu_r h\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r \varepsilon_r}\,(k_0 a)}\right)\right] \text{ [H]}$$

$$L_p = \frac{\eta_0}{2\pi c}\mu_r h\left[\ln\left(\frac{2}{\sqrt{\mu_r \varepsilon_r}}\right) - \gamma\right] - \left(\frac{\eta_0}{2\pi c}\mu_r h\right)\ln(k_0 a)$$

The probe inductance is slightly dependent on frequency.

In practice, we can evaluate it using $\omega = \omega_0 = 2\pi f_0$

Probe Inductance (cont.)

Example

$$\varepsilon_r = 2.94$$

$$h = 0.1524 \;\; [\text{cm}] \qquad (60\,\text{mils})$$

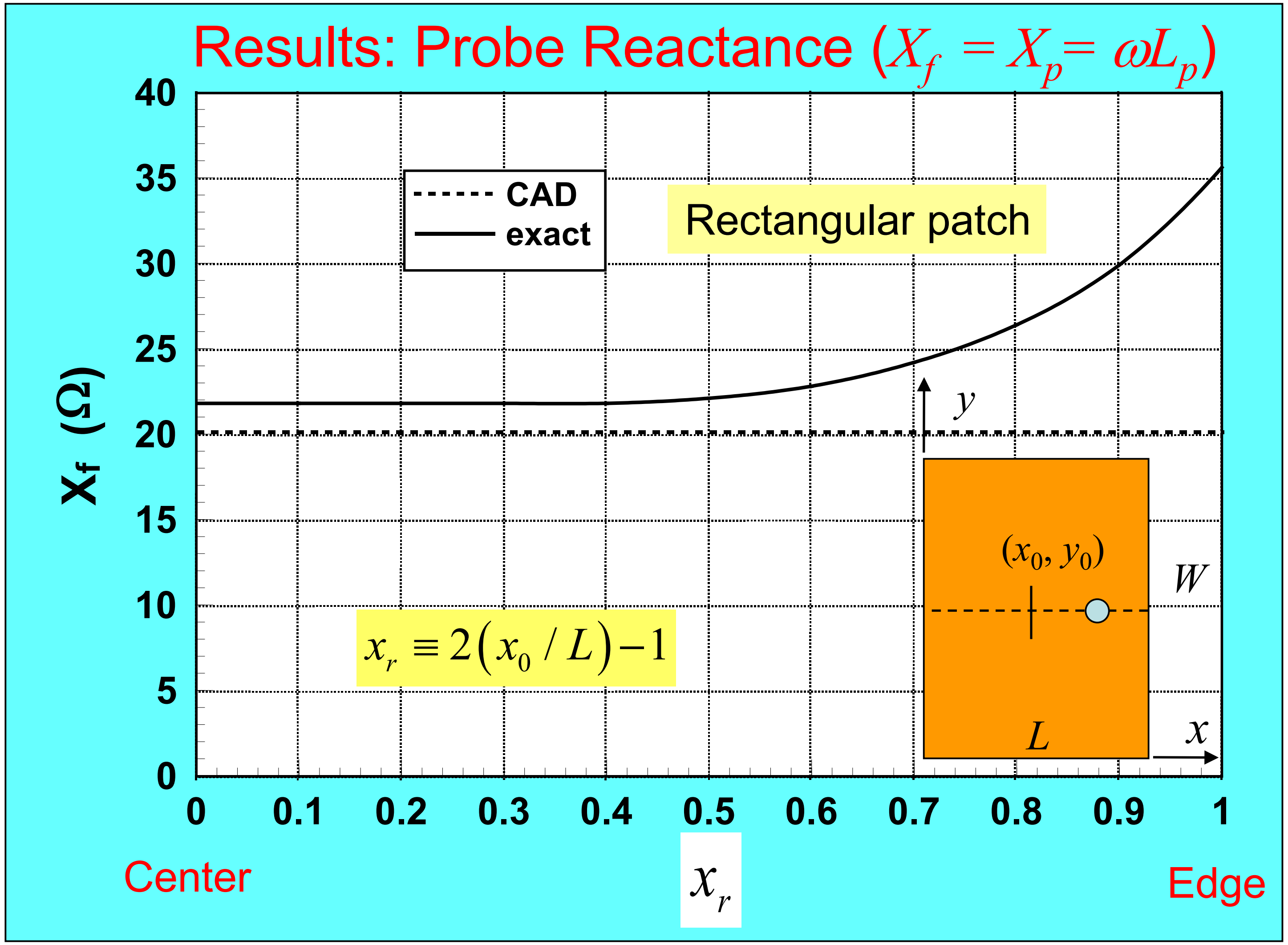
$$a = 0.0635 \;\; [\text{cm}] \qquad (50[\Omega] \;\; \text{SMA Connector})$$

$$f = 2.0 \qquad [\text{GHz}] \quad (\lambda_0 = 14.9896\;[\text{cm}])$$

$$X_p = 12.3 \;\; [\Omega]$$

$$L_p = 9.79 \times 10^{-10} \; [\text{H}] = 0.979 \;\; [\text{nH}]$$

Probe Inductance (cont.)



$\varepsilon_r = 2.2$

$W / L = 1.5$

$h = 0.0254\lambda_0$

$a = 0.5\,\mathrm{mm}$

CAD:

$$X_p = \frac{\eta_0}{2\pi}\mu_r(k_0 h)\left[-\gamma + \ln\left(\frac{2}{\sqrt{\mu_r\varepsilon_r}(k_0 a)}\right)\right]$$

Exact:

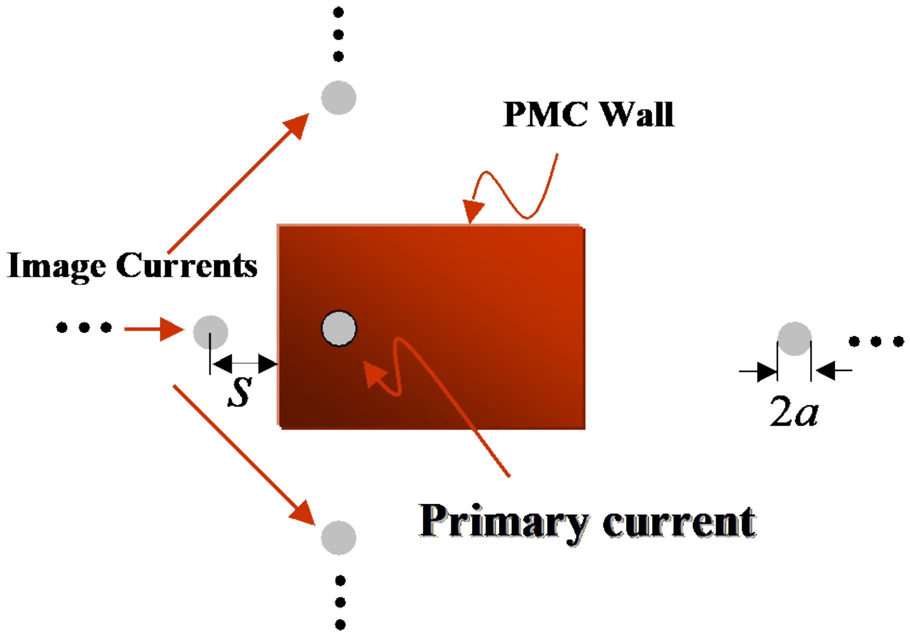
The cavity model for X_p with all infinite modes (excluding the $(1,0)$ term).

The normalized feed location ratio x_r is zero at the center of the patch ($x = L/2$), and is 1.0 at the patch edge ($x = L$).

Image Correction to Probe Inductance

Image Theory

Image theory can be used to improve the simple parallel-plate waveguide model when the probe gets close to the patch edge.



The probe images are reflections of the original probe current about the four PMC walls.

S = distance from probe to left PMC wall.

Using image theory, we have an infinite set of "image probes."

We will just worry about the image that is closest to the original probe current.

Image Correction to Probe Inductance (cont.)

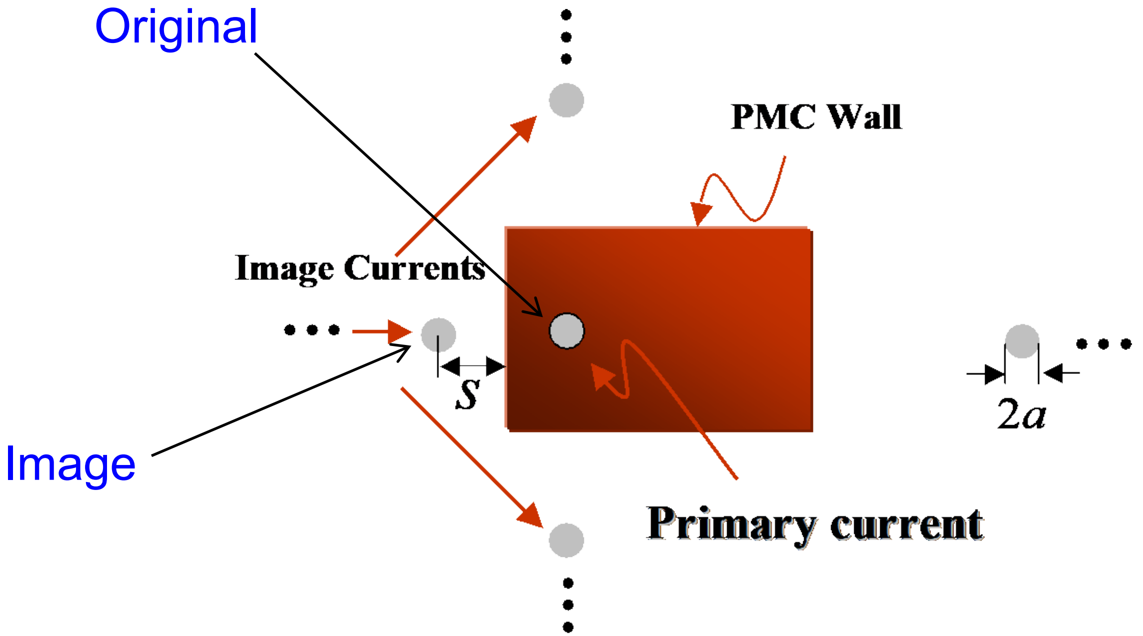
A simple approximate formula is obtained by using two terms: the original probe current in a parallel-plate waveguide and one image. This should be an improvement when the probe is close to an edge.

$$X_p^{\text{two}} = X_p + X_p^{\text{image}}$$

$$X_p^{\text{two}} = -\tfrac{1}{4}\eta\, kh\, J_0(ka)\, Y_0(ka) - \tfrac{1}{4}\eta\, kh\, J_0(ka)\, Y_0(2ks)$$

$$k = k_0\sqrt{\mu_r \varepsilon_r}$$

$$\eta = \eta_0\left(\sqrt{\mu_r / \varepsilon_r}\right)$$



Recall:

$$E_z^+ = -\eta k\left(\frac{I_0}{4}\right) J_0(ka) H_0^{(2)}(k\rho)$$

Image Correction to Probe Inductance (cont.)

