ECE 6345

Spring 2024

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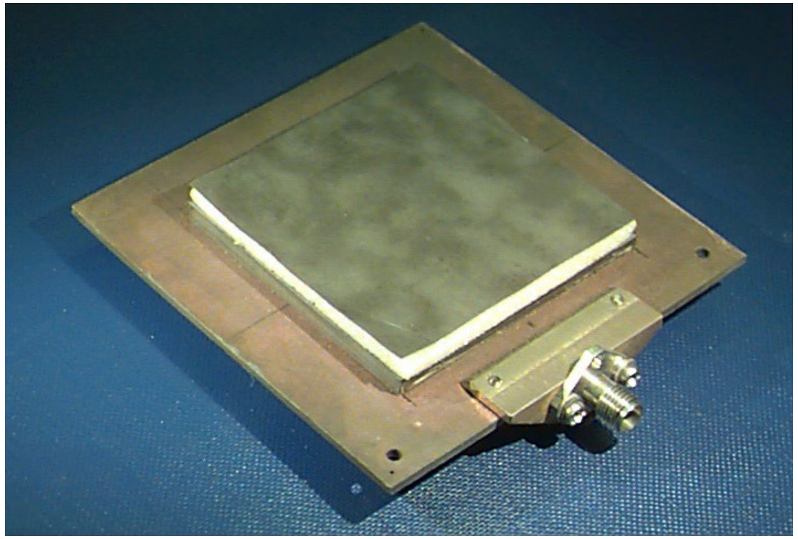
ECE Dept.



Notes 7

Overview

In this set of notes we look at two different models for calculating the radiation pattern of a microstrip antenna:

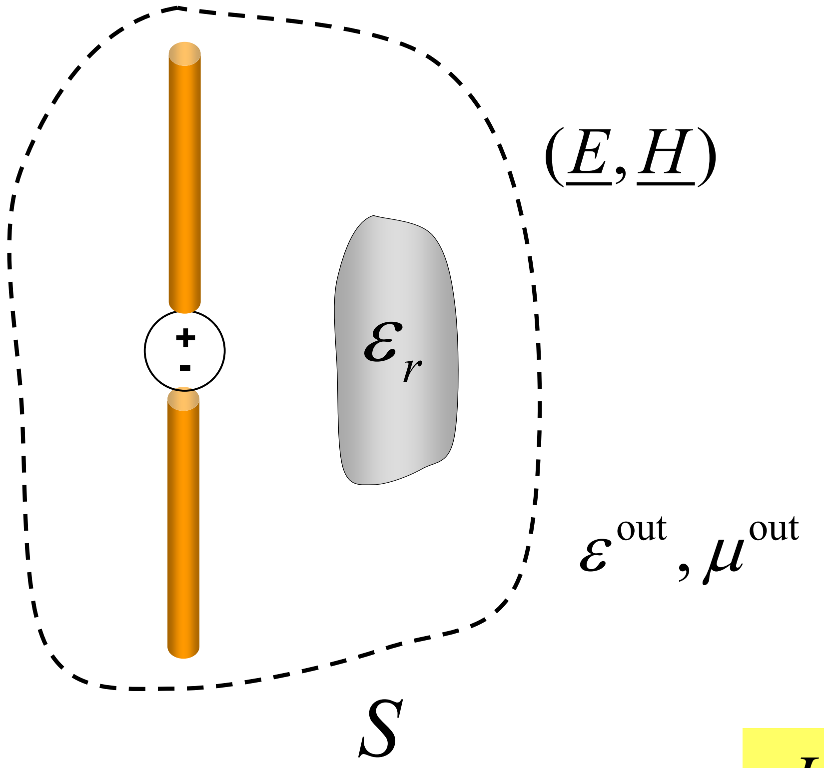
- Electric current model
- Magnetic current model

We also look at two different substrate assumptions:

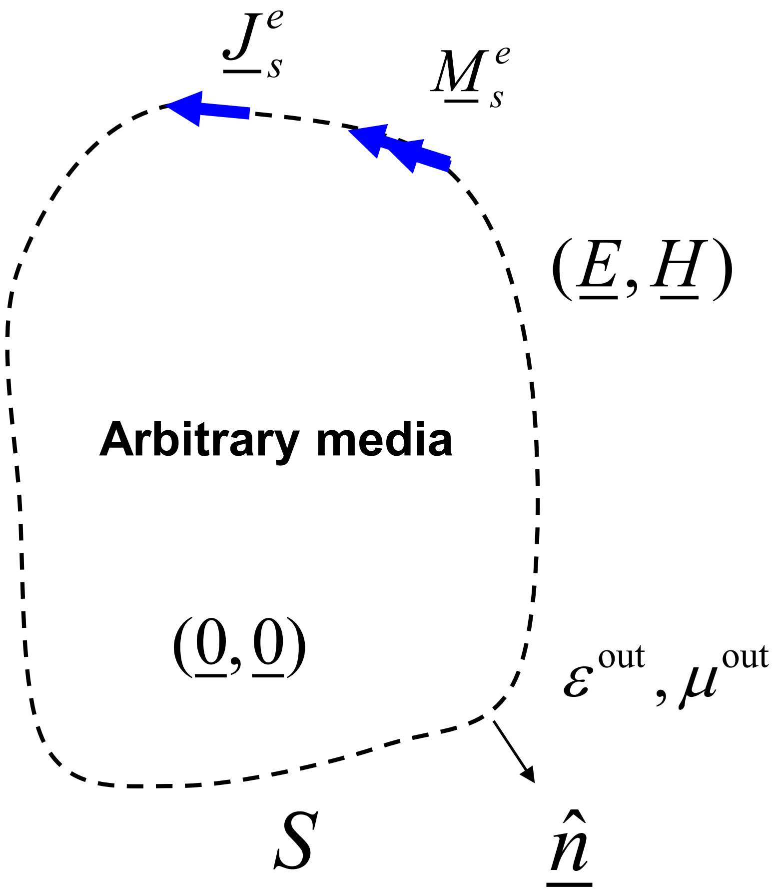
- Infinite substrate
- Truncated substrate (truncated at the edge of the patch).

Review of Equivalence Principle

Original problem:



New problem:

