ECE 6345

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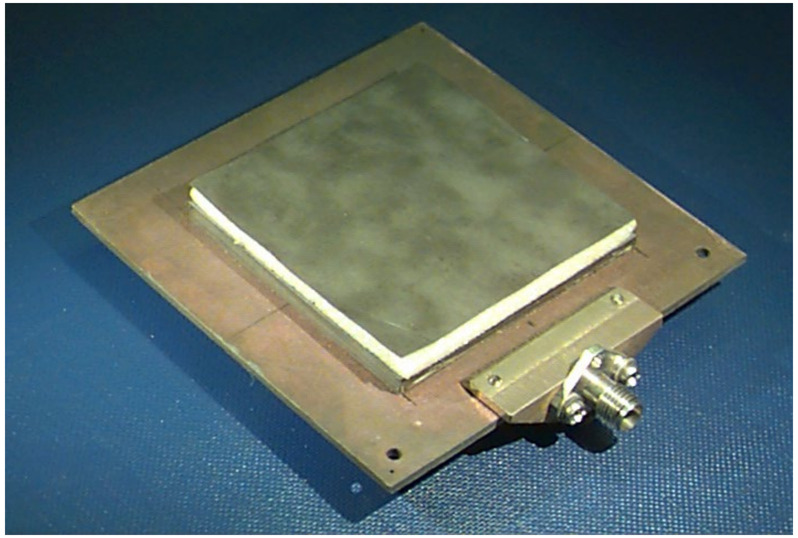
ECE Dept.



Notes 8

Overview

In this set of notes we use reciprocity to calculate the far field of a rectangular patch using the electric-current model, assuming an infinite substrate.

- Review of reciprocity to calculate the far field.
- Far field of horizontal electric dipole in the x direction (hex) on top of a grounded the substrate.
- Far field of dominant mode of rectangular patch, using the electric current model and infinite substrate.

Far-Field

Reciprocity theorem:

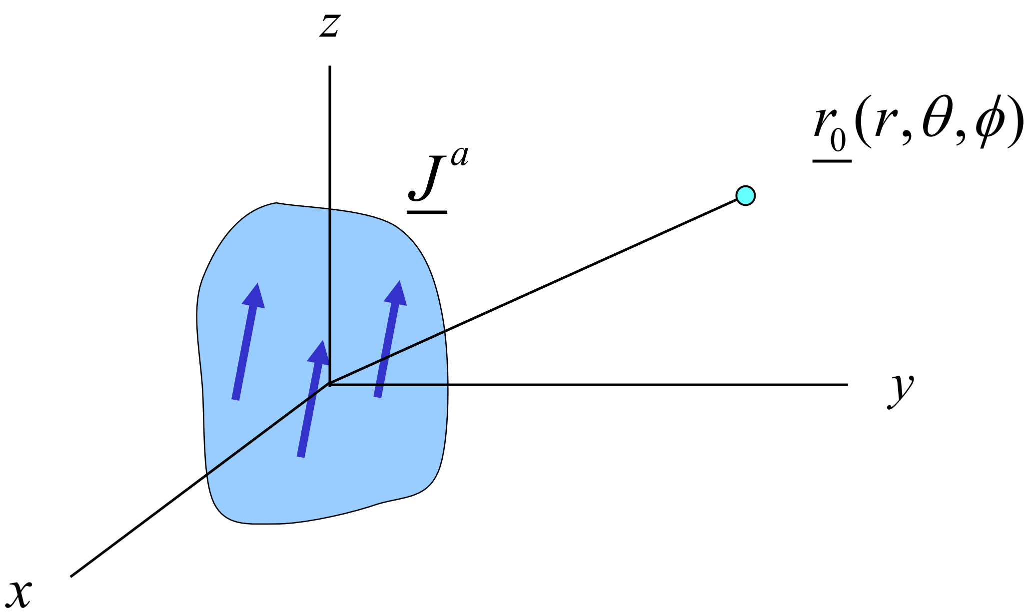
$$< a,b >=< b,a >$$

$$\int_V \left(\underline{E}^a \cdot \underline{J}^b - \underline{H}^a \cdot \underline{M}^b \right) dV = \int_V \left(\underline{E}^b \cdot \underline{J}^a - \underline{H}^b \cdot \underline{M}^a \right) dV$$

Consider an electric current source:

$\underline{J}^a$ is a radiating current source

This source radiates $\underline{E}^{\text{rad}}$

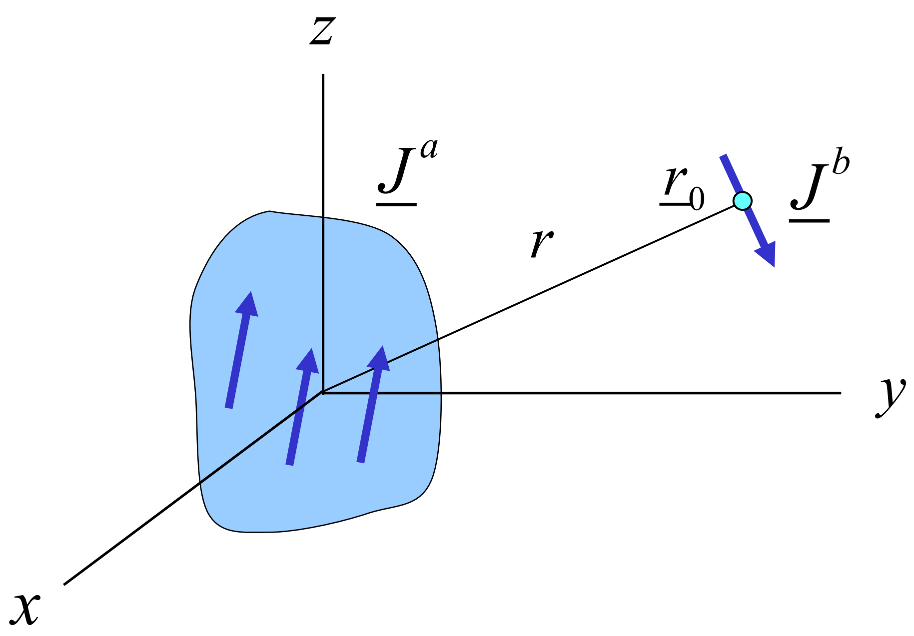


Far-Field (cont.)

Let

$$\underline{J}^b = \underline{\hat{\theta}}\,\delta\left(\underline{r} - \underline{r}_0\right)$$

$\underline{J}^b$ is a "testing" dipole source



$$< a,b > = \int_V \left(\underline{E}^a \cdot \underline{J}^b\right) dV = \underline{E}^a\left(\underline{r}_0\right) \cdot \underline{\hat{\theta}} = E_\theta^{\text{rad}}\left(r,\theta,\phi\right)$$

Far-Field (cont.)

