ECE 6382

Fall 2023

Homework Set #2

Homework problems are from *Mathematical Methods for Physicists*, 7^{th} Ed., by Arfken, Weber, and Harris.

Chapter 11, Section 3, Cauchy's Integral Theorem

Prob. 11.3.3

In addition to parts (a) and (b), also do the following: Evaluate the integral by using the indefinite integral. Should you get the same answer?

Hint for part (a): Consider the parameterization $z = (1-t)z_a + tz_b$, $0 < t < 1$, where z_a and z_b are the lower and upper limits of integration, respectively. Hint for part (b): Consider the parameterization $z = 5e^{i\theta}$, where θ is the variable of integration.

Prob. 11.3.6

Hint: On the horizontal parts, parameterize in terms of x. On the vertical parts, parameterize in terms of y.

Chapter 11, Section 4, Cauchy's Integral Formula

Prob. 11.4.2
Prob. 11.4.3
Prob. 11.4.7

Numerical Integration in the Complex Plane

Prob. N1

Evaluate exactly the following integral, by using the indefinite integral:

$$I = \int_{1+i}^{20+30i} \sin z \, dz \, .$$

Report the answer in rectangular form, keeping at least 6 significant figures for both the real and imaginary parts of the answer. (Note that the indefinite integral at the upper limit will have a very large magnitude, so be careful to evaluate it accurately!)

Prob. N2

Use the parameterization $z = a + (b-a)t,\ 0 \le t \le 1$ to evaluate the following integral by numerically integrating in the variable t from 0 to 1. You may use Matlab or any numerical package that you wish to perform the numerical integration in t for you. Please make sure that your answer is accurate to 6 significant figures.

$$I = \int_{1+i}^{2+3i} \sin\left(z^2\right) dz .$$

Prob. N3

Evaluate numerically the above integral in Prob. N2 by using a straight-line path and the midpoint rule. Use the following number of segments: N = 1, 2, 4, 8, 16, 32, 64. Make a table that summarizes the real and imaginary parts of the result for each N (denoted here as I_N) and the percent error between I_N and the exact result I. The percent error is defined as

$$\text{err\%} \equiv 100\left(\frac{|I - I_N|}{|I|}\right).$$

(It is suggested that you program the midpoint formula in Matlab or use some other programming language to save you time.)

Prob. N4

Repeat Prob. N3 above using 6-point Gaussian quadrature. Make the same type of table, for N = 1, 2, 4, 8, 16, 32, 64.

(It is suggested that you program the 6-point Gaussian Quadrature formula in Matlab or use some other programming language to save you time.)

Conformal Mapping

Prob. C1

We wish to find the electrostatic potential in the upper-half region ($y > 0$) of the (x,y) plane in the problem shown below. The x axis for $x > 1$ is kept at a potential of zero volts. The x axis in the region $x < -1$ is also kept at a potential of zero volts. The x axis in the region -1

$< x < 1$ is kept at a potential of 1 volt. There are insulating gaps (shown as small circles) at $x = 1$ and $x = -1$ (to avoid shorting the three different metal plates).

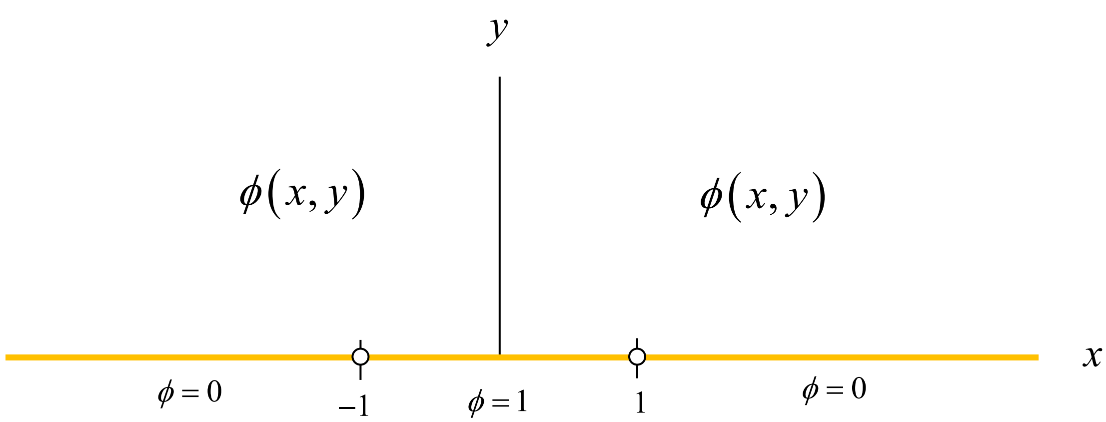
Consider the mapping

$$w = \ln\left(\frac{z-1}{z+1}\right) = \ln(z-1) - \ln(z+1).$$

Show that in the w plane the new problem consists of an infinite horizontal strip region defined by $0 < v < \pi$ and $-\infty < u < \infty$. Draw the figure in the w plane and clearly label the boundary conditions for the new problem in the w plane.

Then find the solution to the potential $\psi(u,v)$ in the w plane. From this, find the solution $\phi(x,y)$ to the original problem shown below.