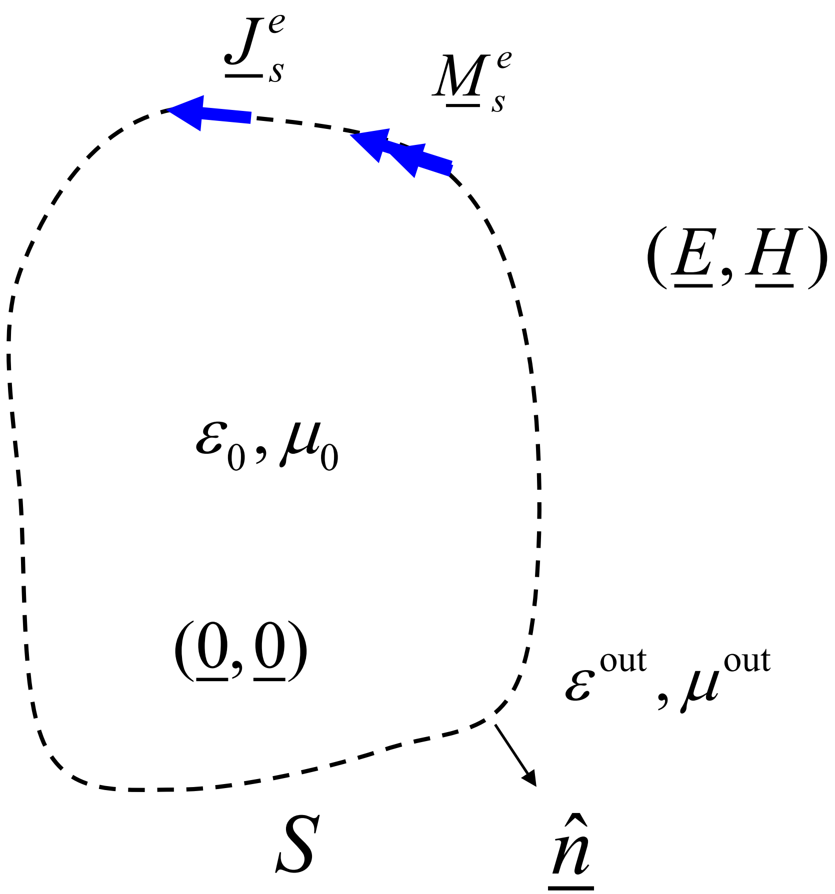


$$\underline{J}_s^e = \underline{\hat{n}} \times \underline{H}$$

$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

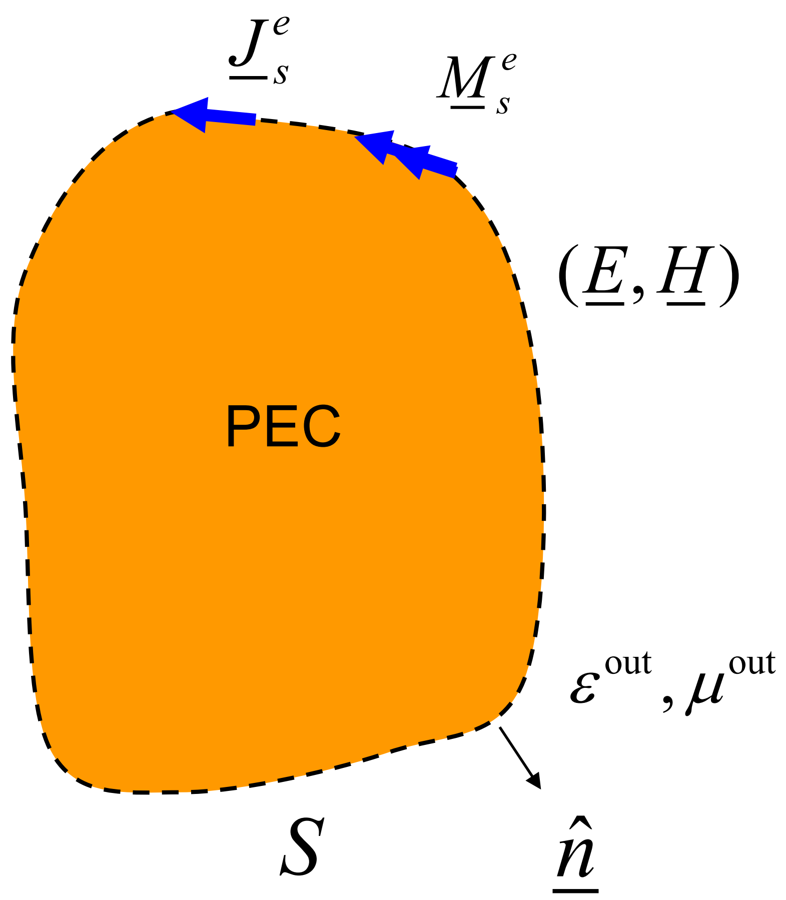
Review of Equivalence Principle (cont.)

A common choice (free space inside):



Review of Equivalence Principle (cont.)

A common choice (PEC inside):



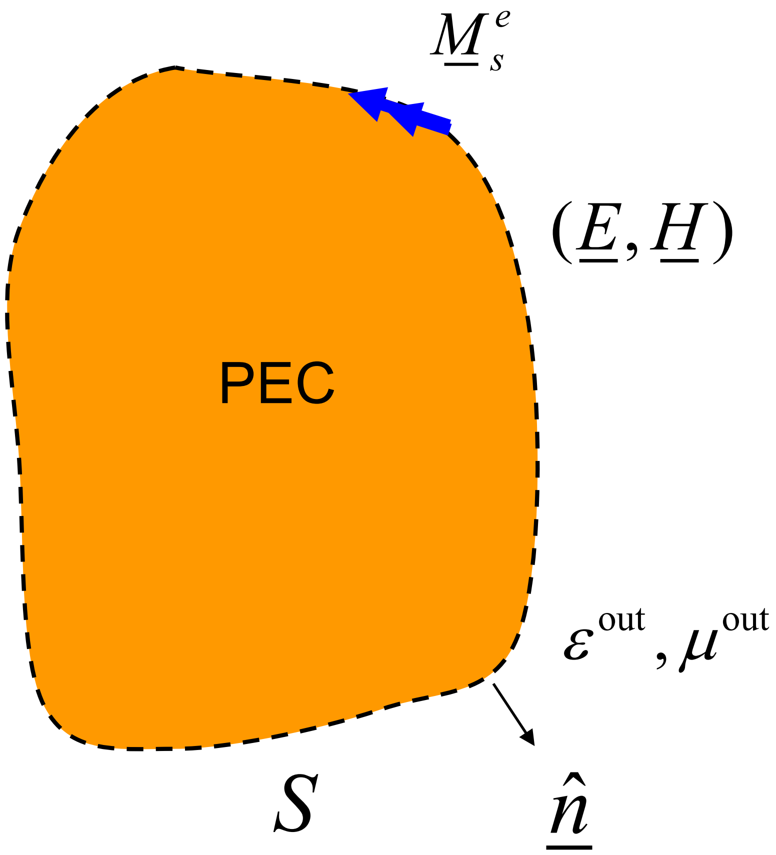
The electric surface current sitting on the PEC object does not radiate, and it can be ignored.

$$\underline{J}_s^e = \underline{\hat{n}} \times \underline{H}$$

$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

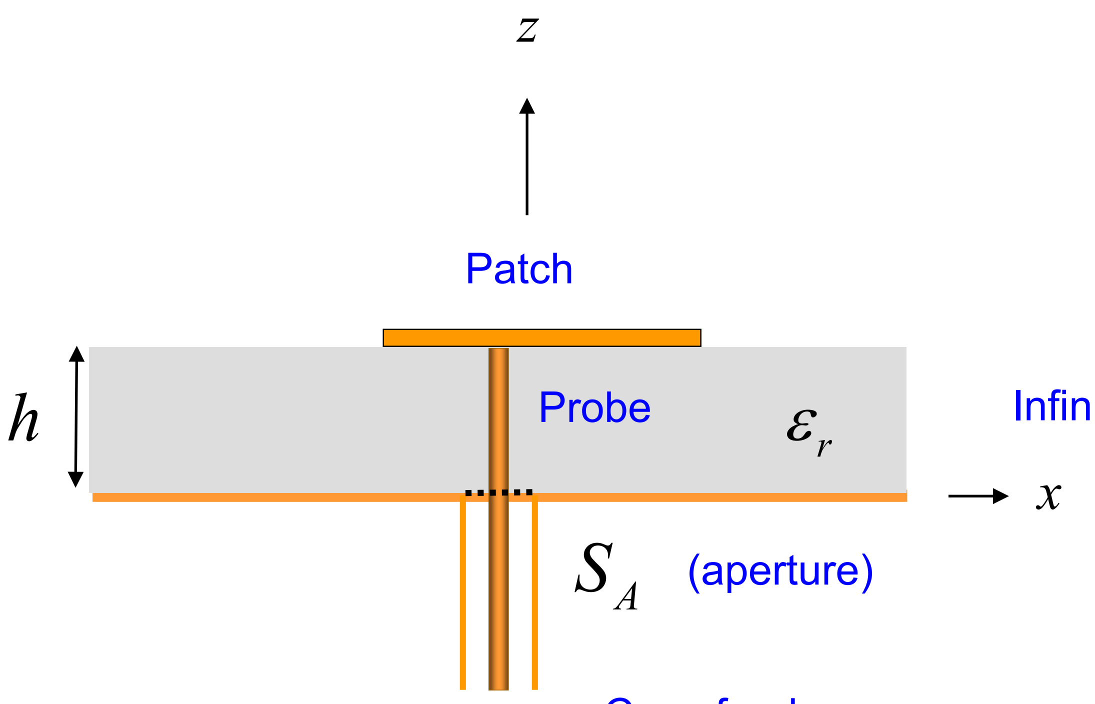
Review of Equivalence Principle (cont.)

Final model with PEC:



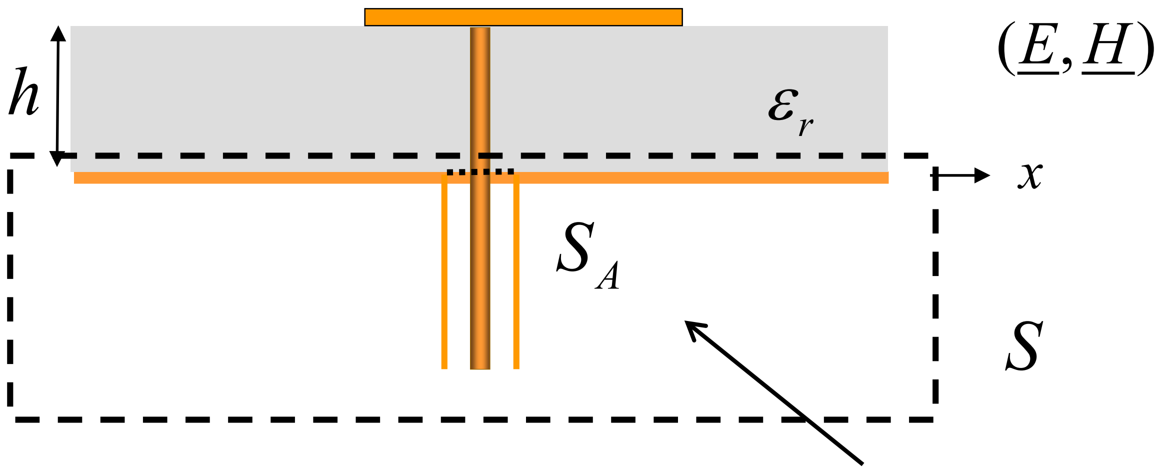
$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

Model of Patch and Feed



(The patch can be of arbitrary shape here.)