As shown on the next plot, an improved probe reactance is obtained by using the following modified CAD formula:

$$X_p^{\text{imp}} = \max\left(X_p^{\text{probe}}, X_p^{\text{two}}\right)$$

"modified CAD formula"

(the improved formula)

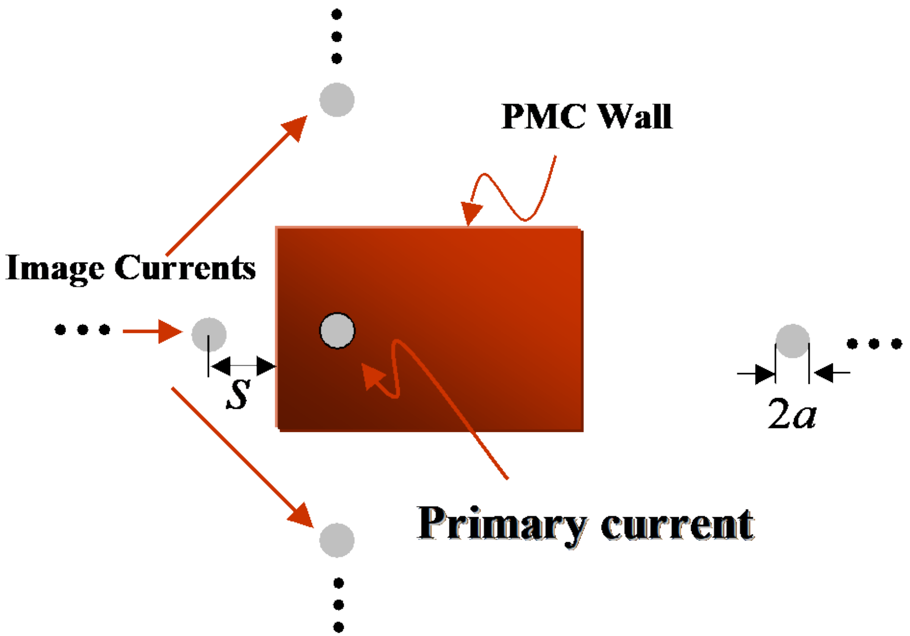
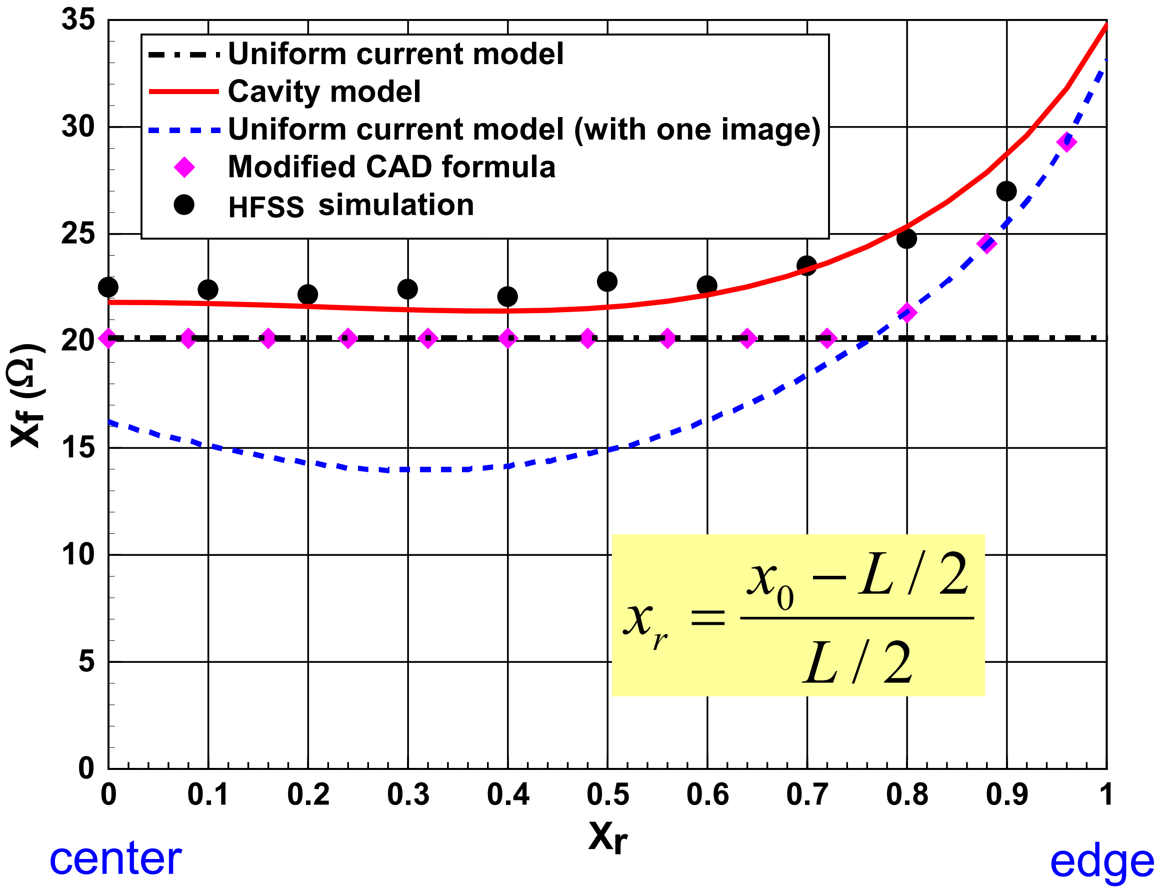


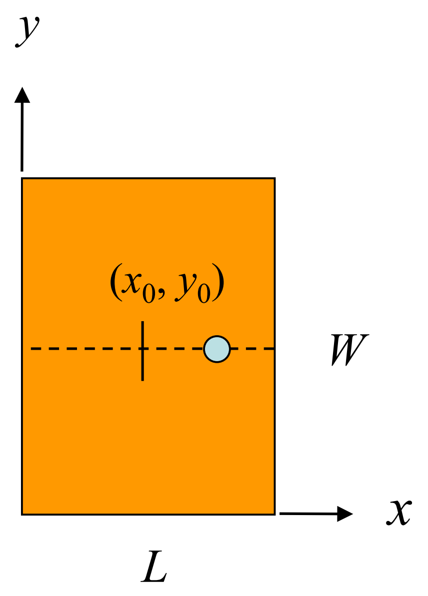
Image Correction to Probe Inductance (cont.)

Results show that the simple improved formula ("modified CAD formula") works fairly well when the probe gets close to an edge.

$$X_p^{\text{imp}} = \max\left(X_p^{\text{probe}}, X_p^{\text{two}}\right)$$



$$x_r = \frac{x_0 - L/2}{L/2}$$



$\varepsilon_r = 2.2$

$W / L = 1.5$

$h = 0.020\lambda_0$

$a = 0.5\,\text{mm}$

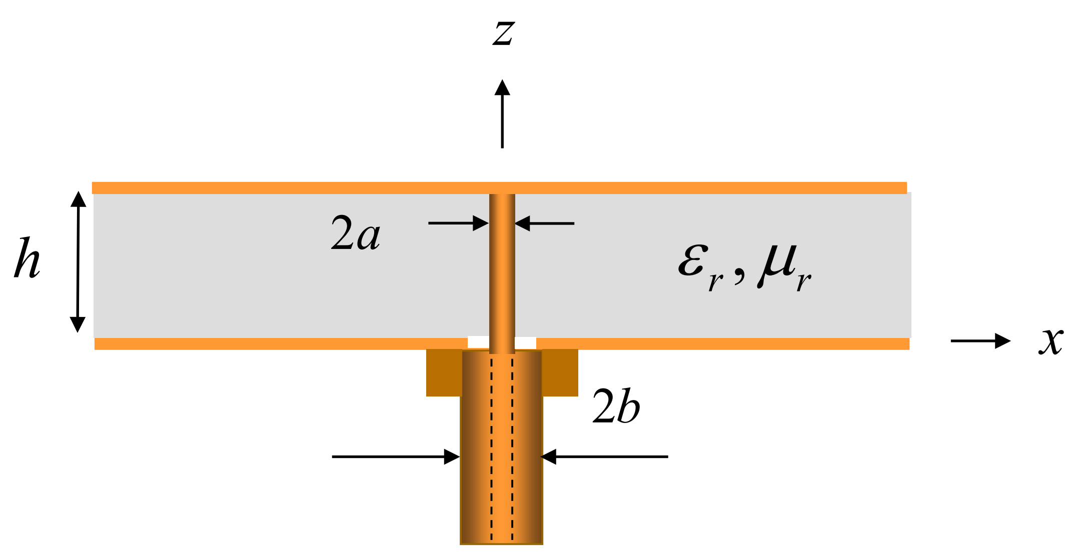
$f = 5.0\,\text{GHz}$

$Z_0 = 50\,\Omega$

Probe Models for Thicker Substrates

This section discusses improved models of the probe inductance of a coaxially-fed patch (accurate for thicker substrates).

- A parallel-plate waveguide model is initially assumed.



Probe Models for Thicker Substrates (cont.)

The following models are investigated:

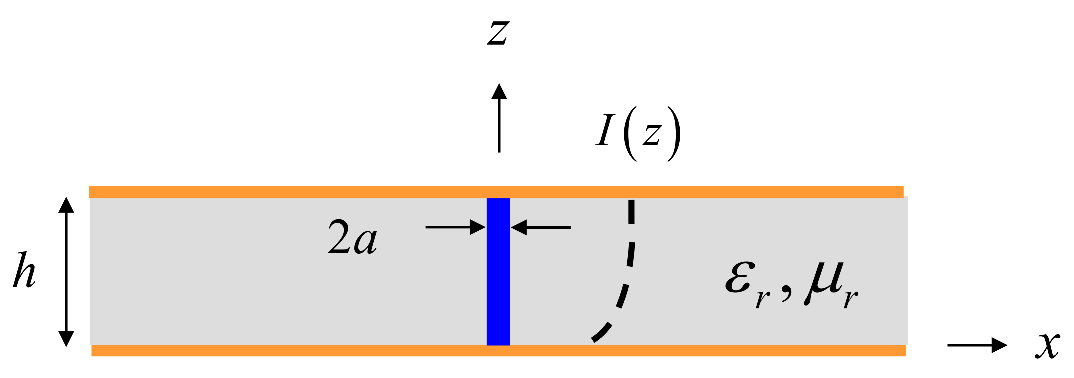
- Cosine-current model
- Frill model

Derivations are given in the Appendix. Even more details may be found in the reference below.