Hence

$$E_{\theta}^{\text{rad}}\left(r,\theta,\phi\right) = \langle a,b \rangle = \langle b,a \rangle$$

$$\langle b,a \rangle = \int_V \left(\underline{E}^{b} \cdot \underline{J}^{a}\right) dV$$

Assume $r \to \infty$ $\quad \underline{E}^{b,\text{inc}} = \underline{E}^{\text{ipw}}$ $\quad$ ipw = "incident plane wave"

Hence, we have:

$$E_{\theta}^{\text{FF}}\left(r,\theta,\phi\right) = \int_V \left(\underline{E}^{\text{ipw}} \cdot \underline{J}^{a}\right) dV$$

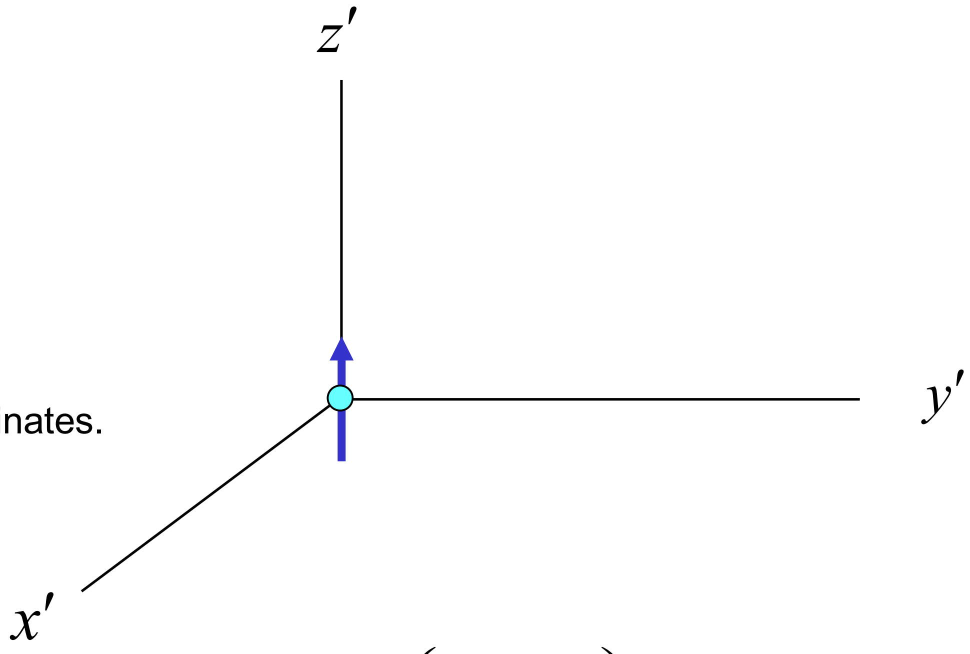
FF = "far field"

Far-Field (cont.)

We need to find $\underline{E}^{\text{ipw}}$

Consider a dipole in free space:

The primed coordinates denote local coordinates.

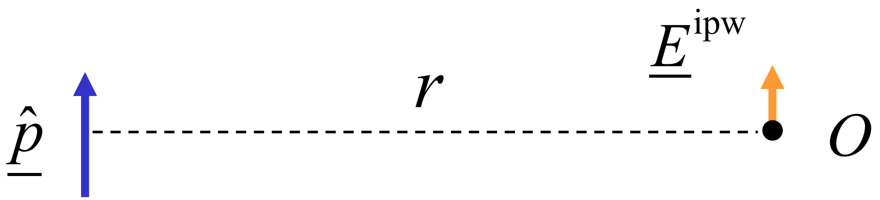


$$\underline{E} \sim \underline{\hat{\theta}}' \left(\frac{j\omega\mu_0}{4\pi r'} \right) \sin\theta' \, e^{-jk_0 r'}$$

At $\theta' = 90^0 \;\; \left(\underline{\hat{\theta}}' = -\underline{\hat{z}}' \right)$ we have: $$\underline{E} \sim -\frac{j\omega\mu_0}{4\pi r'} \; e^{-jk_0 r'} \, \underline{\hat{z}}'$$

Far-Field (cont.)

In general, we have this picture:

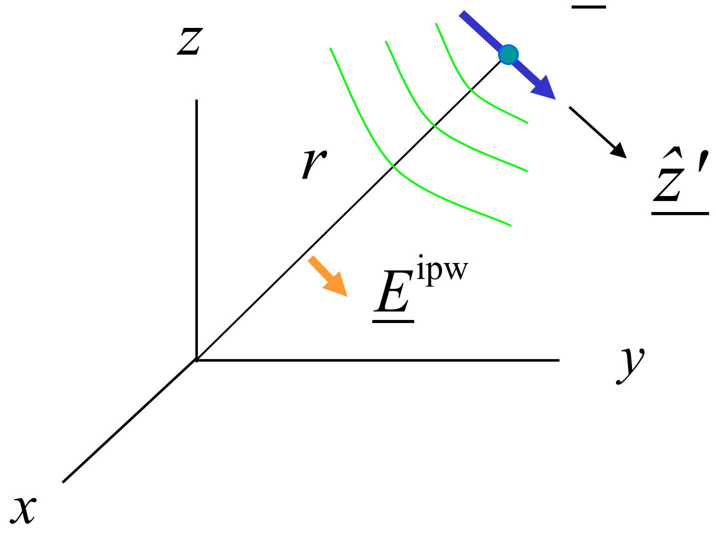


At O : $\underline{E}^{\text{ipw}} \sim \underline{\hat{p}} E_0$

where

$$E_0 = -\frac{j\omega\mu_0}{4\pi r} e^{-jk_0 r}$$

The "testing" dipole source :



At (0, 0, 0): $\underline{E}^{\text{ipw}} \sim \underline{\hat{\theta}} E_0$

Far-Field (cont.)

For an arbitrary observation point $\underline{r} = (x, y, z)$ we have:

$$\underline{E}^{\text{ipw}}\left(x,y,z\right) \sim \underline{\hat{\theta}}\, E_0\, \psi(x,y,z)$$

where

$$\psi(x,y,z) = e^{+j\left(k_x x + k_y y + k_z z\right)} = e^{+j\underline{k}\cdot\underline{r}}$$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$
$$k_z = k_0 \cos\theta$$

$$\underline{k} = \underline{\hat{x}}k_x + \underline{\hat{y}}k_y + \underline{\hat{z}}k_z$$
$$\underline{r} = \underline{\hat{x}}x + \underline{\hat{y}}y + \underline{\hat{z}}z$$

Far-Field (cont.)