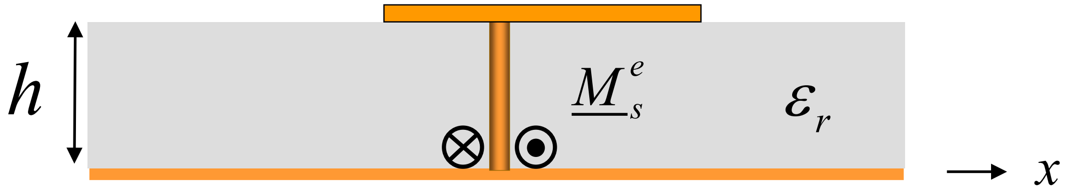
Model of Patch and Feed (cont.)



- Put zero fields
- Put ground plane (no aperture) at $z = 0$

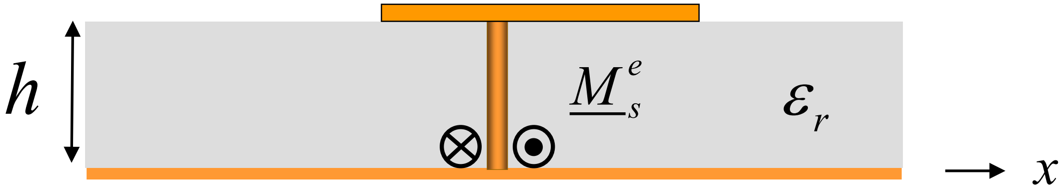
Magnetic frill model:

Aperture: $\underline{M}_s^e = -\hat{\underline{z}} \times \underline{E}$



Model of Patch and Feed (cont.)

Final magnetic frill model:

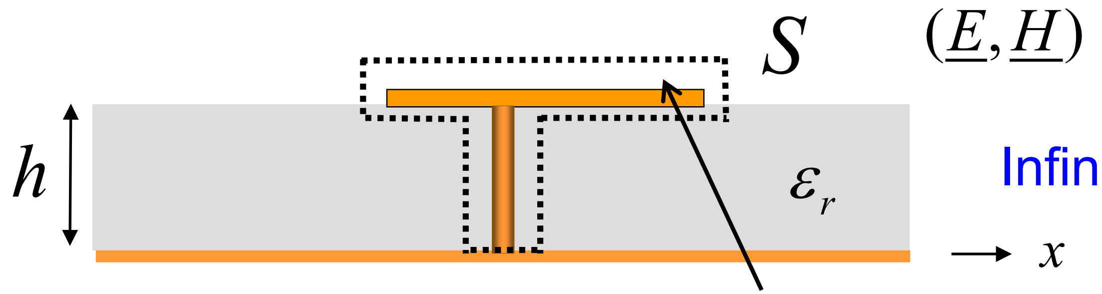


Aperture: $\underline{M}_s^e = -\underline{\hat{z}} \times \underline{E}$

This is essentially an exact model, but not convenient for analytical purposes!

Note: The frill radiates very little direct radiation, but it excites the patch.

Electric Current Model: Infinite Substrate



Infinite substrate

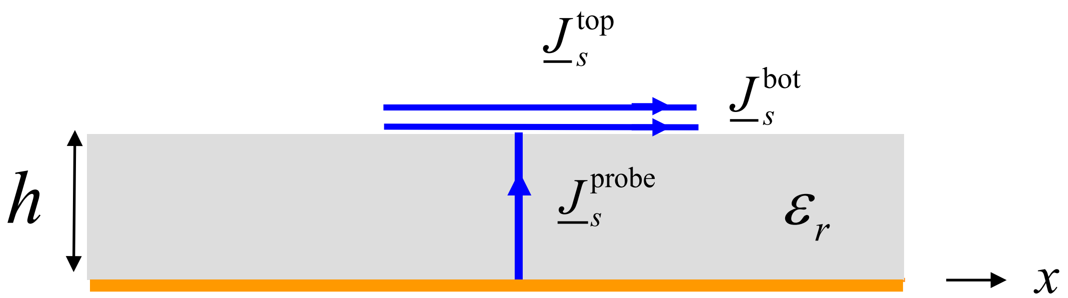
The surface S "hugs" the PEC metal.

- Put zero fields
- Remove patch and probe

$$\underline{M}_s^e = -\hat{\underline{n}} \times \underline{E} = -\hat{\underline{n}} \times \underline{E}_t = \underline{0}$$

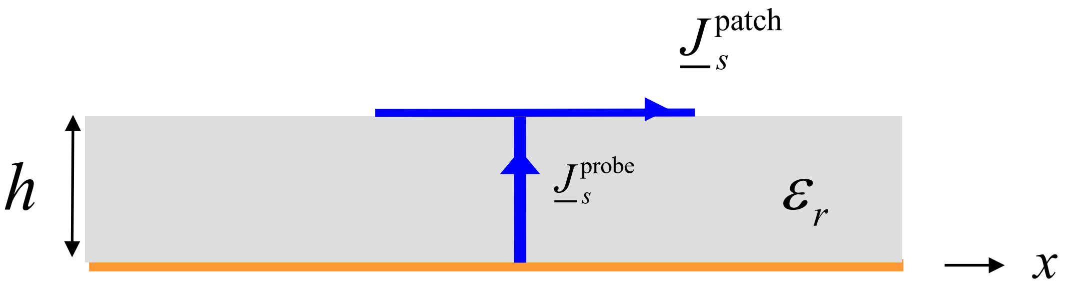
$$\underline{J}_s^e = \hat{\underline{n}} \times \underline{H} = \underline{J}_s$$

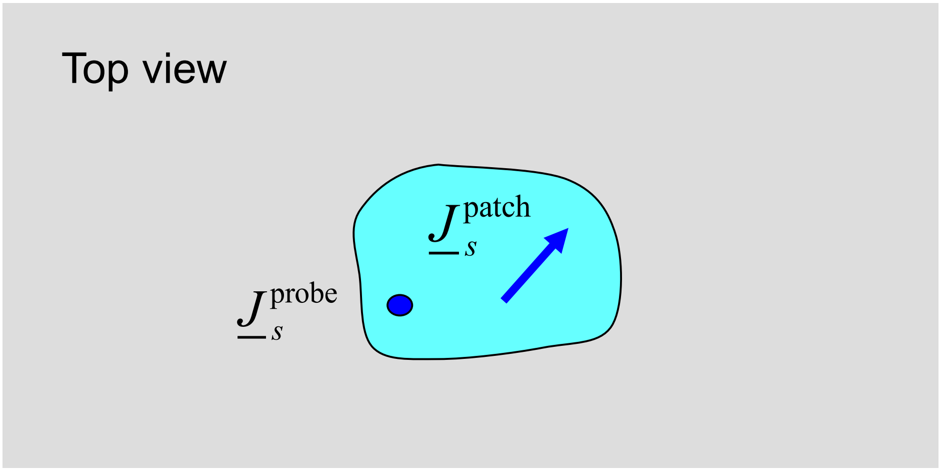
Note:
The frill is ignored (but we keep the electric currents on the structure that are set up by the frill source).



Electric Current Model: Infinite Substrate (cont.)