Reference:

H. Xu, D. R. Jackson, and J. T. Williams, "Comparison of models for the probe Inductance for a parallel plate waveguide and a microstrip patch," *IEEE Trans. Antennas and Propagation*, vol. 53, pp. 3229-3235, Oct. 2005.

Cosine Current Model



$$k = k_0\sqrt{\mu_r \varepsilon_r}$$

$$\eta = \eta_0\left(\sqrt{\mu_r / \varepsilon_r}\right)$$

We assume a tube of surface current (as before) but with a z variation.

$$I(z) = \cos\left[k(z-h)\right]$$

Note: The derivative of the current is zero at the top conductor (PEC).

$$Z_{\text{in}} = \frac{2P_c}{\left|I(0)\right|^2}$$

P_c = complex power radiated by probe current

$$P_c = -\frac{1}{2}\int_{S_p} E_z J_{sz}^{i*}\, dS$$

Cosine Current Model (cont.)

Final result:

$$Z_{\text{in}} = \frac{1}{8}(k_0 h)\eta_0 \left(\frac{1}{\varepsilon_r}\right)\sec^2\left(k_0 h\sqrt{\varepsilon_r}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 \bar{k}_{\rho m}^{-2}(1+\delta_{m0})\,H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

$$X_p = \operatorname{Im} Z_{\text{in}}$$

where

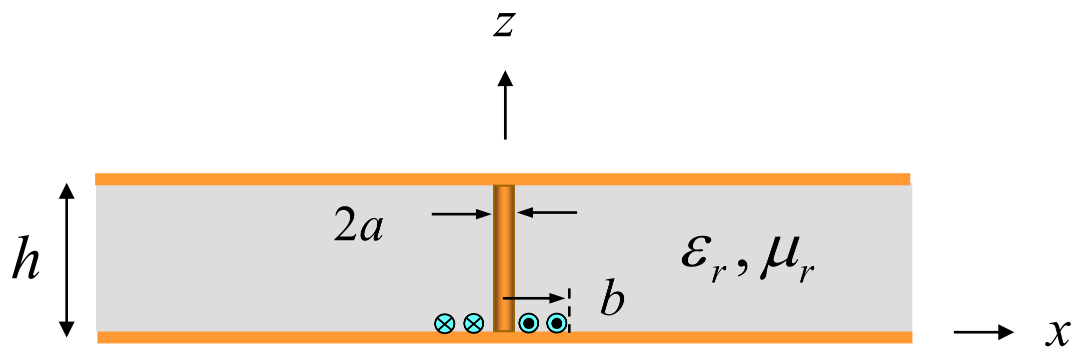
$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2-(m\pi)^2}\right]\sin(kh)$$

$$k_{\rho m} = \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

$$\bar{k}_{\rho m} = k_{\rho m} / k_0 \qquad \delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Note:
The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

Frill Model



A magnetic frill of radius b is assumed on the mouth of the coax.

$$\underline{M}_s = -\underline{\hat{z}} \times \underline{E} = -\underline{\hat{z}} \times \left(\underline{\hat{\rho}}\, E_\rho \right) \quad \Rightarrow \quad M_{s\phi} = -E_\rho$$

Choose:

$$E_\rho = \frac{1}{\rho} \left[\frac{1}{\ln\left(b / a \right)} \right]$$

$$Z_{\text{in}} = \frac{1}{I\left(0\right)}$$

(TEM mode of coax, assuming 1 V)

Frill Model (cont.)

Final result:

$$Y_{\text{in}} = 1/Z_{\text{in}} = j\left(\frac{1}{\eta_0}\right)\left(\frac{1}{k_0 h}\right)\left(\frac{1}{\ln\left(b/a\right)}\right)4\pi\varepsilon_r \sum_{m=0}^{\infty} \frac{H_0^{(2)}(k_{\rho m}b) - H_0^{(2)}(k_{\rho m}a)}{(\bar{k}_{\rho m}^2)(1+\delta_{m0})\,H_0^{(2)}(k_{\rho m}a)}$$

$$X_p = \operatorname{Im} Z_{\text{in}}$$

where

$$k_{\rho m} = \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

$$\bar{k}_{\rho m} = k_{\rho m} / k_0$$

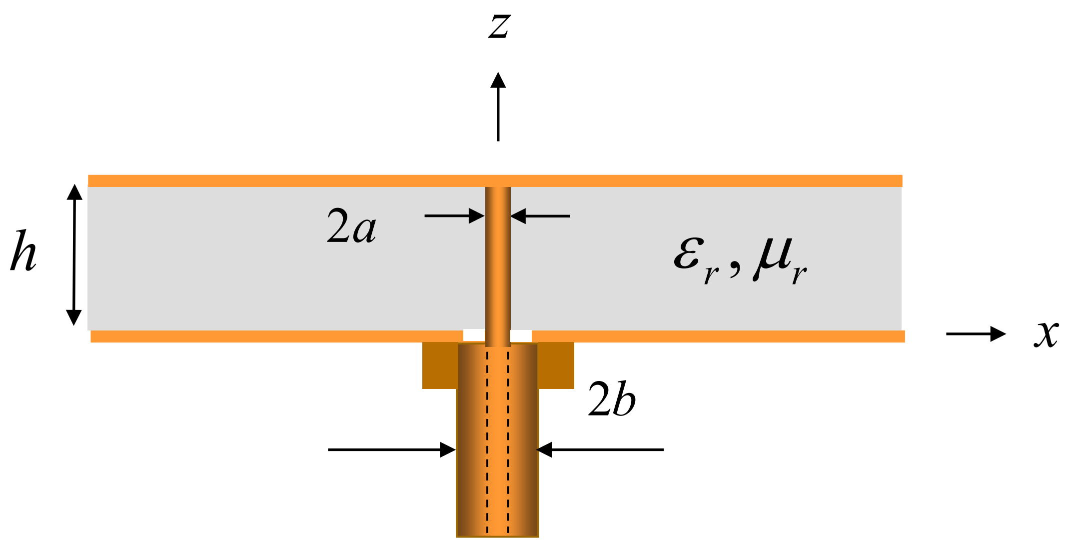
$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

Note:
The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

Comparison of Models

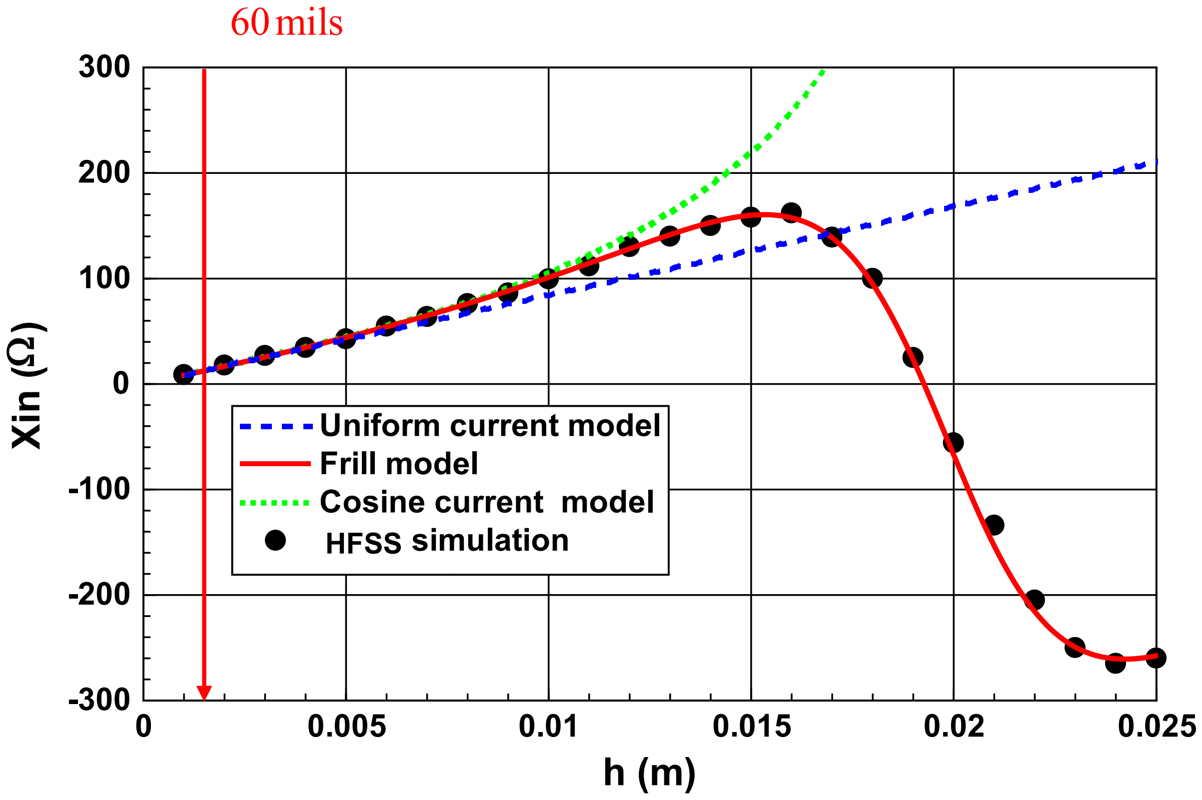
The probe models are compared as the substrate thickness increases.

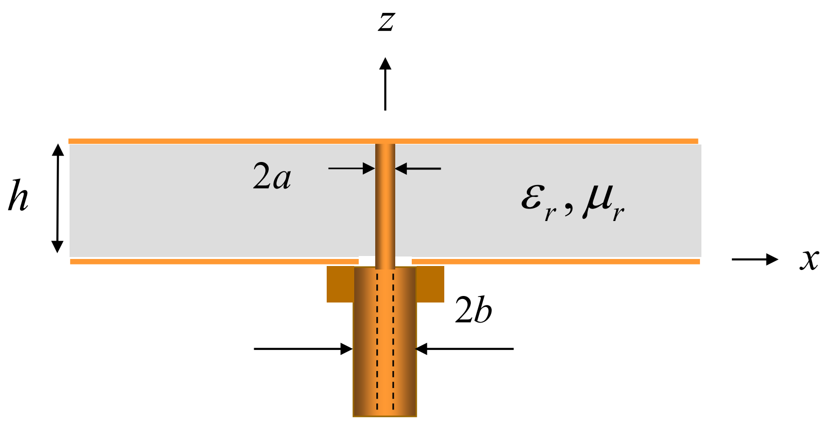
The probe is in an infinite parallel-plate waveguide.