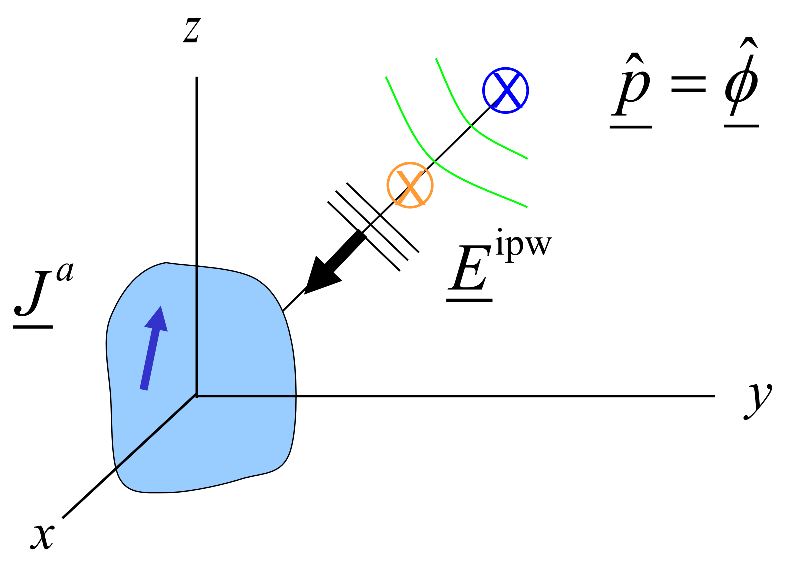
Hence $\underline{E}^{\text{ipw}} \sim \underline{\hat{\theta}}\, E_0\, \psi(x, y, z)$

Similarly, if $\underline{\hat{p}} = \underline{\hat{\phi}}$

then we have:

$$\underline{E}^{\text{ipw}} = \underline{\hat{\phi}}\, E_0 \psi\left(x, y, z\right)$$

The "testing" dipole source:



Far-Field (cont.)

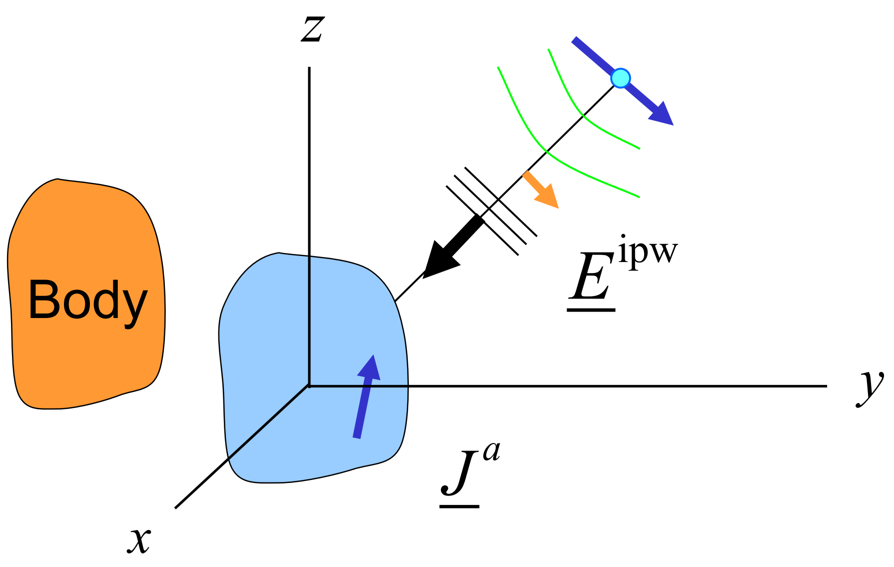
Now consider an object near the radiating current:

$$E_{\theta}^{\mathrm{FF}}(r,0,\phi)=\int_{V}\left(\underline{E}^{b}\cdot\underline{J}^{a}\right)dV$$

We then have:

$$\underline{E}^{b}=\underline{E}^{\mathrm{ipw}}+\underline{E}^{\mathrm{scattered}}$$

The "testing" dipole source:



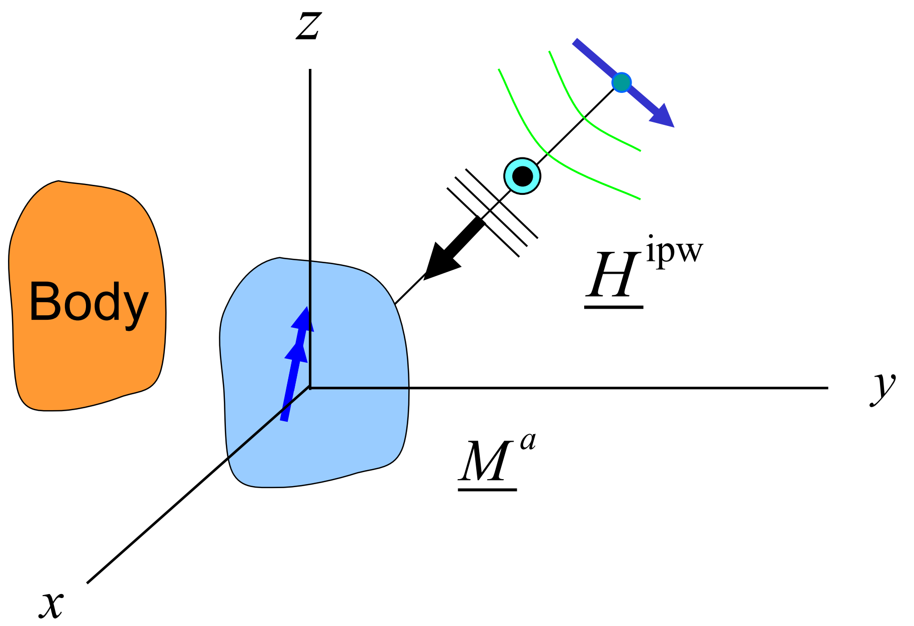
If the "body" is an infinite layered dielectric structure, the scattered field can be calculated exactly.

Far-Field (cont.)

For a magnetic current source:

$$E_{\theta}^{\mathrm{FF}}(r,\theta,\phi) = -\int_V \left(\underline{H}^b \cdot \underline{M}^a \right) dV$$

The "testing" dipole source:



We then have:

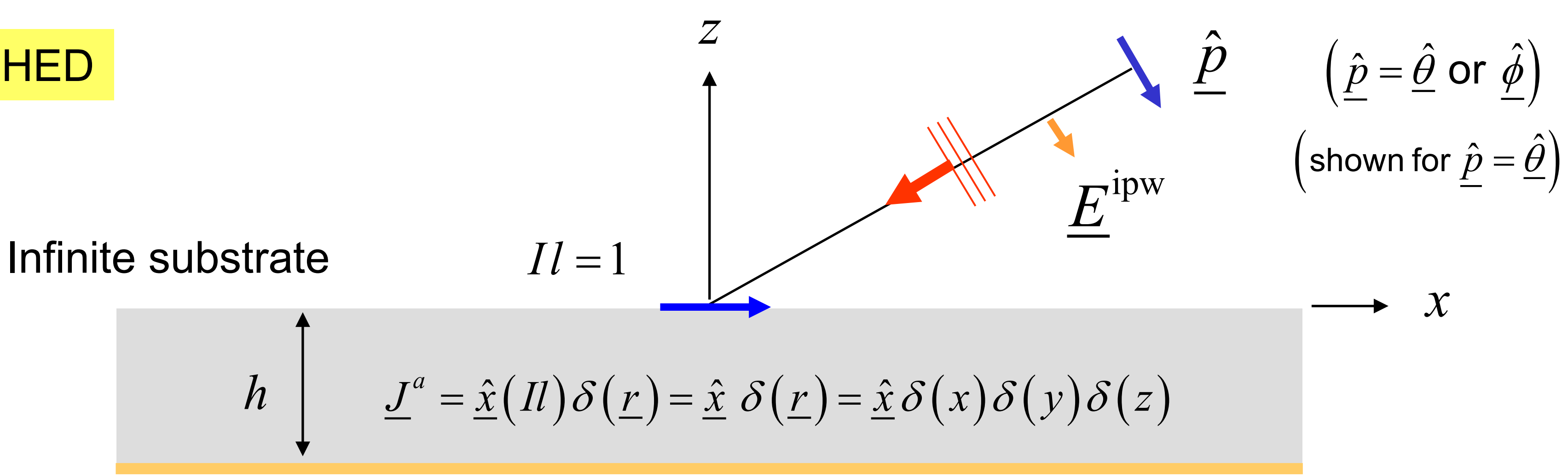
$$\underline{H}^b = \underline{H}^{\mathrm{ipw}} + \underline{H}^{\mathrm{scattered}}$$