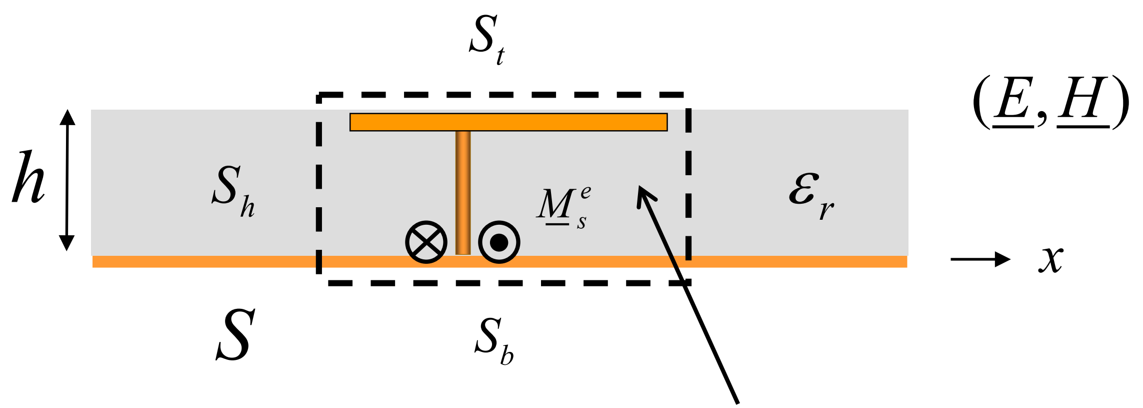
Let $$\underline{J}_s^{\text{patch}} = \underline{J}_s^{\text{top}} + \underline{J}_s^{\text{bot}}$$





Note: The electric currents radiate on/inside an infinite substrate above a ground plane.

Magnetic Current Model: Infinite Substrate



- Put zero fields
- Remove patch, probe, and frill current
- Put substrate and ground plane

$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E} = \underline{0}, \qquad \underline{r} \in S_t, S_b$$

$$\underline{J}_s^e = +\underline{\hat{n}} \times \underline{H} = \underline{0}, \qquad \underline{r} \in S_b$$

$$\underline{J}_s^e \approx \underline{0}, \qquad \underline{r} \in S_t$$ (weak fields)

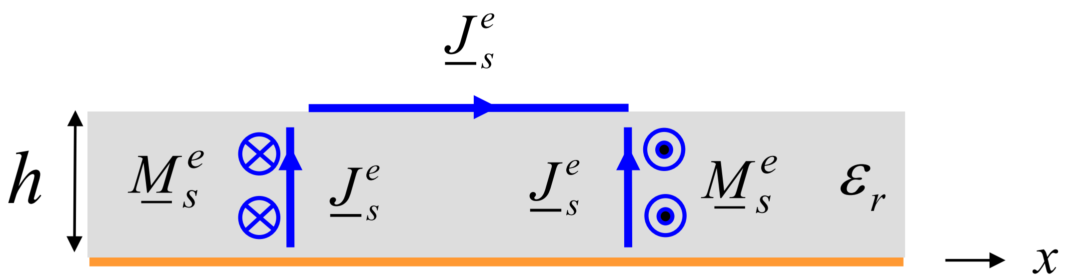
$$\underline{J}_s^e \approx \underline{0}, \qquad \underline{r} \in S_h$$ (approximate PMC)

Approximation

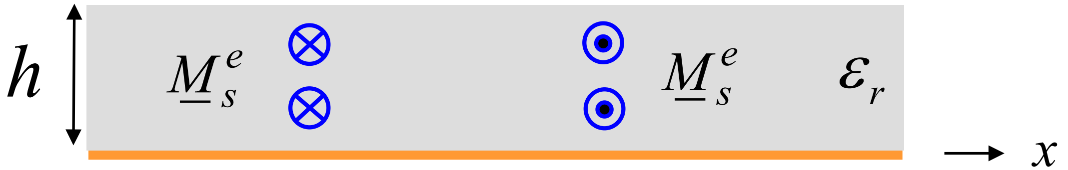
We keep only the magnetic current on the edges.

Magnetic Current Model: Infinite Substrate (cont.)

Exact model:



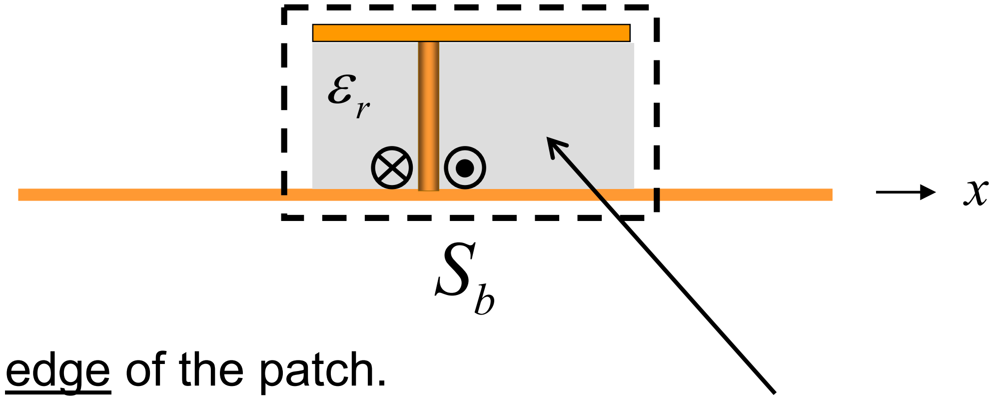
Approximate model:



$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

Note: The magnetic currents radiate inside an infinite substrate above a ground plane.

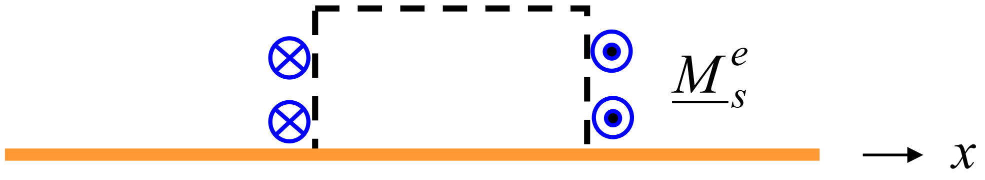
Magnetic Current Model: Truncated Substrate



The substrate is truncated at the edge of the patch.

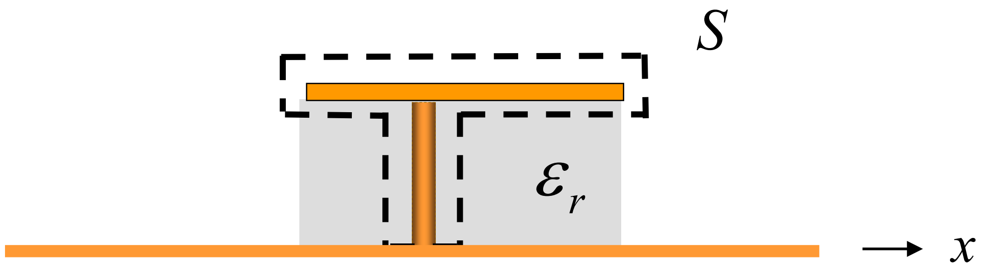
- Put zero fields
- Remove the substrate
- Keep the ground plane

Approximate model:



Note: The magnetic currents now radiate in free space above a ground plane.

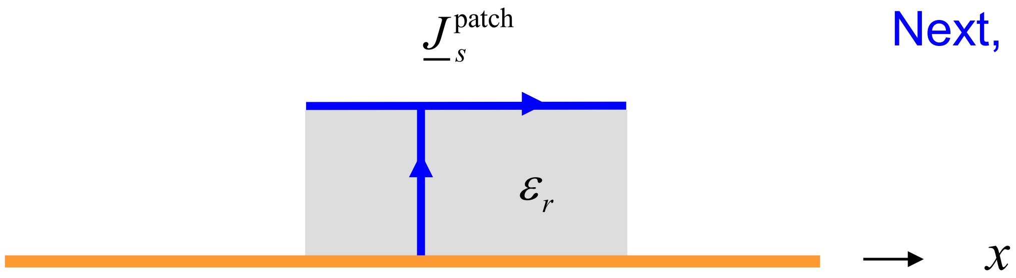
Electric Current Model: Truncated Substrate



The patch and probe are replaced by surface currents, as before.

The substrate is truncated at the edge of the patch.

$$\underline{J}_s^{\text{patch}} = \underline{J}_s^{\text{top}} + \underline{J}_s^{\text{bot}}$$



Next, we replace the dielectric with *polarization currents*.

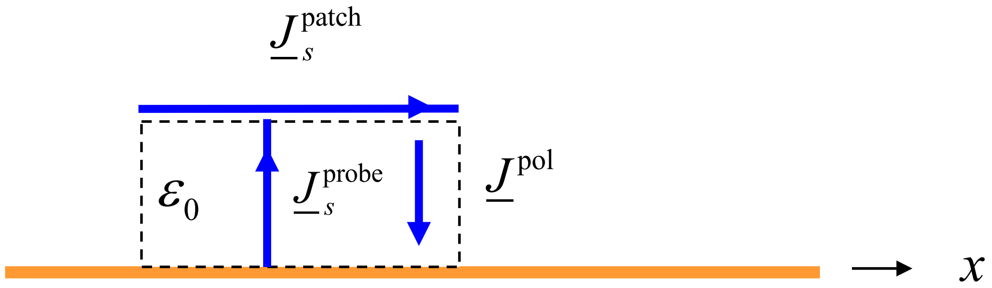
Electric Current Model: Truncated Substrate (cont.)