Comparison of Models (cont.)

Models are compared for increasing substrate thickness.





$Z_0 = 50\,\Omega$

$\varepsilon_r = 2.2$

$a = 0.635\,\text{mm}$

$b = 2.19\ \text{mm}$

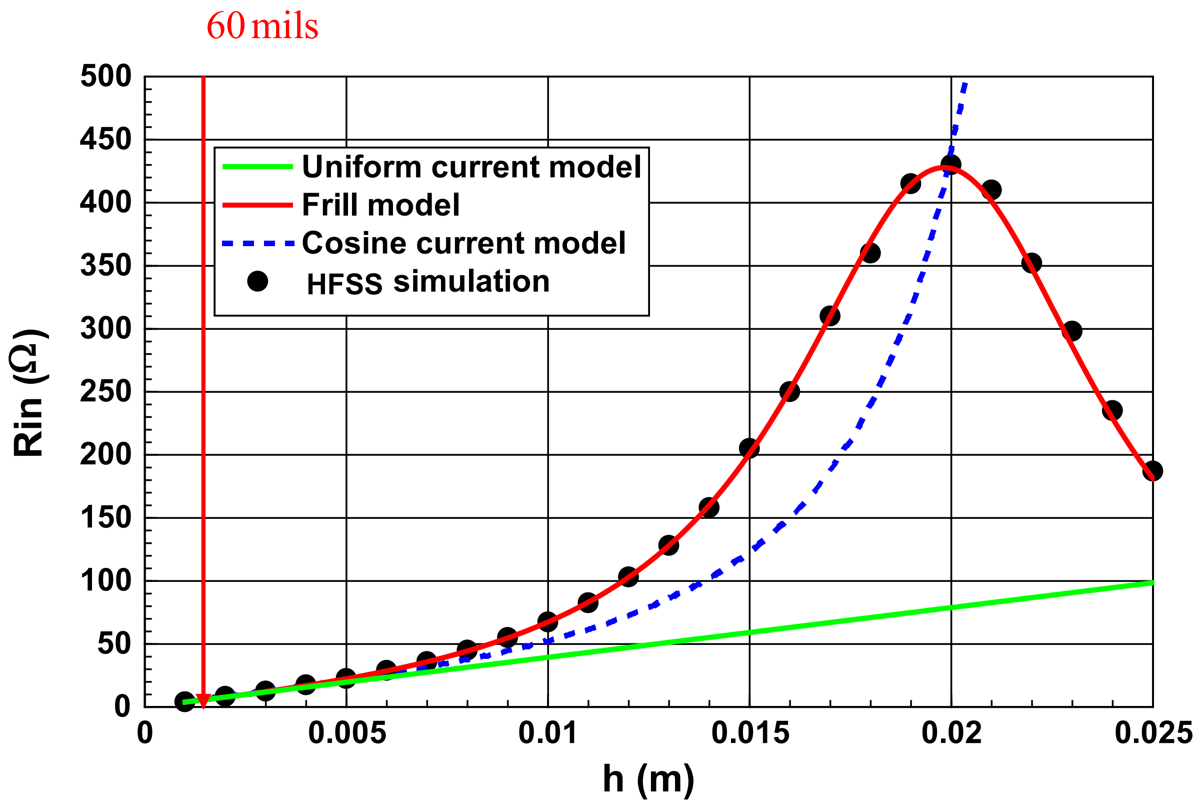
$f = 2.0\ \text{GHz}$

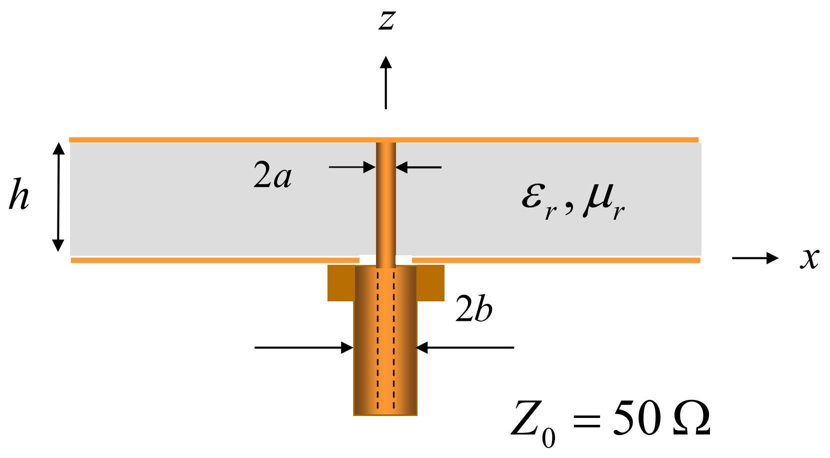
Note: $\lambda_d / 4 = 0.0253\ [\text{m}]$

Note: 60 mils = 1.524 mm

Comparison of Models (cont.)

Note: The real part is ignored when calculating the probe reactance!





$\varepsilon_r = 2.2$

$a = 0.635$ mm

$b = 2.19$ mm

$f = 2.0$ GHz

Note: $\lambda_d / 4 = 0.0253$ [m]

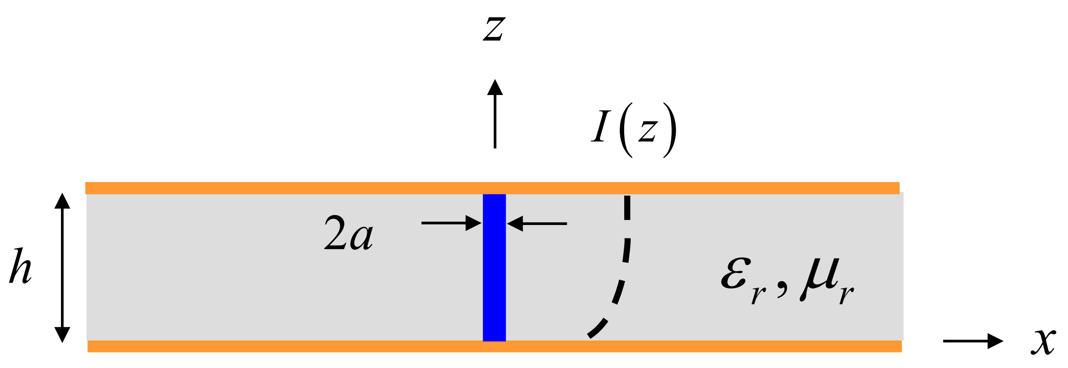
Note: 60 mils = 1.524 mm

Appendix

Next, we investigate each of the improved probe models in more detail:

- Cosine-current model
- Frill model

Cosine Current Model



$k = k_0\sqrt{\mu_r \varepsilon_r}$

$\eta = \eta_0\left(\sqrt{\mu_r / \varepsilon_r}\right)$

We assume a tube of current (as before) but with a z variation.

$$I(z) = \cos\left[k(z-h)\right]$$

Note: The derivative of the current is zero at the top conductor (PEC).

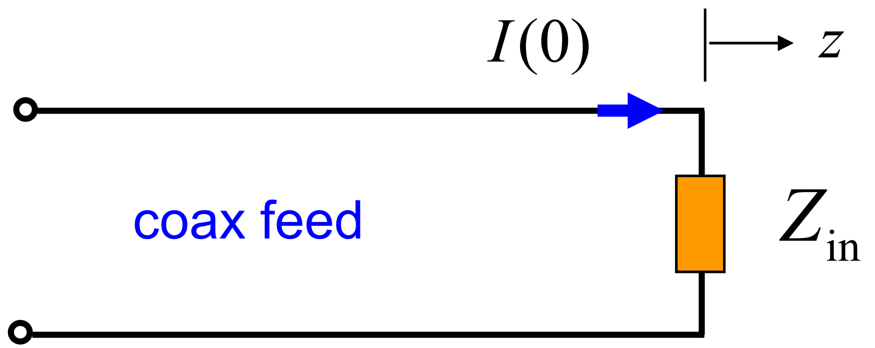
$$Z_{\text{in}} = \frac{2P_c}{\left|I(0)\right|^2}$$

P_c = complex power radiated by probe current

$$P_c = -\frac{1}{2}\int_{S_p} E_z J_{sz}^{i*}\,dS$$

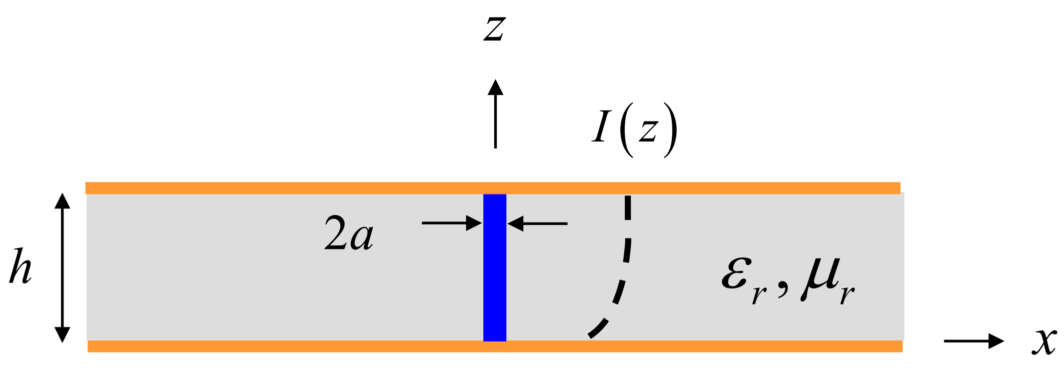
Cosine Current Model (cont.)

Circuit Model:



$$P_c = \frac{1}{2} Z_{\text{in}} \left| I(0) \right|^2 \quad \Rightarrow \quad Z_{\text{in}} = \frac{2P_c}{\left| I(0) \right|^2}$$

Cosine Current Model (cont.)



$$Z_{\text{in}} = \frac{2P_c}{|I(0)|^2}$$

Represent the probe current as:

$$I(z) = \sum_{m=0}^{\infty} I_m \cos\left(\frac{m\pi z}{h}\right)$$

This will allow us to find the fields (and hence the power radiated by the probe current) using a Fourier series approach.