where

$$\underline{H}^{\mathrm{ipw}} = -\frac{1}{\eta_0}\left(\underline{\hat{r}} \times \underline{E}^{\mathrm{ipw}} \right)$$

Electric Dipole

Far field of HED



$$
\begin{aligned}
E_p^{\text{hex}}(r,\theta,\phi) &= <b,a> \\
&= \int_V \left(\underline{E}^b \cdot \underline{J}^a\right) dV \\
&= E_x^b(0,0,0) \\
&= \left[E_x^{\text{ipw}}(0,0,0) + E_x^{\text{rpw}}(0,0,0)\right]
\end{aligned}
$$

"hex" = unit-amplitude horizontal electric dipole (HED) in the x direction.

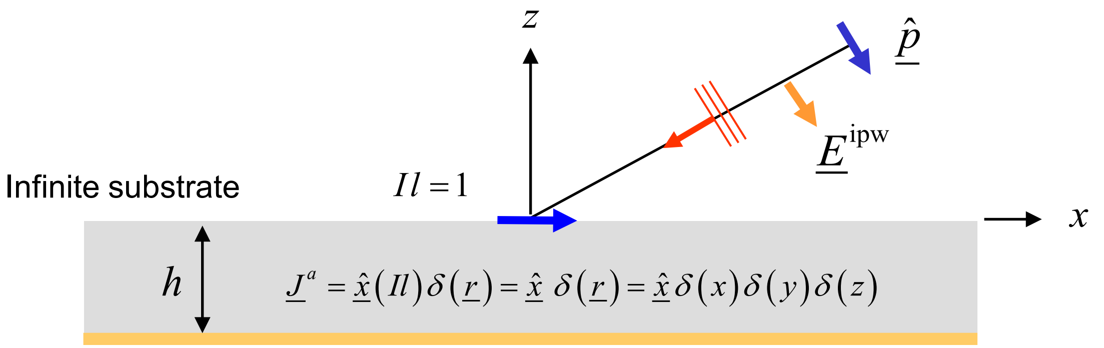
rpw = "reflected plane wave"

Electric Dipole (cont.)

For $\underline{\hat{p}} = \underline{\hat{\theta}}$ TM$_z$

$$\underline{E}^{\text{ipw}} = \underline{\hat{\theta}} E_0 e^{+j\underline{k}\cdot\underline{r}}$$

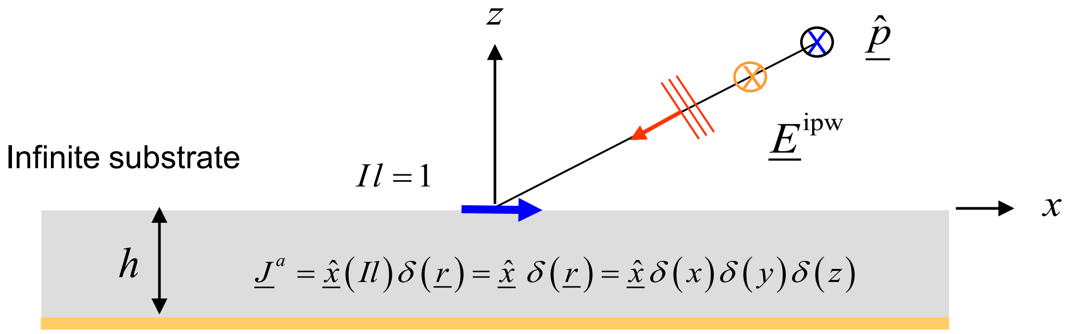
$$E_x^{\text{ipw}} = E_0 \cos\theta \cos\phi \, e^{+j\underline{k}\cdot\underline{r}}$$



For $\underline{\hat{p}} = \underline{\hat{\phi}}$ TE$_z$

$$\underline{E}^{\text{ipw}} = \underline{\hat{\phi}} E_0 e^{+j\underline{k}\cdot\underline{r}}$$

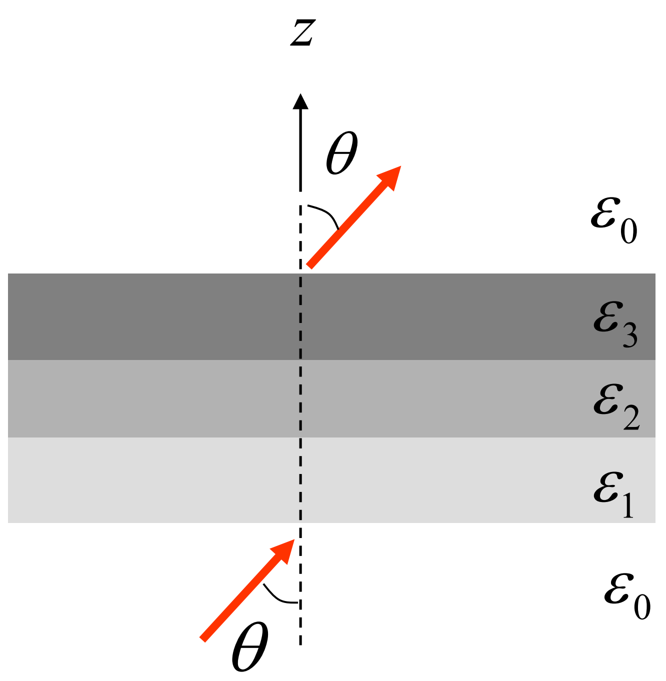
$$\underline{E}_x^{\text{ipw}} = E_0 \left(-\sin\phi\right) e^{+j\underline{k}\cdot\underline{r}}$$