Inside the substrate:

$$\begin{aligned}\nabla \times \underline{H} &= j\omega\varepsilon \underline{E} \\ &= j\omega\left(\varepsilon - \varepsilon_0\right)\underline{E} + j\omega\varepsilon_0\underline{E} \\ &= j\omega\varepsilon_0\left(\varepsilon_r - 1\right)\underline{E} + j\omega\varepsilon_0\underline{E}\end{aligned}$$

$$\Rightarrow \quad \underline{J}^{\text{pol}} = j\omega\varepsilon_0\left(\varepsilon_r - 1\right)\underline{E}$$

In this model we have <u>three</u> separate electric currents.



$$\left(\underline{J}_s^{\text{patch}},\ \underline{J}_s^{\text{probe}},\ \underline{J}^{\text{pol}}\right)$$

Comments on Models

Infinite Substrate

- The electric current model is exact (if we neglect the direct radiation from the frill), but it requires knowledge of the exact patch and probe currents.
- The magnetic current model is approximate. It requires knowledge of the electric field at the patch edge.
- For a rectangular patch, both models are fairly simple if only the (1,0) mode is assumed.
- For a circular patch, the magnetic current model is much simpler (it does not involve Bessel functions).

Comments on Models (cont.)

Truncated Substrate

- The electric current model is exact (if we neglect the frill), but it requires knowledge of the exact patch and probe currents, as well as the field inside the patch cavity (to get the polarization currents). It is a complicated model.
- The magnetic current model is approximate, but is fairly simple. *This is the recommended model for a truncated substrate*.
- For the magnetic current model, the same formulation applies as for the infinite substrate – the substrate is simply taken to be air.

Theorem

The electric and magnetic models yield identical results
at the resonance frequency of the cavity mode.

Assumptions:

1) The electric and magnetic current models are based on the fields of a *single cavity mode corresponding to an ideal lossless cavity with PMC walls*.

2) The probe current is neglected in the electric current model.

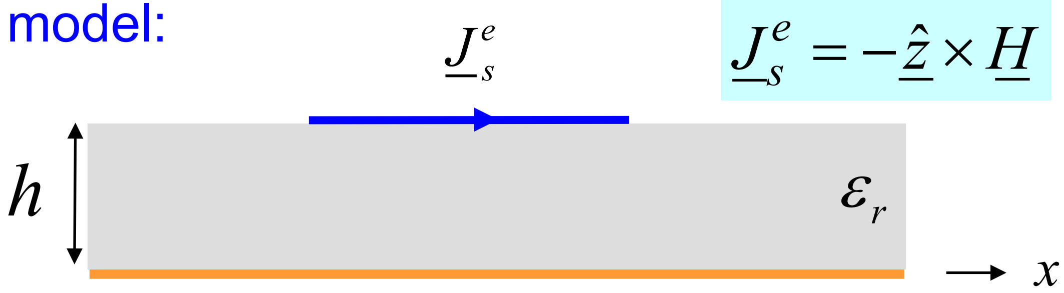
Note: This theorem is true for either infinite or truncated substrates.

D. R. Jackson and J. T. Williams, “A Comparison of CAD Models for Radiation from Rectangular Microstrip Patches,” *Intl. Journal of Microwave and Millimeter-Wave Computer Aided Design*, vol. 1, no. 2, pp. 236-248, April 1991.

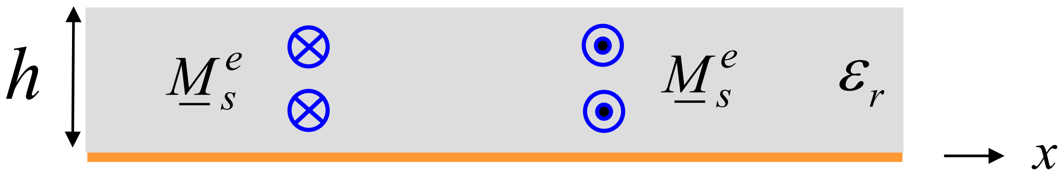
Theorem for Infinite Substrate

Electric-current model:

$$\underline{J}_s^e = -\underline{\hat{z}} \times \underline{H}$$



Magnetic-current model:



$$\underline{M}_s^e = -\underline{\hat{n}} \times \underline{E}$$

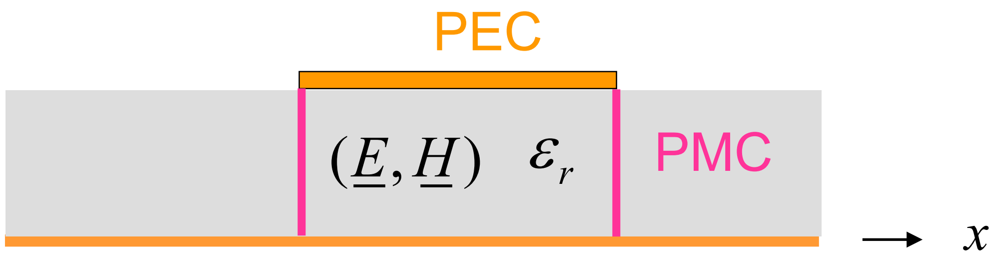
($\underline{E}$, $\underline{H}$) = fields of resonant cavity mode with PMC side walls

Proof (Infinite Substrate)

Proof:

We start with an ideal lossless cavity having PMC walls on the sides. This cavity will support a valid non-zero set of fields at the resonance frequency f_0 of the mode.

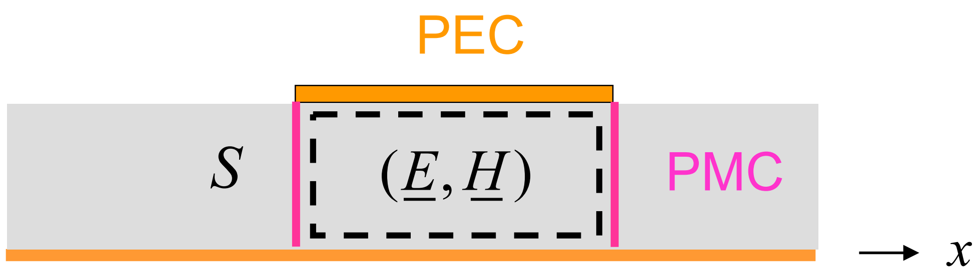
Ideal cavity



At $f = f_0 : \quad \left(\underline{E}, \underline{H}\right) \neq \left(\underline{0}, \underline{0}\right)$

Proof (Infinite Substrate) (cont.)

Proof for infinite substrate:

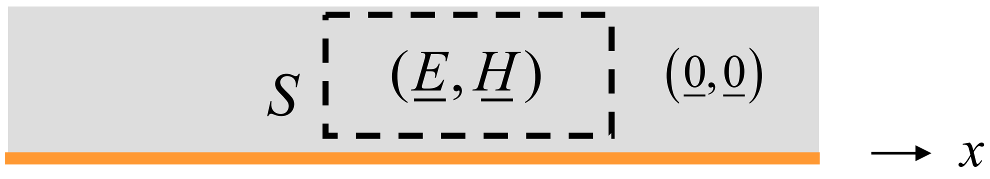


Equivalence principle:

Put $(\underline{0}, \underline{0})$ outside S

Keep $(\underline{E}, \underline{H})$ inside S

The PEC and PMC walls have been removed in the zero field (outside) region. We keep the substrate and ground plane in the outside region.

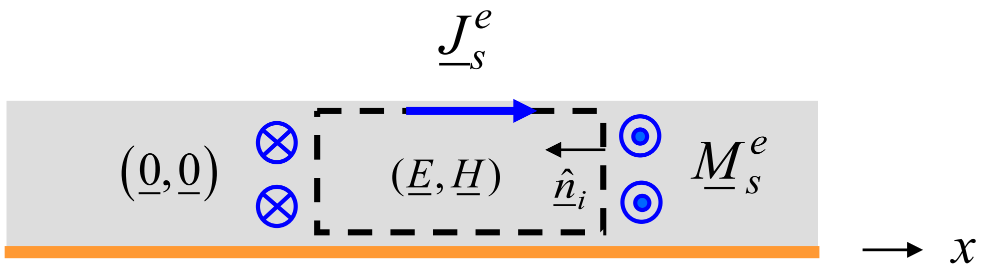


Proof (Infinite Substrate) (cont.)

$$\underline{J}_s^e = \underline{\hat{n}}_i \times \underline{H}$$

$$\underline{M}_s^e = -\underline{\hat{n}}_i \times \underline{E}$$

Note the inward pointing normal $\underline{\hat{n}}_i$



Note: The electric current on the ground has been neglected (it does not radiate).