Cosine Current Model (cont.)

Using Fourier-series theory to find I_m:

$$\int_0^h I(z)\cos\left(\frac{m'\pi z}{h}\right)dz = \sum_{m=0}^{\infty} I_m \int_0^h \cos\left(\frac{m\pi z}{h}\right)\cos\left(\frac{m'\pi z}{h}\right)dz$$

The integral is zero unless $m = m'$.

Hence

$$\int_0^h I(z)\cos\left(\frac{m\pi z}{h}\right)dz = I_m\left[\frac{h}{2}\left(1+\delta_{m0}\right)\right]$$

$$\Rightarrow \quad I_m = \frac{2}{h\left(1+\delta_{m0}\right)}\int_0^h I(z)\cos\left(\frac{m\pi z}{h}\right)dz$$

Cosine Current Model (cont.)

We then have:

$$I_m = \frac{2}{h\left(1+\delta_{mo}\right)} \int_0^h \cos k(z-h) \cos\left(\frac{m\pi z}{h}\right) dz$$

Result:

$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2-(m\pi)^2}\right]\sin(kh)$$

(derivation omitted)

Cosine Current Model (cont.)

Note: We have both E_z and E_ρ.

To see this:

$$\nabla \cdot \underline{E} = 0 \quad \text{(Time-Harmonic Fields)}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = 0$$

so

$$E_\rho \neq 0$$

Cosine Current Model (cont.)

For E_z, we represent the field as follows:

$\rho < a$

$$E_z^- = \sum_{m=0}^{\infty} A_m^- \cos\left(\frac{m\pi z}{h}\right) J_0\left(k_{\rho m}\rho\right)$$

$\rho > a$

$$E_z^+ = \sum_{m=0}^{\infty} A_m^+ \cos\left(\frac{m\pi z}{h}\right) H_0^{(2)}\left(k_{\rho m}\rho\right)$$

where

$$k_{\rho m} = \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

$$= \left(k^2 - k_{zm}^2\right)^{1/2} \qquad k_{zm} \equiv \frac{m\pi}{h}$$

Cosine Current Model (cont.)

At $\rho = a$

$$E_z^+ = E_z^- \qquad \text{(BC \#1)}$$

so

$$A_m^+ H_0^{(2)}\left(k_{\rho m} a\right) = A_m^- J_0\left(k_{\rho m} a\right)$$

Note:
Orthogonality tells us that we can equate each corresponding term (term # m) in the two series on either side of the equation.

Cosine Current Model (cont.)

Also, we have

$$H_{\phi 2} - H_{\phi 1} = J_{sz} \qquad \text{(BC \#2)}$$

where

$$H_{\phi} = \frac{-1}{j\omega\mu}\left(\frac{\partial E_{\rho}}{\partial z} - \frac{\partial E_{z}}{\partial \rho}\right) \qquad \left(E_{\rho} \neq 0\right)$$

To solve for E_{ρ}, use

$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E}$$

Cosine Current Model (cont.)

so $$j\omega\varepsilon E_\rho = \frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}$$

$$E_\rho = -\frac{1}{j\omega\varepsilon}\frac{\partial H_\phi}{\partial z}$$

Hence, we have $$H_\phi = -\frac{1}{j\omega\mu}\left[-\frac{1}{j\omega\varepsilon}\frac{\partial^2 H_\phi}{\partial z^2} - \frac{\partial E_z}{\partial \rho}\right]$$

For the m^{th} Fourier term:

$$H_\phi^{(m)} = -\frac{1}{j\omega\mu}\left[-\frac{1}{j\omega\varepsilon}\left(-k_{zm}^2\right)H_\phi^{(m)} - \frac{\partial E_z^{(m)}}{\partial \rho}\right] \qquad \left(k_{zm} \equiv \frac{m\pi}{h}\right)$$

Cosine Current Model (cont.)

so that

$$k^2 H_\phi^{(m)} - k_{zm}^2 H_\phi^{(m)} = -j\omega\varepsilon \frac{\partial E_z^{(m)}}{\partial \rho}$$

where

$$k^2 - k_{zm}^2 = k_{\rho m}^2$$

Hence

$$H_\phi^{(m)} = -\frac{j\omega\varepsilon}{k_{\rho m}^2} \frac{\partial E_z^{(m)}}{\partial \rho}$$

Cosine Current Model (cont.)

(BC 2) $$H_{\phi 2} - H_{\phi 1} = J_{sz} = \frac{I}{2\pi a}$$

For the m^{th} Fourier term:

$$H_{\phi 2}^{(m)} - H_{\phi 1}^{(m)} = J_{sz}^{(m)} = \frac{I_m}{2\pi a}$$

where

$$H_{\phi}^{(m)} = -\frac{j\omega\varepsilon}{k_{\rho m}^2}\frac{\partial E_z^{(m)}}{\partial \rho}$$

Cosine Current Model (cont.)

Hence

$$\left(\frac{-j\omega\varepsilon}{k_{\rho m}^2}\right)(k_{\rho m})\left[A_m^+ H_0^{(2)\prime}\left(k_{\rho m}a\right) - A_m^- J_0{}'\left(k_{\rho m}a\right)\right] = \frac{I_m}{2\pi a}$$

Using $A_m^+ H_0^{(2)}\left(k_{\rho m}a\right) = A_m^- J_0\left(k_{\rho m}a\right)$ (BC #1)

we then have:

$$A_m^+ H_0^{(2)\prime}(k_{\rho m}a) - A_m^+\left(\frac{H_0^{(2)}(k_{\rho m}a)}{J_0(k_{\rho m}a)}\right) J_0{}'(k_{pm}a) = \left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)$$

Cosine Current Model (cont.)

or

$$A_m^+\left[J_0(k_{\rho m}a)H_0^{(2)\prime}(k_{\rho m}a)-H_0^{(2)}(k_{\rho m}a)J_0{}'(k_{\rho m}a)\right]=\left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)J_0(k_{\rho m}a)$$

or

$$A_m^+\left[-j\left(\frac{2}{\pi k_{\rho m}a}\right)\right]=\left(\frac{I_m}{2\pi a}\right)\left(\frac{k_{\rho m}}{-j\omega\varepsilon}\right)J_0(k_{\rho m}a)$$

(using the Wronskian identity)

$$J_n(x)H_n^{(2)\prime}(x)-J_n{}'(x)H_n^{(2)}(x)=-j\left(\frac{2}{\pi x}\right)$$

Hence

$$A_m^+=I_m\left[-\frac{1}{4}\left(\frac{k_{\rho m}^2}{\omega\varepsilon}\right)J_0\left(k_{\rho m}a\right)\right]$$

Cosine Current Model (cont.)

We now find the complex power radiated by the probe:

$$P_c = \frac{-1}{2} \oint_s \underline{E} \cdot \underline{J}_s^{i*} \, dS$$

$$= \frac{-1}{2} \int_0^{2\pi} \int_0^h E_z(a) \, J_{sz}^{i*} \, a \, dz \, d\phi$$

$$= -\pi a \int_0^h E_z(a) \, J_{sz}^{i*} \, dz$$

$$= -\frac{\pi a}{2\pi a} \int_0^h E_z(a) \, I^*(z) \, dz$$

$$= -\frac{1}{2} \int_0^h \left(\sum_{m=0}^{\infty} A_m^+ H_0^{(2)}(k_{\rho m} a) \cos\left(\frac{m\pi z}{h}\right) \right) \cdot \left(\sum_{m'=0}^{\infty} I_{m'}^* \cos\left(\frac{m'\pi z}{h}\right) \right) dz$$

Cosine Current Model (cont.)

Integrating in z and using orthogonality, we have:

A_m^{+} coefficient

$$P_c = -\frac{1}{2}\sum_{m=0}^{\infty} A_m^{+} I_m^{*} H_0^{(2)}(k_{\rho m}a)\left(\frac{h}{2}\right)\left(1+\delta_{m0}\right)$$

$$= -\left(\frac{h}{4}\right)\sum_{m=0}^{\infty}\left(1+\delta_{m0}\right)H_0^{(2)}(k_{\rho m}a)\left[I_m\left(-\frac{1}{4}\right)\left(\frac{k_{\rho m}^2}{\omega\varepsilon}\right)J_0(k_{\rho m}a)\right]I_m^{*}$$

Hence, we have:

$$P_c = +\frac{h}{16}\left(\frac{1}{\omega\varepsilon}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 k_{\rho m}^2\left(1+\delta_{m0}\right)H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

Cosine Current Model (cont.)

$$Z_{\text{in}} = \frac{2P_c}{\cos^2(kh)}$$

Therefore, we have:

$$Z_{\text{in}} = \frac{h}{8}\left(\frac{1}{\omega\varepsilon}\right)\sec^2(kh)\sum_{m=0}^{\infty}\left|I_m\right|^2 k_{\rho m}^2\left(1+\delta_{m0}\right)H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

Note:
The wavenumber $k_{\rho m}$ is chosen to be a positive real number or a negative imaginary number.

Define:

$$\bar{k}_{\rho m} \equiv \frac{k_{\rho m}}{k_0} = \left(\varepsilon_r\mu_r - \left(\frac{m\pi}{k_0 h}\right)^2\right)^{1/2}$$

Recall: $k_{\rho m} = \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2} = \left(k^2 - k_{zm}^2\right)^{1/2}$

Cosine Current Model (cont.)

Also, use $$\frac{k_0^2}{\omega\varepsilon} = k_0 \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\varepsilon} = k_0 \frac{\eta_0}{\varepsilon_r}$$

We then have:

$$Z_{\text{in}} = \frac{1}{8}(k_0 h)\eta_0\left(\frac{1}{\varepsilon_r}\right)\sec^2\left(k_0 h\sqrt{\varepsilon_r}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 \bar{k}_{\rho m}^{-2}(1+\delta_{m0})\,H_0^{(2)}(k_{\rho m}a)\,J_0(k_{\rho m}a)$$

The probe reactance is: $$X_p = \text{Im}(Z_{\text{in}})$$

Cosine Current Model (cont.)