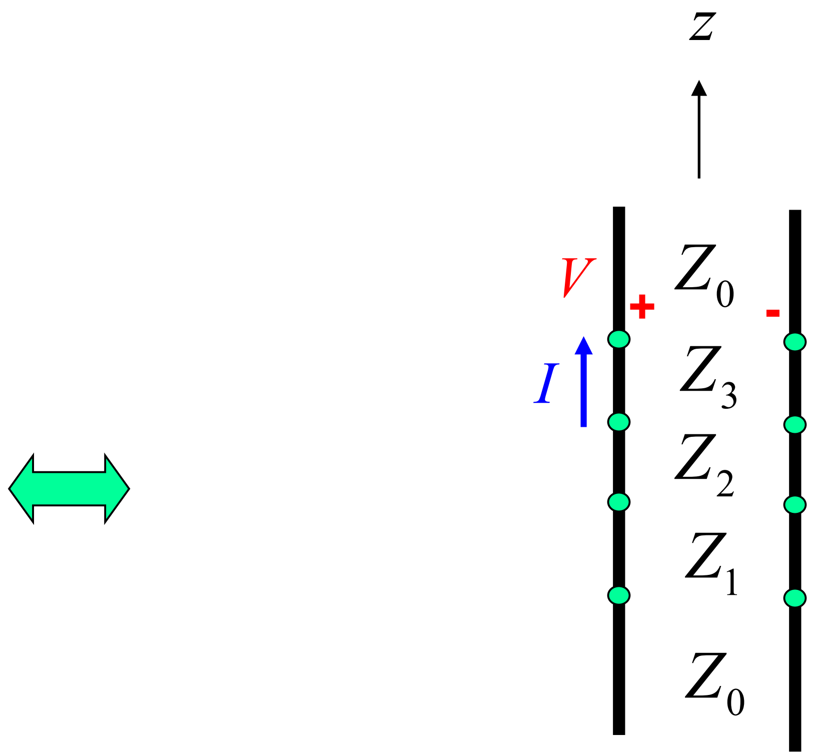


Transverse Equivalent Network (TEN)

$$E_x(x,y,z) = \psi_t(x,y)\, V(z)$$
$$H_y(x,y,z) = \psi_t(x,y)\, I(z)$$

$$\psi_t(x,y) = e^{j(k_x x + k_y y)}$$



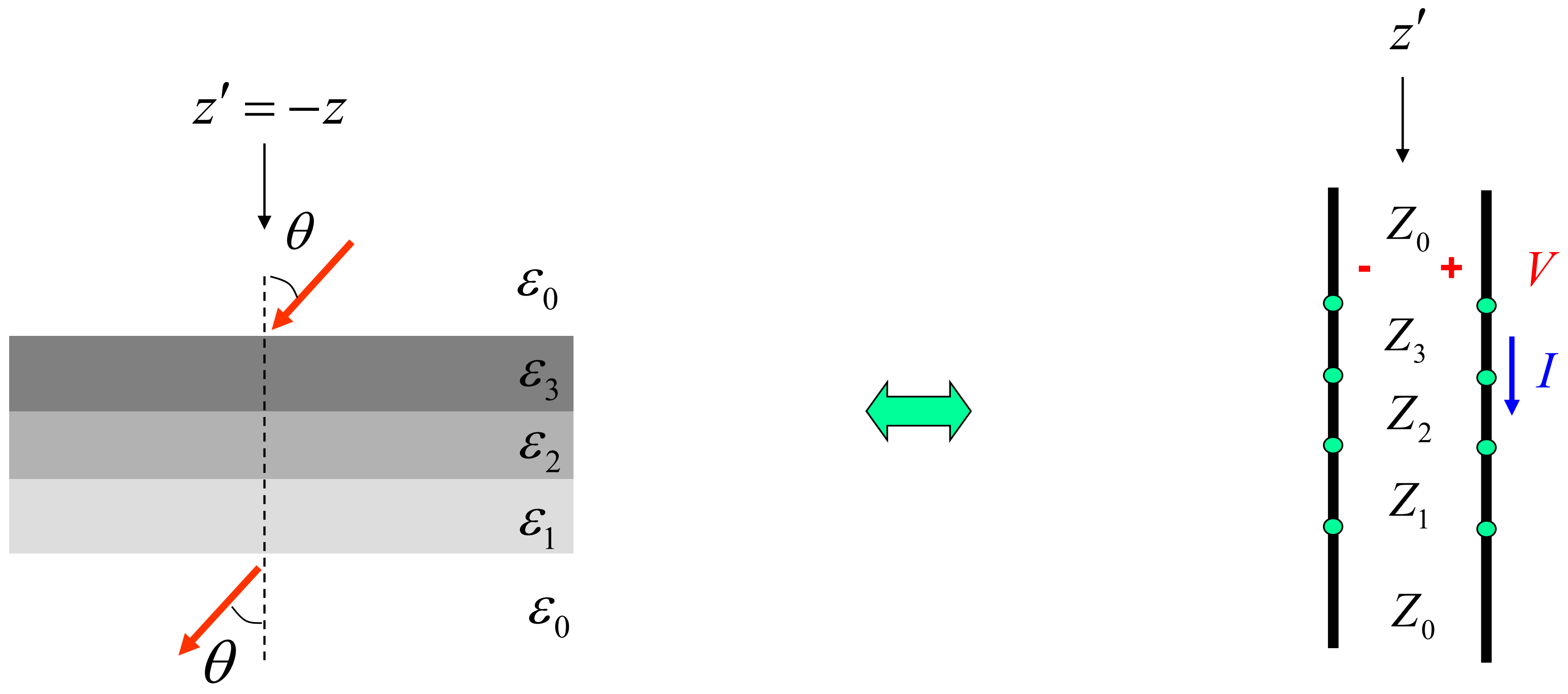


TEN (cont.)

For a wave traveling in the $z' = -z$ direction, we use:

$$E_x(x, y, z') = \psi_t(x, y)\; V(z')$$
$$-H_y(x, y, z') = \psi_t(x, y)\; I(z')$$

This is the situation for our incident wave.



TEN (cont.)

$$Z_i^{\mathrm{TM}} = \frac{k_{zi}}{\omega \varepsilon_i}$$

$$Z_i^{\mathrm{TE}} = \frac{\omega \mu_i}{k_{zi}}$$

where

$$k_{zi} = \sqrt{k_i^2 - k_x^2 - k_y^2}$$

and

$$k_{zi} = k_i \cos \theta_i$$

or

$$k_{zi} = k_i \sqrt{1 - \sin^2 \theta_i}$$

TEN (cont.)

From Snell's law: $(1)\sin\theta = n_i \sin\theta_i \qquad \left(n_i = \sqrt{\mu_{ri}\varepsilon_{ri}}\right)$

Hence, we have:

$$k_{zi} = k_i \cos\theta_i = k_i\sqrt{1-\sin^2\theta_i} = k_i\sqrt{1-\left(\frac{\sin\theta}{n_i}\right)^2} = k_0 n_i\sqrt{1-\frac{\sin^2\theta}{n_i^{\,2}}} = k_0\sqrt{n_i^{\,2}-\sin^2\theta}$$

Define:

$$N_i(\theta) \equiv \sqrt{n_i^2 - \sin^2\theta}$$

Then we have:

$$k_{zi} = k_0 N_i\left(\theta\right)$$

TEN (cont.)