Proof (Infinite Substrate) (cont.)

Exterior Fields:

$$\underline{E}^{+}\left[\underline{J}_s^e\right]+\underline{E}^{+}\left[\underline{M}_s^e\right]=\underline{0}$$

$$\underline{J}_s^e=\underline{\hat{n}}_i\times\underline{H}=-\underline{\hat{z}}\times\underline{H}=\underline{J}_s^{\text{patch}}=\underline{J}_s^J$$

(The equivalent electric current is the same as the electric current in the electric current model.)

$$\begin{aligned}\underline{M}_s^e&=-\underline{\hat{n}}_i\times\underline{E}\\&=+\underline{\hat{n}}\times\underline{E}\\&=-\left(-\underline{\hat{n}}\times\underline{E}\right)\\&=-\underline{M}_s^M\end{aligned}$$

(The equivalent magnetic current is the negative of the magnetic current in the magnetic current model.)

Proof (Infinite Substrate) (cont.)

Hence

$$\underline{E}^{+}\left[\underline{J}_{s}^{J}\right]+\underline{E}^{+}\left[-\underline{M}_{s}^{M}\right]=\underline{0}$$

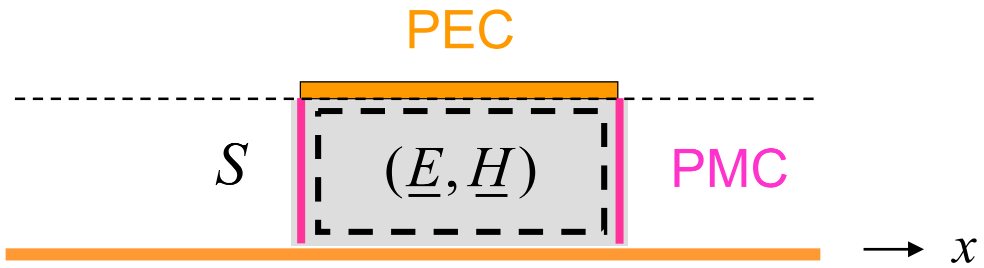
or

$$\underline{E}^{+}\left[\underline{J}_{s}^{J}\right]=\underline{E}^{+}\left[\underline{M}_{s}^{M}\right]$$

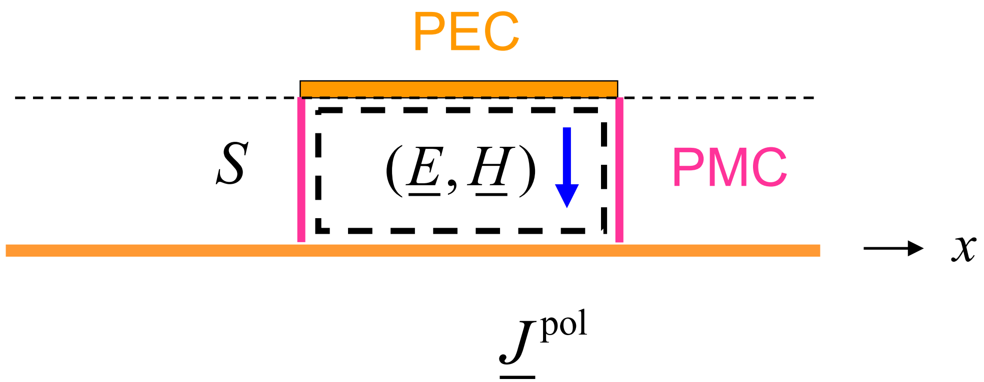
Proof complete.

Theorem for Truncated Substrate

Proof for truncated model:



Replace the dielectric with polarization current:

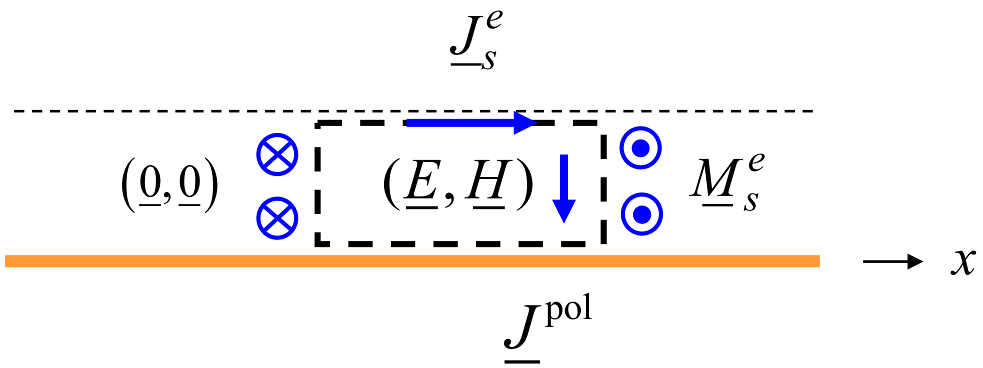


Proof (Truncated Substrate)

Apply equivalence principle as before:

$$\underline{J}_s^e = \underline{J}_s^{\text{patch}}$$

$$\underline{M}_s^e = -\underline{M}_s^M$$



$$\underline{E}^+\left[\underline{J}_s^{\text{patch}}\right] + \underline{E}^+\left[\underline{J}^{\text{pol}}\right] + \underline{E}^+\left[-\underline{M}_s^M\right] = 0$$

or

$$\underline{E}^+\left[\underline{J}_s^{\text{patch}}\right] + \underline{E}^+\left[\underline{J}^{\text{pol}}\right] = \underline{E}^+\left[\underline{M}_s^M\right]$$

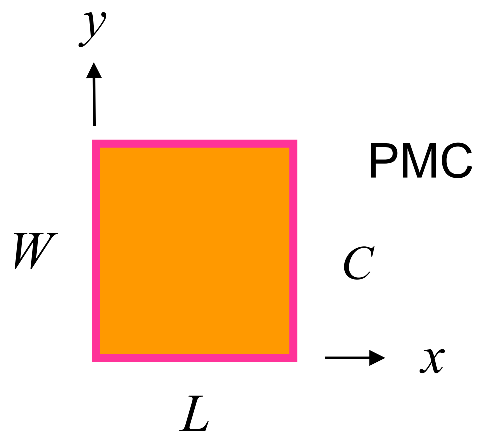
Hence, we have:

$$\underline{E}^+\left[\underline{J}_s^J\right] = \underline{E}^+\left[\underline{M}_s^M\right]$$

Proof complete.

Rectangular Patch

Ideal cavity model:



$$\frac{\partial E_z}{\partial n} = 0 \Big|_C$$

$$\nabla^2 E_z + k^2 E_z = 0$$

Let $E_z(x, y) = X(x) Y(y)$

$$X''Y + XY'' + k^2 XY = 0$$

Divide by $X(x)Y(y)$: $\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0$

so $\frac{X''}{X} = -\left(k^2 + \frac{Y''}{Y}\right)$

Rectangular Patch (cont.)

Hence

$$\frac{X''(x)}{X(x)} = \text{constant} \equiv -k_x^2$$

General solution: $X(x) = A\sin k_x x + B\cos k_x x$

Boundary condition: $X'(0) = k_x\, A\cos(k_x 0) - k_x B\sin(k_x 0) = k_x\, A = 0$

$\Rightarrow$ $A = 0$

Choose $B = 1$

$$X(x) = \cos(k_x x)$$

Boundary condition: $X'(L) = -k_x \sin\left(k_x L\right) = 0$

$\Rightarrow$ $k_x = \dfrac{m\pi}{L}$

Rectangular Patch (cont.)

Therefore, we have: $$X(x) = \cos\left(\frac{m\pi x}{L}\right)$$

Returning to the Helmholtz equation, $$-k_x^2 + \frac{Y''}{Y} + k^2 = 0$$

so $$\frac{Y''}{Y} = \text{constant} = -\left(k^2 - k_x^2\right) \equiv -k_y^2$$

Following the same procedure as for the $X(x)$ function, we have:

$$Y(y) = \cos\left(\frac{n\pi y}{W}\right)$$

Hence $$E_z^{(m,n)}(x,y) = \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{W}\right)$$

Rectangular Patch (cont.)