Thin substrate approximation

$$Z_{\text{in}} = \frac{1}{8}(k_0 h)\eta_0 \left(\frac{1}{\varepsilon_r}\right)\sec^2\left(k_0 h\sqrt{\varepsilon_r}\right)\sum_{m=0}^{\infty}\left|I_m\right|^2 \bar{k}_{\rho m}^{-2}(1+\delta_{m0})\, H_0^{(2)}(k_{\rho m}a)\, J_0(k_{\rho m}a)$$

$k_0 h << 1$: Keep only the $m = 0$ term: $\bar{k}_{\rho 0}^{\,2} = \varepsilon_r \mu_r$

$$\left(\bar{k}_{\rho m} = \frac{k_{\rho m}}{k_0} = \sqrt{\varepsilon_r \mu_r - \left(\frac{m\pi}{k_0 h}\right)^2}\right)$$

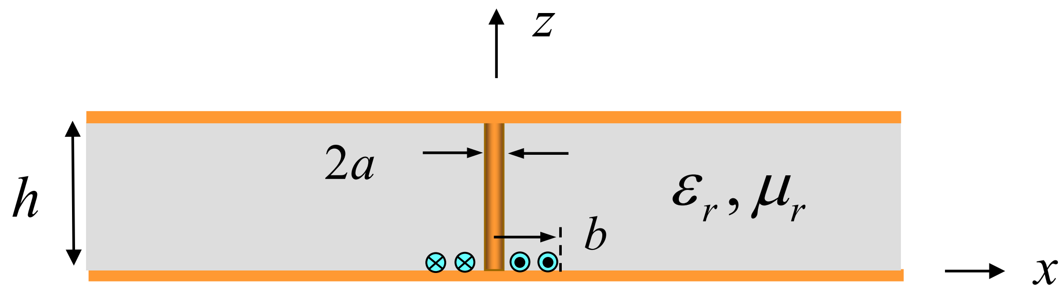
$$I_m = \left(\frac{2}{1+\delta_{mo}}\right)\left[\frac{(kh)}{(kh)^2 - (m\pi)^2}\right]\sin(kh) \approx 1$$

The result is

$$Z_{\text{in}} \approx \frac{1}{4}\eta_0 (k_0 h)\mu_r J_0(ka) H_0^{(2)}(ka)$$

(same as previous result using uniform model)

Frill Model



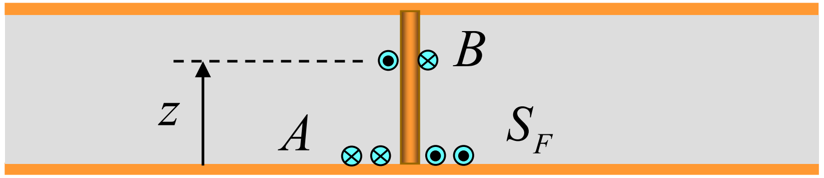
1V frill

$$M_{s\phi} = -\frac{1}{\rho}\left[\frac{1}{\ln(b/a)}\right]$$

$$Z_{\text{in}} = \frac{1}{I(0)}$$

To find the current $I(z)$, use reciprocity.

Introduce a ring of magnetic current $K = 1$ [V] in the ϕ direction at z (the testing current "B").



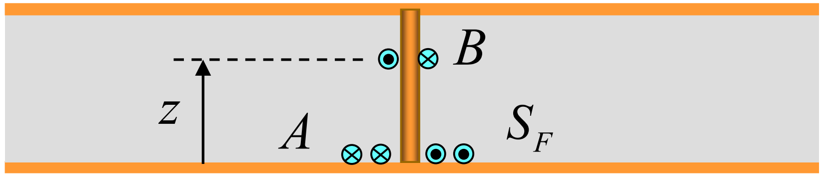
$$I(z) = \int_V \underline{H}^a \cdot \underline{M}^b \, dV = -\langle A, B \rangle = -\langle B, A \rangle$$

$$= \int_V \underline{H}^b \cdot \underline{M}^a \, dV$$

$$= \int_{S_F} \underline{H}^b \cdot \underline{M}_s \, dS$$

Ampere's law: $\int_V \underline{H}^a \cdot \underline{M}^b \, dV = K \int_0^{2\pi} H_\phi^a(a,z)\, a\, d\phi = KI(z) = I(z)$

Frill Model (cont.)



$$I(z) = \int\limits_{S_F} \underline{H}^b \cdot \underline{M}_s \, dS$$

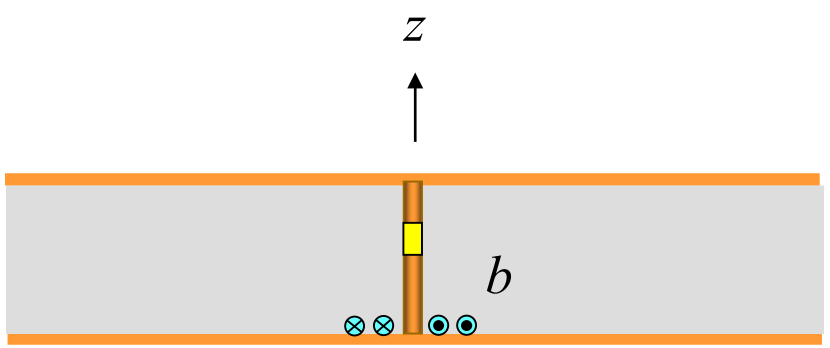
$$= \int\limits_0^{2\pi} \int\limits_a^b H_\phi^b(\rho, 0) \, M_{s\phi} \, \rho \, d\rho \, d\phi$$

$$= 2\pi \int\limits_a^b H_\phi^b(\rho, 0) \, M_{s\phi} \, \rho \, d\rho$$

$$= 2\pi \int\limits_a^b H_\phi^b(\rho, 0) \left[-\frac{1}{\rho} \frac{1}{\ln(b/a)} \right] \rho \, d\rho$$

$$= -\frac{2\pi}{\ln(b/a)} \int\limits_a^b H_\phi^b(\rho, 0) \, d\rho$$

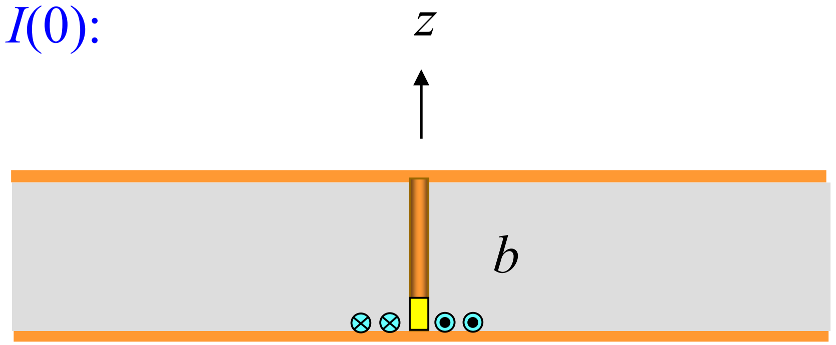
Frill Model (cont.)



The magnetic current ring B may be replaced by a 1V gap source of zero height (by the equivalence principle).

$$I(z) = -\frac{2\pi}{\ln(b/a)} \int_a^b H_\phi^{\text{gap}}(\rho, 0)\, d\rho$$

Let $z \to 0$ to get $I(0)$:

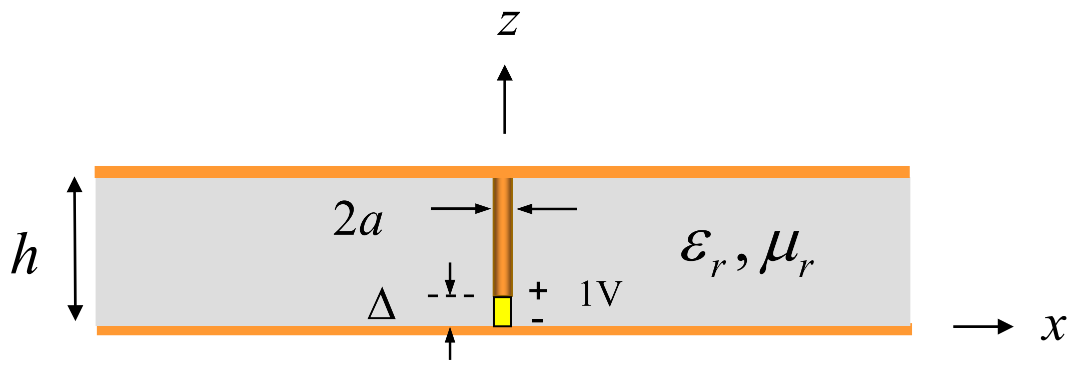


$$I(0) = -\frac{2\pi}{\ln(b/a)} \int_a^b H_\phi^{\text{gap}}(\rho, 0)\, d\rho$$

The magnetic field of the gap source at $z = 0$ is then calculated using a gap-source model (shown next).

Frill Model (cont.)

Calculation of $H_{\phi}^{\text{gap}}(\rho, 0)$



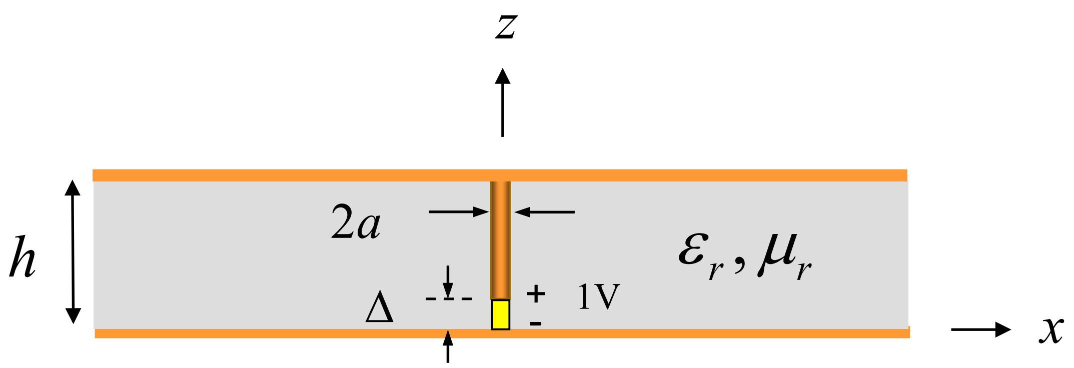
$$E_z(z,\rho) = \sum_{m=0}^{\infty} B_m H_0^{(2)}(k_{\rho m}\rho)\cos\left(\frac{m\pi z}{h}\right)$$

$$E_z(z,a) = \begin{cases} -1/\Delta, & 0 < z < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Note: Δ can be taken to approach zero at the end.

Frill Model (cont.)