Also, we have:

$$Z_i^{\mathrm{TM}} = \frac{k_{zi}}{\omega\varepsilon_i} = \frac{k_0 N_i(\theta)}{\omega\varepsilon_i} = \frac{k_0 N_i(\theta)}{\omega\varepsilon_0\varepsilon_{ri}} = \eta_0 \frac{N_i(\theta)}{\varepsilon_{ri}}$$

$$Z_i^{\mathrm{TE}} = \frac{\omega\mu_i}{k_{zi}} = \frac{\omega\mu_i}{k_0 N_i(\theta)} = \frac{\omega\mu_0\mu_{ri}}{k_0 N_i(\theta)} = \eta_0 \frac{\mu_{ri}}{N_i(\theta)}$$

General Medium

$$Z_i^{\mathrm{TM}} = \frac{\eta_0}{\varepsilon_{ri}} N_i(\theta)$$

$$Z_i^{\mathrm{TE}} = \frac{\eta_0\mu_{ri}}{N_i(\theta)}$$

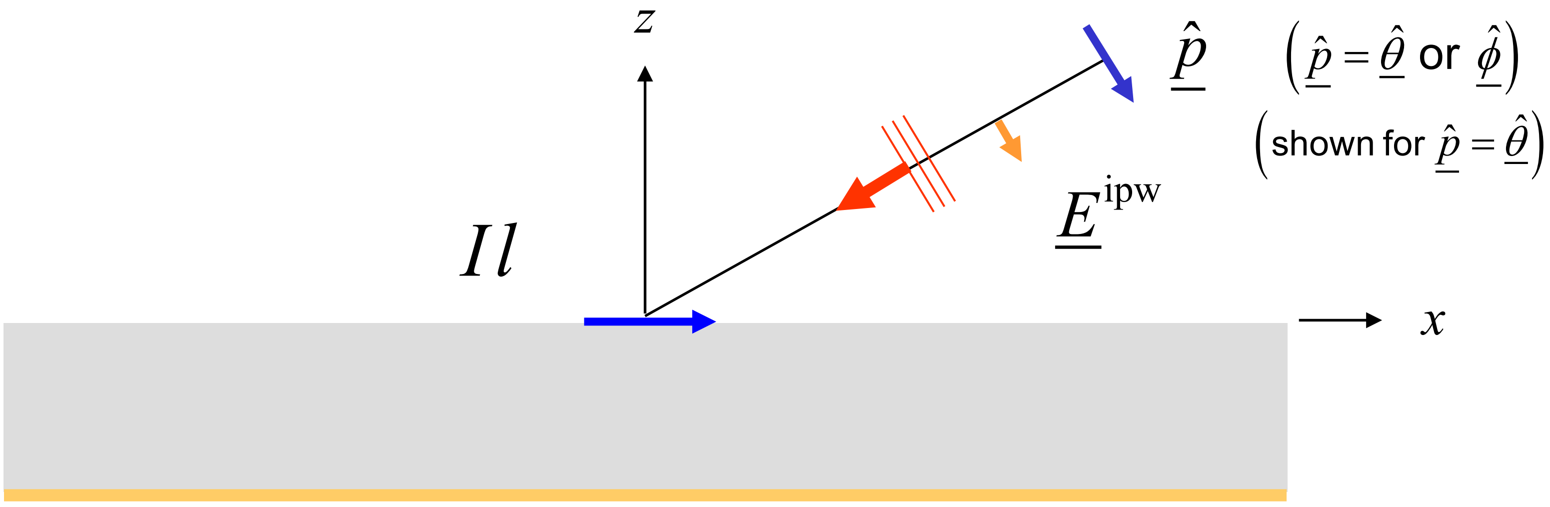
Free-space

$$N_i(\theta) \equiv \sqrt{n_i^2 - \sin^2\theta} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$Z_0^{\mathrm{TM}} = \eta_0 \cos\theta$$

$$Z_0^{\mathrm{TE}} = \frac{\eta_0}{\cos\theta}$$

Electric Dipole Source



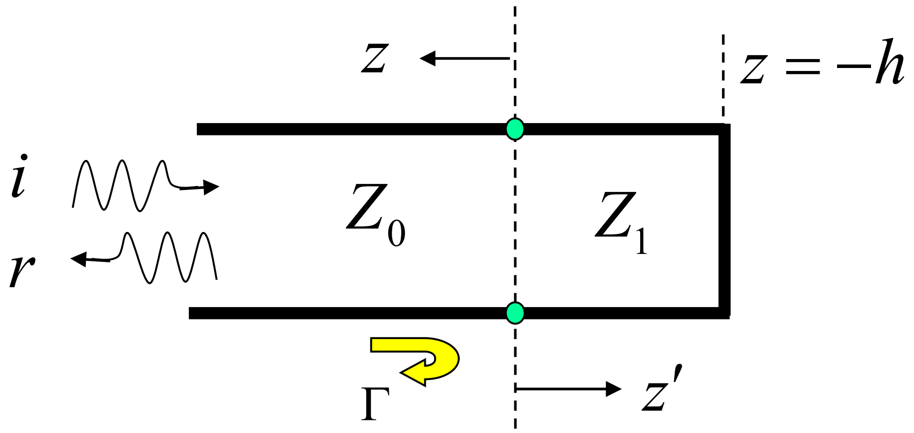
$$E_p^{\mathrm{hex}}(r,\theta,\phi)=\left[E_x^{\mathrm{ipw}}(0,0,0)+E_x^{\mathrm{rpw}}(0,0,0)\right]$$

$$p=\theta:\quad E_x^{\mathrm{ipw}}=E_0\cos\theta\cos\phi$$

$$p=\phi:\quad E_x^{\mathrm{ipw}}=E_0(-\sin\phi)$$

$$E_0=\frac{-j\omega\mu_0}{4\pi r}e^{-jk_0 r}$$

Electric Dipole Source: TEN



$$E_x \leftrightarrow V(z)$$

$$V^r(0) = V^i(0)\,\Gamma$$

so

$$E_x^{\text{rpw}}(0,0,0) = E_x^{\text{ipw}}(0,0,0)\;\Gamma$$

Electric Dipole Source: TEN (cont.)

The radiated field is then:

$$E_p^{\text{hex}}\left(r,\theta,\phi\right)=E_x^b\left(0,0,0\right)=E_x^{\text{ipw}}\left(0,0,0\right)\,(1+\Gamma)$$

$$p=\theta: \qquad \text{TM}_z \qquad \Gamma=\Gamma^{\text{TM}}$$

$$p=\phi: \qquad \text{TE}_z \qquad \Gamma=\Gamma^{\text{TE}}$$

We then have:

$$E_\theta^{\text{hex}}\left(r,\theta,\phi\right)=E_0\cos\theta\cos\phi\left[1+\Gamma^{\text{TM}}\right]$$

$$E_\phi^{\text{hex}}\left(r,\theta,\phi\right)=E_0\left(-\sin\phi\right)\left[1+\Gamma^{\text{TE}}\right]$$

Electric Dipole Source: TEN (cont.)