Using $$-k_x^2 - k_y^2 + k^2 = 0$$

we have $$k = k_{mn} = \sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{W}\right)^2}$$

where $$k_{mn} = \omega_{mn}\sqrt{\mu\varepsilon}$$

Hence $$\omega_{mn} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{W}\right)^2}$$

or $$f_{mn} = \frac{c}{2\pi\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{W}\right)^2}$$

Rectangular Patch (cont.)

The current on the patch (bottom surface):

$$\underline{J}_s^{\text{patch}} = \underline{\hat{n}} \times \underline{H} = -\underline{\hat{z}} \times \underline{H}$$

$$\begin{aligned} \underline{H} &= \frac{-1}{j\omega\mu} \nabla \times \underline{E} \\ &= \frac{-1}{j\omega\mu} \nabla \times \left(\underline{\hat{z}}\, E_z \right) \\ &= \frac{-1}{j\omega\mu} \left[\cancel{\left(\nabla \times \underline{\hat{z}} \right)} E_z - \underline{\hat{z}} \times \nabla E_z \right] \end{aligned}$$

so

$$\underline{H} = \frac{1}{j\omega\mu} \left(\underline{\hat{z}} \times \nabla E_z \right)$$

Rectangular Patch (cont.)

Hence, we have:

$$\underline{J}_s^{\,\text{patch}} = -\frac{1}{j\omega\mu}\,\underline{\hat{z}}\times\left(\underline{\hat{z}}\times\nabla E_z\right) \quad\Rightarrow\quad \underline{J}_s^{\,\text{patch}} = \frac{1}{j\omega\mu}\nabla E_z$$

$$\underline{J}_s^{\,\text{patch}} = \frac{1}{j\omega\mu}\left[\underline{\hat{x}}\left(-\frac{m\pi}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{W}\right) + \underline{\hat{y}}\left(-\frac{n\pi}{W}\right)\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right)\right]$$

Dominant (1,0) Mode:

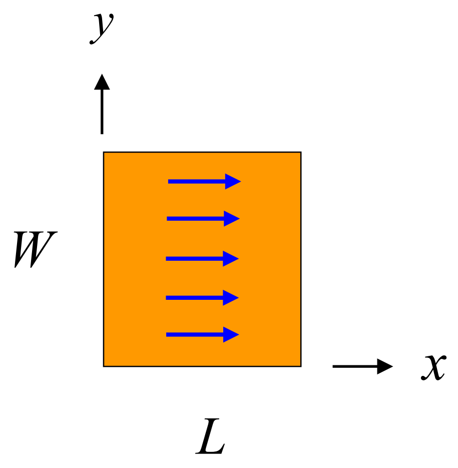
$$E_z(x,y) = \cos\left(\frac{\pi x}{L}\right)$$

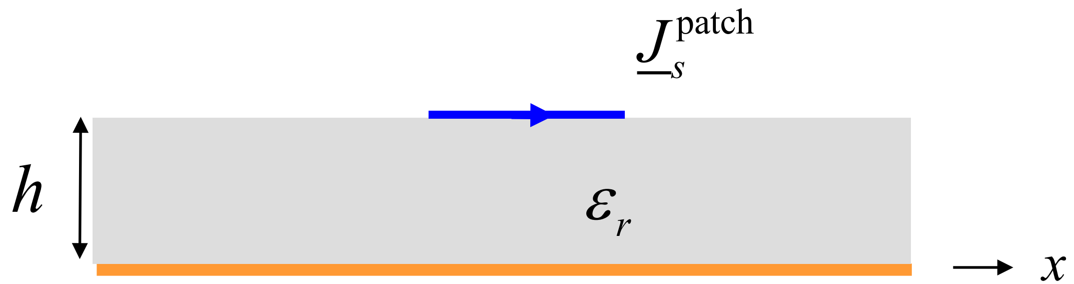
$$\underline{J}_s(x,y) = -\underline{\hat{x}}\left(\frac{1}{j\omega\mu}\right)\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right)$$

Radiation Model for (1,0) Mode

Electric-current model:

$$\underline{J}_s^{\text{patch}} = -\underline{\hat{x}}\left(\frac{\pi}{j\omega\mu L}\right)\sin\left(\frac{\pi x}{L}\right)$$



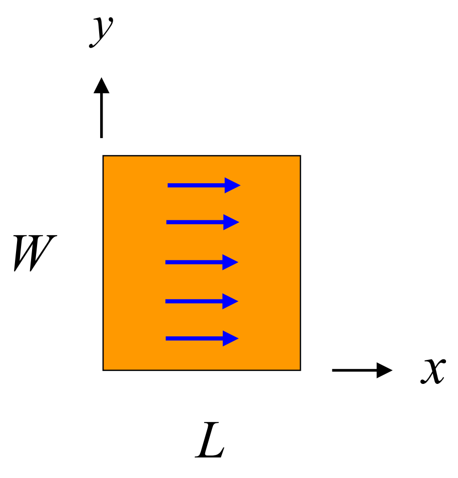


Radiation Model for (1,0) Mode (cont.)

Magnetic-current model:

$$\underline{M}_s^M = -\underline{\hat{n}} \times \underline{E}$$

$$= -\underline{\hat{n}} \times \left[\underline{\hat{z}} \cos\left(\frac{\pi x}{L} \right) \right]$$



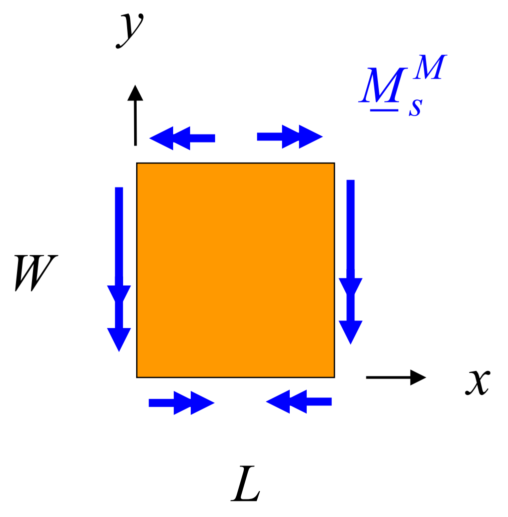
$$\underline{\hat{n}} = \begin{cases} \underline{\hat{x}} & x = L \\ -\underline{\hat{x}} & x = 0 \\ \underline{\hat{y}} & y = W \\ -\underline{\hat{y}} & y = 0 \end{cases}$$

Radiation Model for (1,0) Mode (cont.)

Hence, we have:

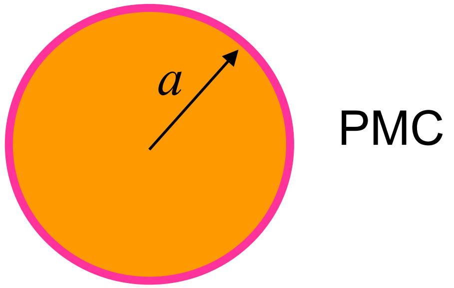
$$\underline{M}_s^M = \begin{cases} \underline{\hat{y}}\cos\pi = -\underline{\hat{y}} & x = L \\ -\underline{\hat{y}}\cos 0 = -\underline{\hat{y}} & x = 0 \\ -\underline{\hat{x}}\cos\left(\dfrac{\pi x}{L}\right) & y = W \\ \underline{\hat{x}}\cos\left(\dfrac{\pi x}{L}\right) & y = 0 \end{cases}$$

radiating edges



The non-radiating edges do not contribute to the far-field pattern in the principal planes.