Calculation of $H_{\phi}^{\text{gap}}(\rho, 0)$



For $\rho = a$:

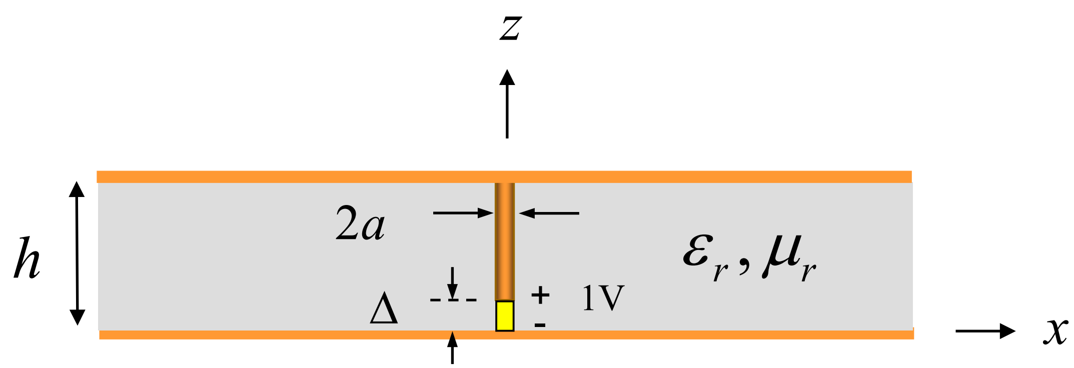
$$E_z(z,a) = \sum_{m=0}^{\infty} B_m H_0^{(2)}(k_{\rho m} a)\cos\left(\frac{m\pi z}{h}\right) = \begin{cases} -1/\Delta, & 0 < z < \Delta \\ 0, & \text{otherwise}. \end{cases}$$

From Fourier series analysis (details omitted):

$$B_m = \frac{-2}{h(1+\delta_{m0}) H_0^{(2)}(k_{\rho m} a)} \operatorname{sinc}\left(\frac{m\pi\Delta}{h}\right)$$

Frill Model (cont.)

Calculation of $H_\phi^{\text{gap}}(\rho,0)$



The magnetic field is found from E_z, with the help of the magnetic vector potential A_z (the field is TM$_z$):

$$H_\phi^{\text{gap}} = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} \quad (z=0)$$

Use: $A_z(z,\rho) = \sum_{m=0}^{\infty} A_m H_0^{(2)}(k_{\rho m}\rho)\cos\left(\frac{m\pi z}{h}\right)$

We then have: $E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z$

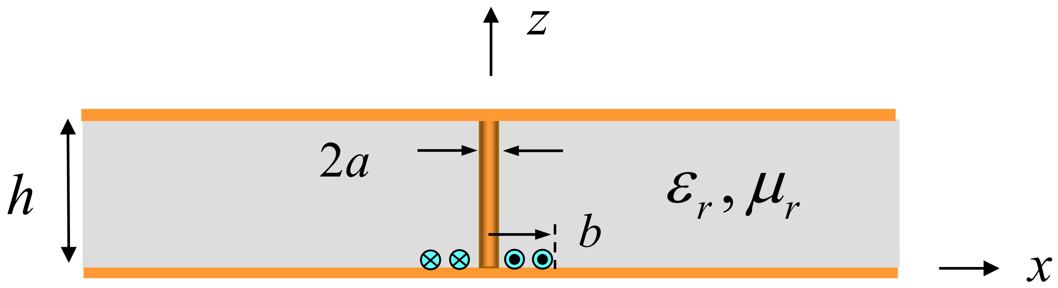
Setting $\rho = a$ allows us to solve for the coefficients A_m from the B_m coefficients.

$$B_m H_0^{(2)}(k_{\rho m} a) = \frac{1}{j\omega\mu\varepsilon}\left(-\left(\frac{m\pi}{h}\right)^2 + k^2\right) A_m H_0^{(2)}(k_{\rho m} a)$$

Frill Model (cont.)

$$Z_{\text{in}} = \frac{1}{I(0)}$$

$$I(0) = -\frac{2\pi}{\ln(b/a)} \int_a^b H_\phi^{\text{gap}}(\rho, 0)\, d\rho$$



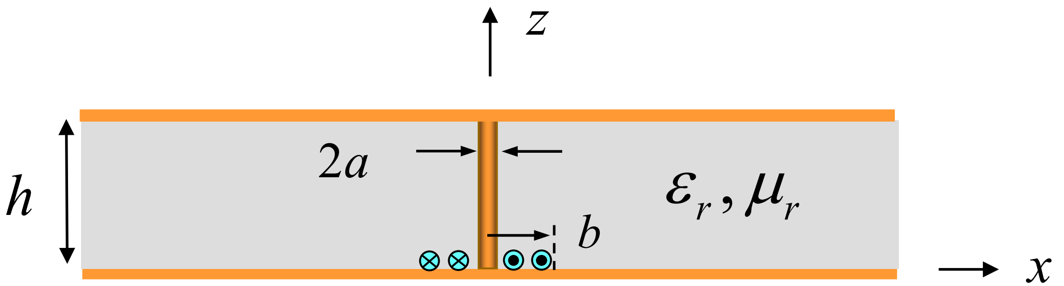
This yields:

$$Y_{\text{in}} = j\left(\frac{1}{h\eta}\right)\left(\frac{1}{\ln(b/a)}\right) 4\pi k \sum_{m=0}^{\infty} \frac{H_0^{(2)}(k_{\rho m} b) - H_0^{(2)}(k_{\rho m} a)}{(k_{\rho m}^2)(1+\delta_{m0})\, H_0^{(2)}(k_{\rho m} a)}$$

$$k = k_0 \sqrt{\varepsilon_r \mu_r} \qquad \eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Frill Model (cont.)

Final result:

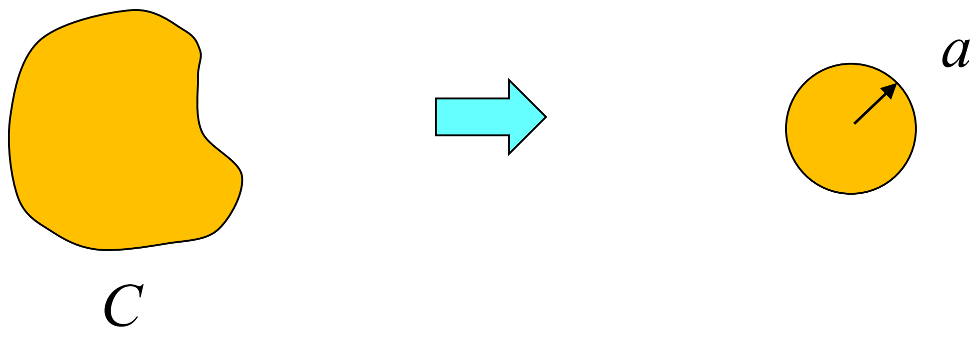


$$Y_{\text{in}} = 1/Z_{\text{in}} = j\left(\frac{1}{\eta_0}\right)\left(\frac{1}{k_0 h}\right)\left(\frac{1}{\ln(b/a)}\right)4\pi\varepsilon_r \sum_{m=0}^{\infty} \frac{H_0^{(2)}(k_{\rho m} b) - H_0^{(2)}(k_{\rho m} a)}{(\bar{k}_{\rho m}^2)(1+\delta_{m0})\, H_0^{(2)}(k_{\rho m} a)}$$

Effective Radius Approximation

Here we derive the equivalent radius approximation for a flat strip.

Start by considering a probe of arbitrary cross section:



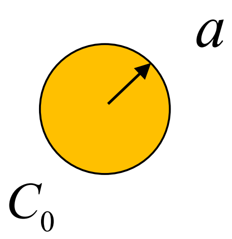
Assume : $I_0 = 1\,\text{A}$

We wish to find the effective radius a of the round probe wire that best models the probe wire with contour C.

Effective Radius Approximation (cont.)

Approach: Equate complex power being radiated by the two objects.

Round wire:



ε, μ

Assume : $I_0 = 1\,\mathrm{A}$

For the round probe wire: $$E_z = -\eta k \left(\frac{1}{4}\right) J_0(ka) H_0^{(2)}(k\rho), \quad \rho \geq a$$

$$P_c = -\frac{1}{2}\int_{C_0} E_z J_{sz}^{*} dl = -\frac{1}{2}\int_0^{2\pi} E_z J_{sz}^{*}\, a\, d\phi = -\frac{1}{2} E_z J_{sz}^{*}\, a\left(2\pi\right)$$

$$= -\frac{1}{2}\left(-\eta k\left(\frac{1}{4}\right) J_0(ka) H_0^{(2)}(ka)\right)\left(\frac{1}{2\pi a}\right)^{*} a\left(2\pi\right)$$

Effective Radius Approximation (cont.)

$$P_c = -\frac{1}{2}\left(-\eta k\left(\frac{1}{4}\right)J_0(ka)H_0^{(2)}(ka)\right)\left(\frac{1}{2\pi a}\right)^* a\left(2\pi\right)$$

Use $\eta k = \omega\mu$

$$P_c = \frac{1}{8}\omega\mu\, J_0(ka)H_0^{(2)}(ka)$$

$$H_0^{(2)}(x) \equiv J_0\left(x\right) - jY_0\left(x\right)$$

$$J_0\left(x\right) \approx 1 \quad \left(x << 1\right)$$

Assume that the radius is small compared with a wavelength:

$$P_c \approx -j\frac{1}{8}\omega\mu\, Y_0\left(ka\right)$$

Next, use

$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right], \quad \gamma = 0.5772156$$

Effective Radius Approximation (cont.)

We then have

$$P_c \approx -j\frac{1}{8}\omega\mu\left(\frac{2}{\pi}\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]\right)$$

or

$$P_c \approx -j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]$$

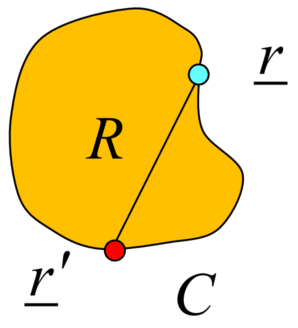
Next, we consider the arbitrary-shaped wire.

Effective Radius Approximation (cont.)