For the TM reflection coefficient we have:

$$\Gamma^{\mathrm{TM}} = \frac{Z_{\mathrm{in}}^{\mathrm{TM}} - Z_0^{\mathrm{TM}}}{Z_{\mathrm{in}}^{\mathrm{TM}} + Z_0^{\mathrm{TM}}}$$

where

$$Z_{\mathrm{in}}^{\mathrm{TM}} = jZ_1^{\mathrm{TM}} \tan(k_{z1}h)$$

so

$$Z_{\mathrm{in}}^{\mathrm{TM}} = j\left(\frac{\eta_0 N_1(\theta)}{\varepsilon_r}\right)\tan\left(k_0 N_1(\theta) h\right)$$

For the air region, we have:

$$Z_0^{\mathrm{TM}} = \eta_0 \frac{N_0(\theta)}{1} = \eta_0 \sqrt{1-\sin^2\theta} = \eta_0 \cos\theta$$

Electric Dipole Source: TEN (cont.)

Note that $$1+\Gamma^{\mathrm{TM}} = 1 + \frac{Z_{\mathrm{in}}^{\mathrm{TM}} - Z_0^{\mathrm{TM}}}{Z_{\mathrm{in}}^{\mathrm{TM}} + Z_0^{\mathrm{TM}}} = \frac{2Z_{\mathrm{in}}^{\mathrm{TM}}}{Z_{\mathrm{in}}^{\mathrm{TM}} + Z_0^{\mathrm{TM}}}$$

After simplifying, we obtain the following results:

$$1+\Gamma^{\mathrm{TM}}(\theta) = \frac{2}{1 - j\left(\dfrac{\varepsilon_r \cos(\theta)}{N_1(\theta)}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Similarly, for the TE case we have:

$$1+\Gamma^{\mathrm{TE}}(\theta) = \frac{2}{1 - j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Electric Dipole Source: Final Results

Hence, we have the following final results:

$$E_\theta^{\text{hex}}(r,\theta,\phi) = E_0 \cos\phi \; G(\theta)$$

$$E_\phi^{\text{hex}}(r,\theta,\phi) = E_0 \left(-\sin\phi\right) \; F(\theta)$$

$$G(\theta) = \cos\theta \left(1 + \Gamma^{\text{TM}}(\theta)\right)$$

$$F(\theta) = 1 + \Gamma^{\text{TE}}(\theta)$$

$$E_0 = \left(\frac{-j\omega\mu_0}{4\pi r}\right) e^{-jk_0 r}$$

Electric Dipole: Final Results (cont.)

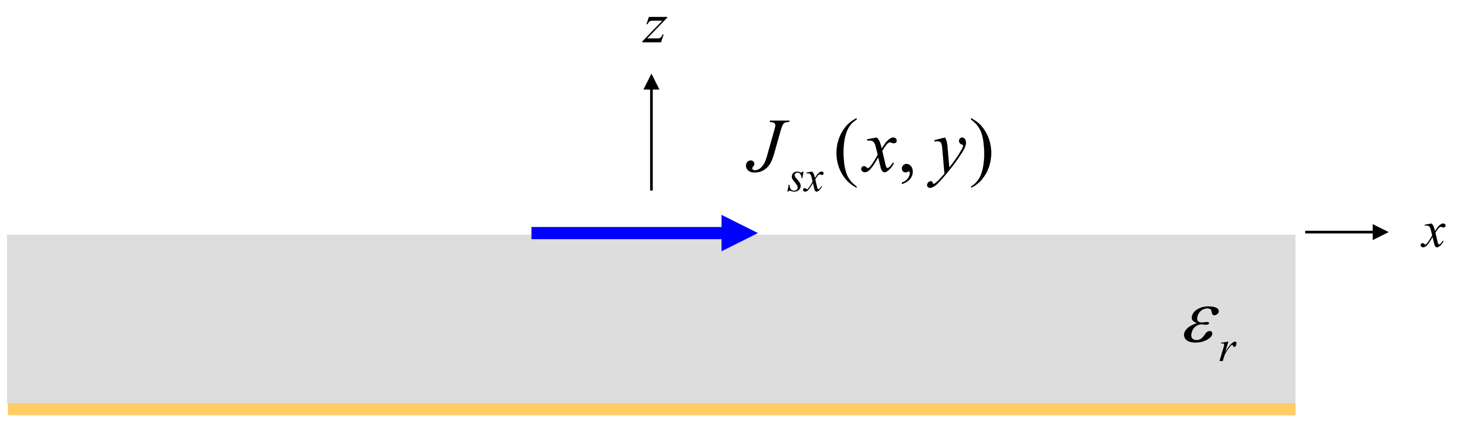
Results for a y-directed electric dipole:

$$E_{\theta}^{\text{hey}}(r,\theta,\phi) = E_0 \sin\phi \; G(\theta)$$

$$E_{\phi}^{\text{hey}}(r,\theta,\phi) = E_0 \cos\phi \; F(\theta)$$

We don't need this for modeling the radiation from the TM_{10} mode, however.

Far Field of Patch Current



$$E_p^{\text{patch}}(r,\theta,\phi) = \langle b, a \rangle$$

$$= \int_S J_{sx}(x,y)\, E_x^{\text{pw}}(x,y,z)\, dS$$

$$= E_x^{\text{pw}}(0,0,0) \int_S J_{sx}(x,y)\, e^{j(k_x x + k_y y)} dS$$

$$= E_x^{\text{pw}}(0,0,0)\, \tilde{J}_{sx}(k_x, k_y)$$

$$E_x^{\text{pw}}(x,y,z) = E_x^{\text{ipw}}(x,y,z) + E_x^{\text{rpw}}(x,y,z)$$

Far Field of Patch Current (cont.)

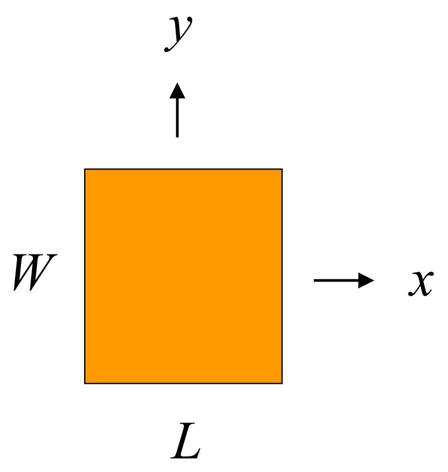
Hence, we have:

$$E_p^{\text{patch}}(r,\theta,\phi) = E_p^{\text{hex}}(r,\theta,\phi)\,\tilde{J}_{sx}(k_x,k_y)$$

$$k_x = k_0 \sin\theta \cos\phi$$

$$k_y = k_0 \sin\theta \sin\phi$$

hex = horizontal electric dipole in the x direction



The origin is at the center of the patch

Assume:

$$J_{sx}^{(1,0)}(x,y) = A_{10}^{J} \cos\left(\frac{\pi x}{L}\right)$$

Far Field of Patch Current (cont.)

For this patch current we have the following Fourier transform:

$$\tilde{J}_{sx}^{(1,0)}\left(k_x,k_y\right)=\int_S J_{sx}^{(1,0)}(x,y)\,e^{j(k_x x+k_y y)}dS$$

$$=\int_{-L/2}^{L/2}\cos\left(\frac{\pi x}{L}\right)e^{jk_x x}dx\int_{-W/2}^{W/2}e^{jk_y y}\;dy$$

$$=\left[\frac{\pi}{2}L\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]\left[W\mathrm{sinc}\left[k_y\frac{W}{2}\right]\right]$$

where $\mathrm{sinc}\left(x\right)\equiv\dfrac{\sin\left(x\right)}{x}$

Far Field of Patch Current (cont.)

