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Modified Bessel Functions
and
Kelvin Functions
Modified Bessel differential equation:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0 \]

This comes from the Bessel differential equation:

\[ z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0 \] \hspace{1cm} \text{(Bessel DE)}

Set \( z = ix; \ dz = idx \):

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0 \] \hspace{1cm} \text{(modified Bessel DE)}

The modified Bessel functions are Bessel functions of imaginary argument. \( \left( J_{\nu}(ix), Y_{\nu}(ix) \right) \)
Modified Bessel Function of the First Kind

Definition:

\[ I_\nu(x) \equiv (-i)^\nu J_\nu(ix) \quad (I_\nu \text{ is a real function of } x.) \]

To see that \( I_\nu \) is a real function, use the Frobenius solution for \( J_\nu \):

\[
J_\nu(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!} \left(\frac{ix}{2}\right)^{\nu+2k}
\]

Note: \( (i)^{2k} = (-1)^k \)

Frobenius series solution for \( I_\nu \):

\[
I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!(\nu+k)!} \left(\frac{x}{2}\right)^{\nu+2k}
\]

For \( \nu = n, \) an integer, \( I_{-n}(x) = I_n(x) \quad \left\{ I_{-n}(x) = (i)^{-n} J_{-n}(ix) = (i)^{-n} (-1)^n J_n(ix) = (i)^{-n} (-1)^n \frac{I_n(x)}{(-i)^n} = I_n(x) \right\} \)
Second Solution of Modified Bessel Equation

For $\nu \neq n$, the modified Bessel function of the 2nd kind is defined as:

$$K_\nu(x) \equiv \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi}$$

For $\nu = n$ (an integer):

$$K_n(x) \equiv \lim_{\nu \to n} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi}$$
The modified Bessel functions are related to the regular Bessel functions as

\[ I_\nu(x) \equiv (-i)^\nu J_\nu(ix) \]

\[ K_\nu(x) \equiv \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi} = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \quad \text{(proof omitted*)} \]

**Note:** The added factors in front ensure that the functions are real.

*comes from: \[ H_\nu^{(1)}(z) \equiv J_\nu(z) + iY_\nu(z) \quad \text{&} \quad Y_\nu(z) \equiv \frac{J_\nu(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)} \]
For small arguments we have:

\[ I_{\pm \nu} (x) \xrightarrow{x \to 0} \frac{1}{(\pm \nu)!} \left( \frac{x}{2} \right)^{\pm \nu} \]

\[ K_0 (x) \xrightarrow{x \to 0} - \ln \left( \frac{x}{2} \right) - \gamma \]

\[ K_\nu (x) \xrightarrow{x \to 0} \frac{(\nu - 1)!}{2} \left( \frac{x}{2} \right)^{-\nu} \]
Large Argument Approximations

For large arguments we have:

\[ I_ν(x) \xrightarrow{x \to \infty} \frac{e^x}{\sqrt{2\pi x}} \quad \text{exponentially grows!} \]

\[ K_ν(x) \xrightarrow{x \to \infty} \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{exponentially decays!} \]
Plots of Modified Bessel Functions for Real Arguments

The $I_n$ functions increase exponentially. They are finite at $x = 0$. 
The $K_n$ functions decrease exponentially. They are infinite at $x = 0$. 
Some recurrence relations are:

\[ I_{v-1}(x) - I_{v+1}(x) = \frac{2v}{x} I_v(x) \]
\[ I_{v-1}(x) + I_{v+1}(x) = 2I'_v(x) \]
\[ K_{v-1}(x) - K_{v+1}(x) = -\frac{2v}{x} K_v(x) \]
\[ K_{v-1}(x) + K_{v+1}(x) = -2K'_v(x) \]
A Wronskian identity is:

\[
W \left[ I_v, K_v \right] = I_v(x)K'_v(x) - I'_v(x)K_v(x) = -\frac{1}{x}
\]
The Kelvin functions are defined as

\[
\begin{align*}
\text{Ber}_v(x) & \equiv \text{Re}\left( J_v\left(x e^{i3\pi/4}\right) \right) \\
\text{Bei}_v(x) & \equiv \text{Im}\left( J_v\left(x e^{i3\pi/4}\right) \right) \\
\text{Ker}_v(x) & \equiv \text{Re}\left( K_v\left(x e^{i\pi/4}\right) \right) \\
\text{Kei}_v(x) & \equiv \text{Im}\left( K_v\left(x e^{i\pi/4}\right) \right)
\end{align*}
\]

**Note:** These are important for studying the fields inside of a conducting wire.
Kelvin Functions (cont.)

The Ber functions increase exponentially. They are finite at $x = 0$. 

![Graph showing Kelvin functions Ber0, Ber1, and Ber2]
The Bei functions increase exponentially. They are finite at $x = 0$. 
Normalizing makes it more obvious that the Ber and Bei functions increase exponentially and also oscillate.

\[
\frac{\text{Ber}_0(x)}{1/\sqrt{x} e^{x/\sqrt{2}}}
\]
Normalizing makes it more obvious that the Ber and Bei functions increase exponentially and also oscillate.

\[
\frac{\text{Bei}_0(x)}{\frac{1}{\sqrt{x}} e^{x/\sqrt{2}}}
\]
The Ker functions decay exponentially.
They are infinite at $x = 0$. 

$\text{Ker}_0$  
$\text{Ker}_1$
The Kei functions decay exponentially. They are infinite at $x = 0$. 

![Graph showing the behavior of Kei functions](image)
Normalizing makes it more obvious that the Ker and Kei functions decrease exponentially and also oscillate.

\[
\frac{\text{Ker}_0 (x)}{\frac{1}{\sqrt{x}} e^{-x/\sqrt{2}}}
\]
Normalizing makes it more obvious that the Ker and Kei functions decrease exponentially and also oscillate.

\[
\frac{\text{Kei}_0(x)}{\frac{1}{\sqrt{x}} e^{-x/\sqrt{2}}}
\]