Arbitrary-shaped wire:

ε, μ

$$R = \left|\underline{r} - \underline{r}'\right|$$



Assume : $I_0 = 1\,\mathrm{A}$

$$P_c = -\frac{1}{2}\int_C J_{sz}^*(l)\,E_z(l)\,dl$$

l = distance measured along boundary

where

$$E_z(l) = \int_C J_{sz}(l')\left[-\eta k\left(\frac{1}{4}\right)H_0^{(2)}(kR)\right]dl'$$

Effective Radius Approximation (cont.)

$$P_c = -\frac{1}{2}\int_C J_{sz}^*(l)\int_C J_{sz}(l')\left[-\eta k\left(\frac{1}{4}\right)H_0^{(2)}(kR)\right]dl'\,dl$$

Denote $J_{sz}(l) = f(l)$ Assume : $I_0 = 1\,\mathrm{A}$

$$P_c = \frac{1}{8}\omega\mu\int_C\int_C f(l')\,f^*(l)\,H_0^{(2)}(kR)\,dl'\,dl$$

so

$$P_c \approx \frac{1}{8}\omega\mu\int_C\int_C f(l')\,f^*(l)\left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right)+\gamma\right]\right)dl'\,dl$$

Effective Radius Approximation (cont.)

$$P_c \approx \frac{1}{8}\omega\mu \int_C \int_C f(l') f^*(l) \left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right) + \gamma \right] \right) dl'\, dl$$

Note that

$$\int_C f(l')\, dl' = 1 \quad \text{(1A on object)} \qquad \int_C f(l)\, dl = 1 \quad \text{(1A on object)}$$

so that

$$P_c \approx \frac{1}{8}\omega\mu \int_C \int_C f(l') f^*(l) \left(-j\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right) \right] \right) dl'\, dl - j\gamma\left(\frac{1}{4\pi}\omega\mu \right)$$

Effective Radius Approximation (cont.)

Equate the two complex powers:

$$P_c \approx -j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right]$$ (round wire)

$$P_c \approx -j\frac{1}{4\pi}\omega\mu\int\limits_C\int\limits_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl - j\gamma\left(\frac{1}{4\pi}\omega\mu\right)$$ (arbitrary wire)

$$-j\frac{1}{4\pi}\omega\mu\left[\ln\left(\frac{ka}{2}\right)+\gamma\right] = -j\frac{1}{4\pi}\omega\mu\int\limits_C\int\limits_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl - j\gamma\left(\frac{1}{4\pi}\omega\mu\right)$$

or

$$-j\frac{1}{4\pi}\omega\mu\ln\left(\frac{ka}{2}\right) = -j\frac{1}{4\pi}\omega\mu\int\limits_C\int\limits_C f(l')\,f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

Effective Radius Approximation (cont.)

$$-j\frac{1}{4\pi}\omega\mu\ln\left(\frac{ka}{2}\right)=-j\frac{1}{4\pi}\omega\mu\int_C\int_C f(l')f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

or

$$\ln\left(\frac{ka}{2}\right)=\int_C\int_C f(l')f^*(l)\ln\left(\frac{kR}{2}\right)dl'\,dl$$

or

$$\ln(k)+\ln a-\ln 2=\int_C\int_C f(l')f^*(l)\left(\ln k+\ln R-\ln 2\right)dl'\,dl$$

or

$$\ln a=\int_C\int_C f(l')f^*(l)\ln R\,dl'\,dl \qquad \left(\int_C f(l')\,dl'=\int_C f(l)\,dl=1=1\right)$$

Effective Radius Approximation (cont.)

The general result (applicable to any arbitrary-shaped wire) is therefore:

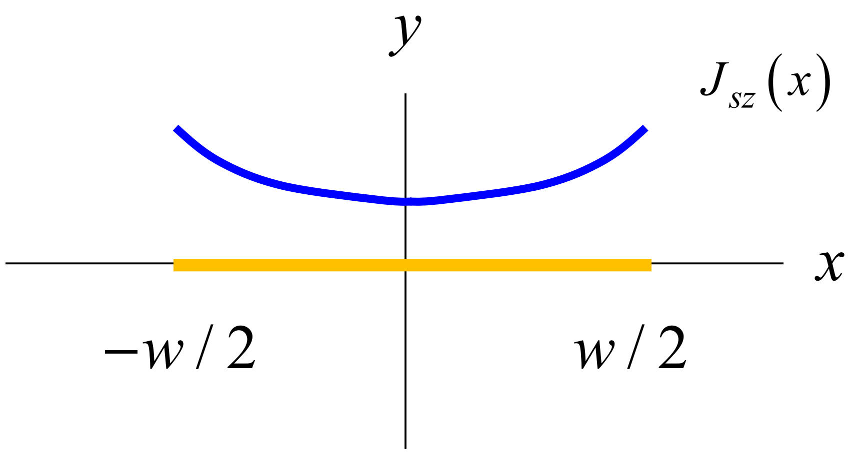
$$\ln a = \int_C \int_C f(l') f^*(l) \ln R(l, l') \, dl' \, dl$$

$$f(l) = J_{sz}(l) \quad \left(I_0 = 1\,[\mathrm{A}]\right)$$

We next evaluate this for a flat strip.

Effective Radius Approximation (cont.)

Strip model:

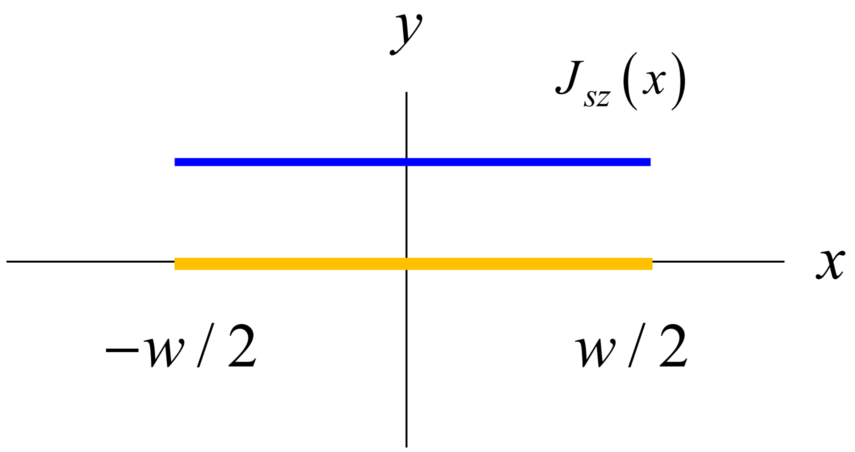


$$J_{sz}(x) = f(x)$$

$$\ln a = \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} f(x') f^*(x) \ln|x - x'| \, dx' dx$$

Effective Radius Approximation (cont.)

Uniform current model:



$$J_{sz}(x) = f(x) = 1/w$$

$$\ln a = \frac{1}{w^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \ln|x - x'| \, dx' dx$$

Effective Radius Approximation (cont.)

$$\ln a = \frac{1}{w^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \ln|x - x'| \, dx' dx$$

Use

$$s = x / w$$

$$t = x' / w$$

$$x - x' = ws - wt = w(s - t)$$

We then have

$$\ln a = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(w|s - t|\right) dt \, ds$$

Effective Radius Approximation (cont.)

$$\ln a = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(w \left| s - t \right| \right) dt\, ds$$

Therefore, we have:

$$\ln a = \ln w \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} dt\, ds + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left| s - t \right| \right) dt\, ds$$

or

$$\ln a = \ln w + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left| s - t \right| \right) dt\, ds$$

Effective Radius Approximation (cont.)

$$\ln a = \ln w + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds$$

Define:

$$A_2 \equiv \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds$$

We then have:

$$\ln a = \ln w + A_2$$

or

$$a = w e^{A_2}$$

or

$$w = e^{-A_2} a$$

Effective Radius Approximation (cont.)

We have:

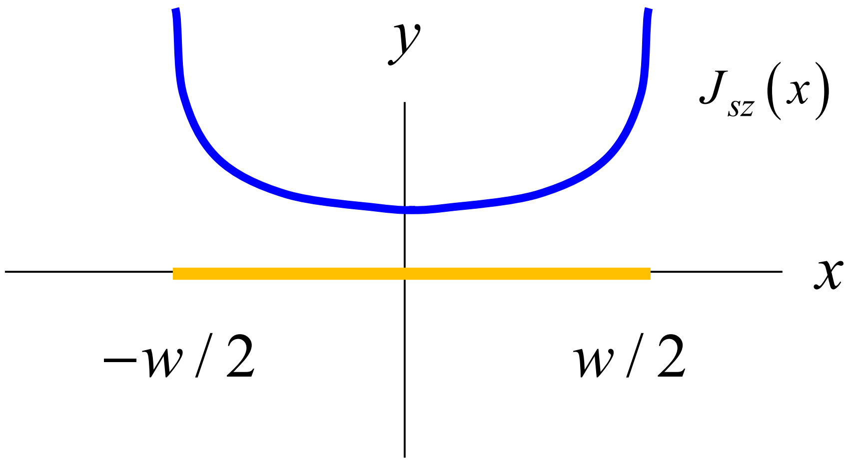
$$A_2 \equiv \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \ln\left(\left|s-t\right|\right) dt\, ds = -\frac{3}{2}$$

We then have

$$w = e^{3/2} a$$

Effective Radius Approximation (cont.)

Maxwell current model:



$$J_{sz}(x) = \frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}}$$

(This corresponds to 1A.)

Effective Radius Approximation (cont.)

$$\ln a = \frac{1}{\pi^2} \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - x'^2}} \ln|x - x'| \, dx' dx$$

Use

$$s = x / w$$

$$t = x' / w$$

$$x - x' = ws - wt = w(s - t)$$

$$\ln a = \frac{w^2}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (ws)^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (wt)^2}} \ln\left(w|s - t|\right) ds dt$$

Effective Radius Approximation (cont.)

$$\ln a = \frac{w^2}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (ws)^2}} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - (wt)^2}} \ln\left(w|s-t|\right) ds dt$$

$$\ln a = \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds dt$$

$$+ \ln w \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1/\pi}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1/\pi}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} ds dt$$

$$\left(\int_{-w/2}^{w/2} \frac{1/\pi}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}} = 1 \right)$$

This (separable) double integral equals 1.

Effective Radius Approximation (cont.)

$$\ln a = \ln w + \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds dt$$

Define

$$A_2 \equiv \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(|s-t|\right) ds dt$$

We then have

$$\ln a = \ln w + A_2$$

or

$$a = w e^{A_2}$$

or

$$w = e^{-A_2} a$$

Effective Radius Approximation (cont.)

We have:

$$A_2 \equiv \frac{1}{\pi^2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - s^2}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} \ln\left(\left|s-t\right|\right) ds\,dt = -\ln 4$$

We then have

$$w = e^{-(-\ln 4)} a$$

or

$$w = 4a$$