Circular Patch



$$\nabla^2 E_z + k^2 E_z = 0$$

$$E_z = \begin{pmatrix} J_n(k_\rho \rho) \\ \cancel{Y_n(k_\rho \rho)} \end{pmatrix} \begin{pmatrix} \cos(n\phi) \\ \sin(n\phi) \end{pmatrix} \cos\left(\frac{m\pi z}{h}\right)$$

set $m = 0$ ⇨

$$k_\rho = \left(k^2 - k_z^2\right)^{1/2}$$

$$= \left(k^2 - \left(\frac{m\pi}{h}\right)^2\right)^{1/2}$$

$$= k$$

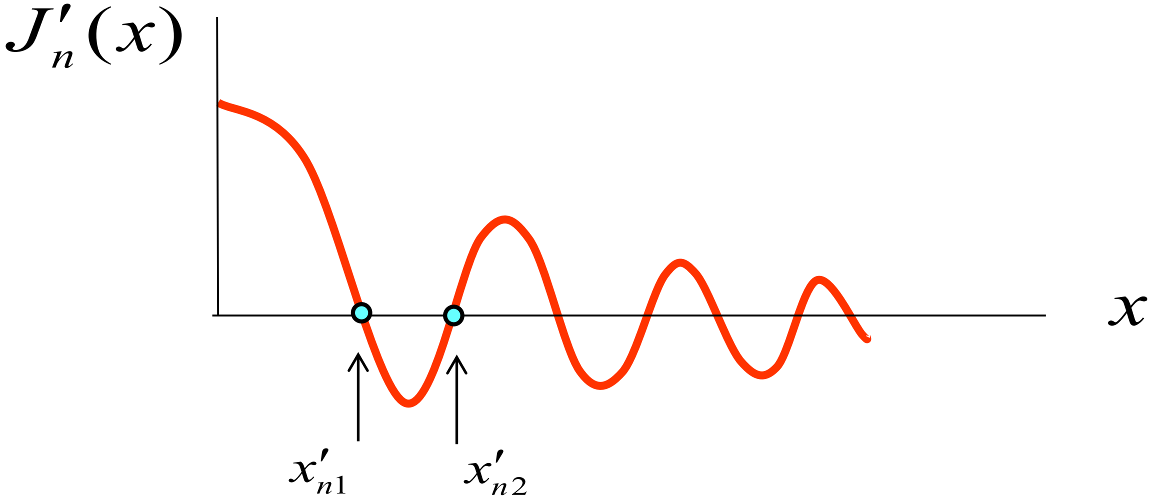
Circular Patch (cont.)

Note: $\cos\phi$ and $\sin\phi$ modes are degenerate (same resonance frequency).

Choose $\cos\phi$: $E_z = \cos(n\phi) J_n(k\rho)$

$$\left.\frac{\partial E_z}{\partial \rho} = 0\right|_{\rho=a}$$

$$J_n'(ka) = 0$$



Circular Patch (cont.)

Hence $$ka = x'_{np}$$

so $$f_{np} = \frac{c}{2\pi\sqrt{\varepsilon_r}} x'_{np}$$

Dominant mode (lowest frequency) is TM_{11}: $(n, p) = (1,1)$

$$x'_{11} = 1.841$$

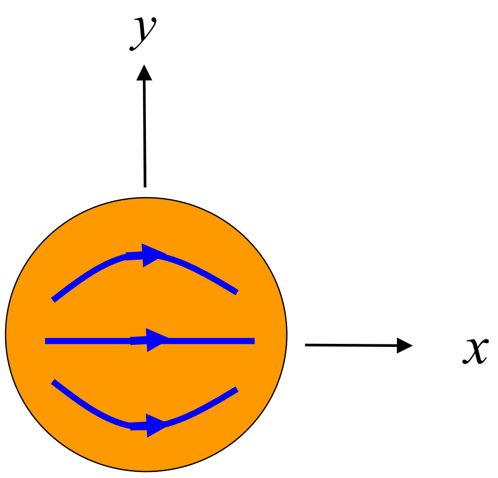
$$E_z^{(1,1)}(\rho,\phi) = \cos\phi \, J_1(k\rho)$$

Circular Patch (cont.)

Electric current model:

$$\underline{J}_s^J = \frac{1}{j\omega\mu}\nabla E_z = \frac{1}{j\omega\mu}\left(\underline{\hat{\rho}}\frac{\partial E_z}{\partial \rho} + \underline{\hat{\phi}}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\right)$$

$$= \frac{1}{j\omega\mu}\left[\underline{\hat{\rho}}\, k\cos n\phi\; J_n'(k\rho) + \underline{\hat{\phi}}\frac{1}{\rho}(-n)\sin n\phi\; J_n(k\rho)\right]$$



TM_{11} mode:

$$n = 1,\; p = 1$$

$$\underline{J}_s^J = -\frac{1}{j\omega\mu}\left[\underline{\hat{\rho}}\, k\cos\phi\; J_1'(k\rho) - \underline{\hat{\phi}}\frac{1}{\rho}\sin\phi\; J_1(k\rho)\right]$$

Very complicated!

Circular Patch (cont.)

Magnetic current model:

$$\underline{M}_s^M = -\underline{\hat{n}} \times \underline{E}$$
$$= -\underline{\hat{\rho}} \times \left(\underline{\hat{z}} E_z \right)$$
$$= \underline{\hat{\phi}} E_z$$

so

$$\underline{M}_s^M = \underline{\hat{\phi}} \cos n\phi \, J_n(ka)$$

TM$_{11}$: $n = 1, \; p = 1$

$$\underline{M}_s^M = \underline{\hat{\phi}} \cos \phi \, J_1(ka)$$

Circular Patch (cont.)

Note:

$$V(\phi) = -h\,E_z(\phi)\Big|_{\rho=a} = -h\cos\phi\; J_1(ka)$$

At $\phi = 0$ we have: $V(0) \equiv V_0 = -h\,J_1(ka)$

Hence, we have:

$$\underline{M}_s^M = \hat{\underline{\phi}}\cos\phi\,J_1(ka) = \hat{\underline{\phi}}\cos\phi\left(-\frac{V_0}{h}\right)$$

so

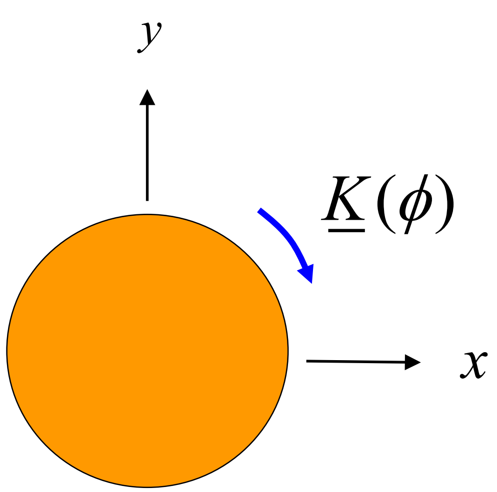
$$\underline{M}_s^M = -\hat{\underline{\phi}}\left(\frac{V_0}{h}\right)\cos\phi$$

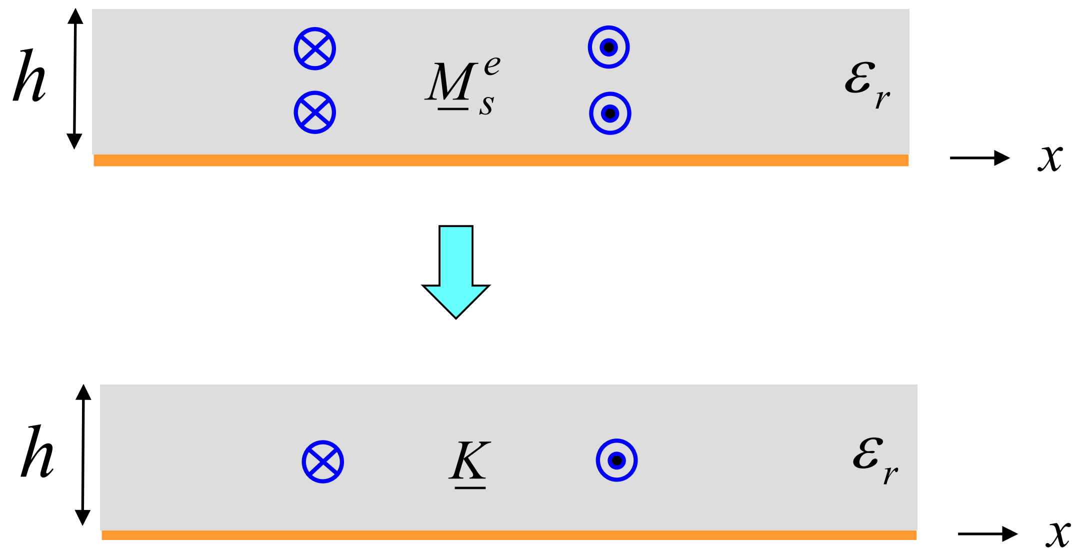
Circular Patch (cont.)

Ring approximation: $\underline{K} = \underline{\hat{\phi}} K_\phi$

$$K_\phi = \int_0^h M_{s\phi}^M \, dz = h \, M_{s\phi}^M = h \cos\phi \left(-\frac{V_0}{h} \right) = -V_0 \cos\phi$$

$$K_\phi(\phi) = -V_0 \cos\phi$$





The ring of magnetic current is put in the middle of the substrate.