Hence, we have:

$$\tilde{J}_{sx}^{(1,0)}\left(k_x,k_y\right)=A_{10}^{J}\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left[k_y\frac{W}{2}\right]\left[\frac{\cos\left(k_x\dfrac{L}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2-\left(k_x\dfrac{L}{2}\right)^2}\right]$$

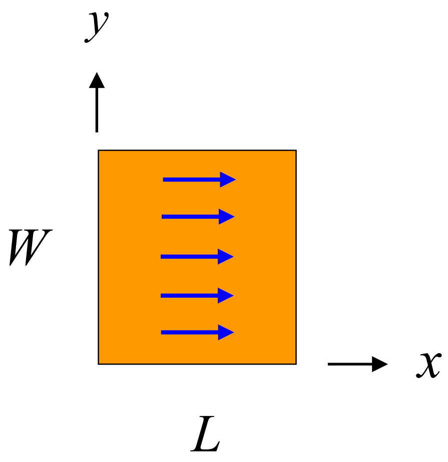
Summary

$$E_p^{\text{patch}}\left(r,\theta,\phi\right)=E_p^{\text{hex}}\left(r,\theta,\phi\right)\tilde{J}_{sx}^{(1,0)}(k_x,k_y)$$

$(p=\theta \text{ or } \phi)$

$$k_x = k_0 \sin\theta\cos\phi$$
$$k_y = k_0 \sin\theta\sin\phi$$

$$\tilde{J}_{sx}^{(1,0)}\left(k_x,k_y\right)=A_{10}^J\left(\frac{\pi}{2}WL\right)\operatorname{sinc}\left[k_y\frac{W}{2}\right]\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(k_x\frac{L}{2}\right)^2}\right]$$



$$E_\theta^{\text{hex}}(r,\theta,\phi)=E_0\cos\phi\; G(\theta)$$
$$E_\phi^{\text{hex}}(r,\theta,\phi)=E_0\left(-\sin\phi\right)\; F(\theta)$$

$$E_0=\left(\frac{-j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}$$

$$G(\theta)=\cos\theta\left(1+\Gamma^{\text{TM}}(\theta)\right)$$
$$F(\theta)=1+\Gamma^{\text{TE}}(\theta)$$

$$1+\Gamma^{\text{TM}}\left(\theta\right)=\frac{2}{1-j\left(\dfrac{\varepsilon_r\cos(\theta)}{N_1(\theta)}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

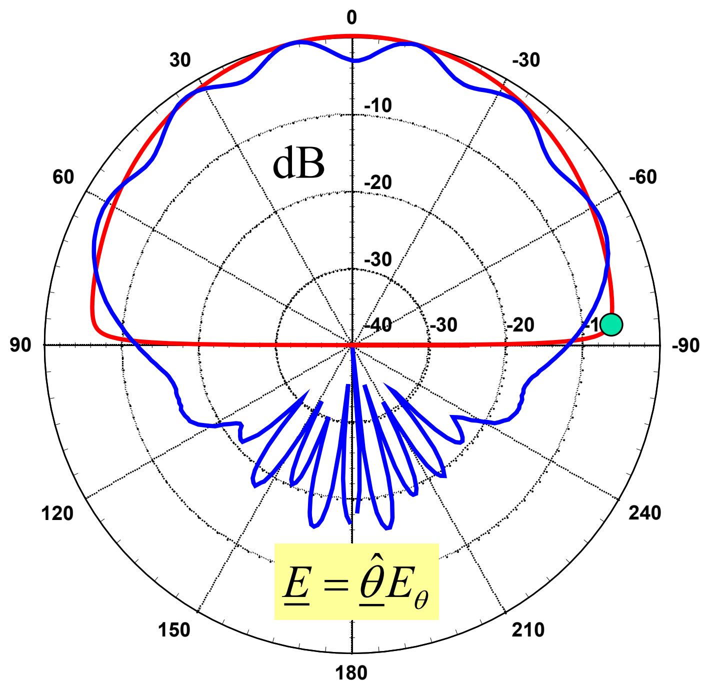
$$1+\Gamma^{\text{TE}}\left(\theta\right)=\frac{2}{1-j\left(\dfrac{N_1(\theta)\sec\theta}{\mu_r}\right)\cot\left(k_0 h N_1(\theta)\right)}$$

Assumption: $J_{sx}^{(1,0)}\left(x,y\right)=A_{10}^J\cos\left(\dfrac{\pi x}{L}\right)$

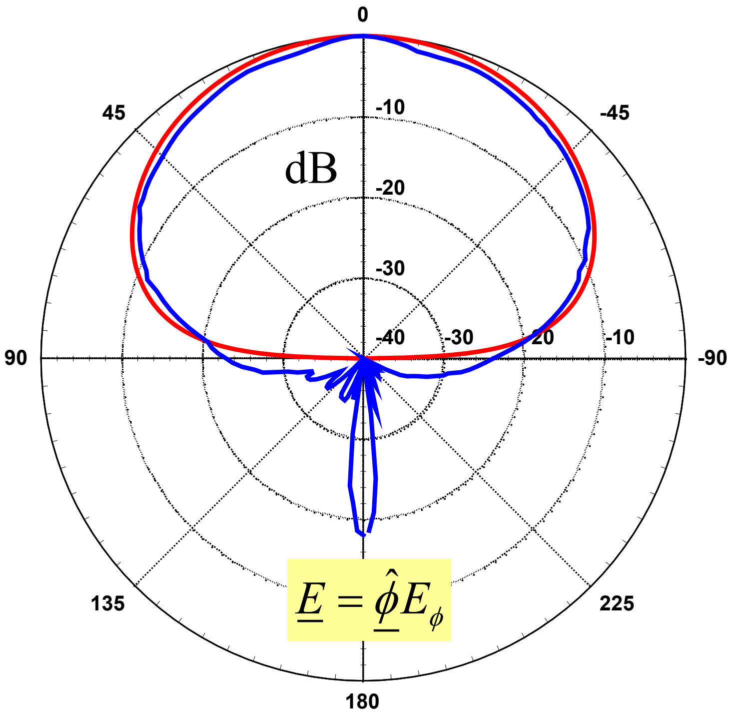
$$N_1(\theta)\equiv\sqrt{n_1^2-\sin^2\theta}\qquad n_1=\sqrt{\mu_r\varepsilon_r}$$

Summary

E-plane pattern ($\phi = 0^{o}$)



H-plane pattern ($\phi = 90^{o}$)



Red: infinite substrate and ground plane

Blue: 1 meter diameter ground plane

Comments:

- The E plane is broader than the H plane.
- The E-plane pattern "tucks in" and tends to zero at the horizon due to the presence of the infinite substrate (green dot). (As the substrate gets thinner, the tuck-in point approaches 90°.)