Note: Assume the principal branch, so that that the argument of a positive real number is zero, and the argument of a negative real number is π.



Prob. C2

We wish to find the electrostatic potential in the first quadrant of the (x,y) plane in the problem shown below. The positive y axis is kept at a potential of zero volts, and the positive x axis for $x > 1$ is kept at a potential of one volt. The part of the x axis in the region $0 < x < 1$ is a "perfect magnetic conductor" (PMC) wall, meaning that the electric flux must be parallel to the boundary in this region (no electric flux can enter the PMC boundary). In this region the potential obeys a Neumann boundary condition.

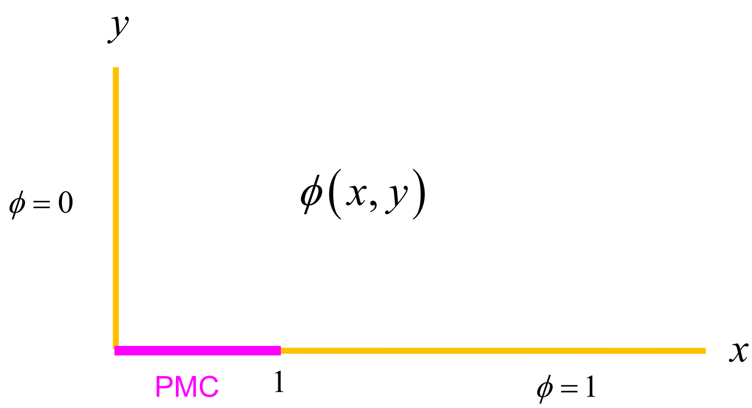
Use the mapping

$$z = \sin w$$

and show the new problem in the w plane. Draw the problem in the w plane and label the boundary conditions for each part of the problem. (Hint: consider what happens to the sin function when $w = u$, $w = iv$, or $w = \pi/2 + iv$.)

Then find the solution to the potential $\psi(u,v)$ in the w plane. From this, find the solution $\phi(x,y)$ to the original problem shown below.

Then find the magnitude of the electric field at any point (x, y) in the z plane, where $z = x + iy$. Do this by first finding the magnitude of the electric field vector in the w plane at the corresponding point (u, v), where $w = u + iv$

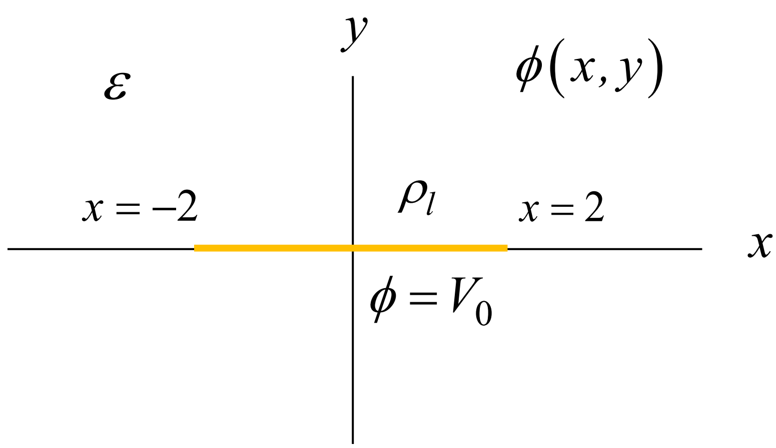


Prob. C3

Consider the strip problem considered in Notes 5, where the solution in the w plane was

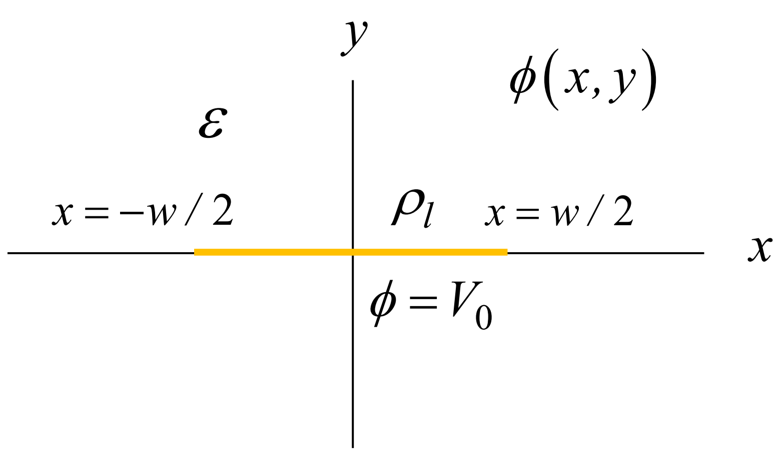
$$\phi(x, y) = 1 - \ln(R(x, y)).$$

Generalize this to give the solution for the potential when the strip is at V_0 volts (instead of 1 volt) and the total charge on the strip (for a one-meter length in the z direction) is ρ_l. This problem is shown below. Hint: Remember that the total charge on the conductor is the same in the z and w planes. Can you find the total charge on the conductor in the w plane? See the class notes for a discussion of this.



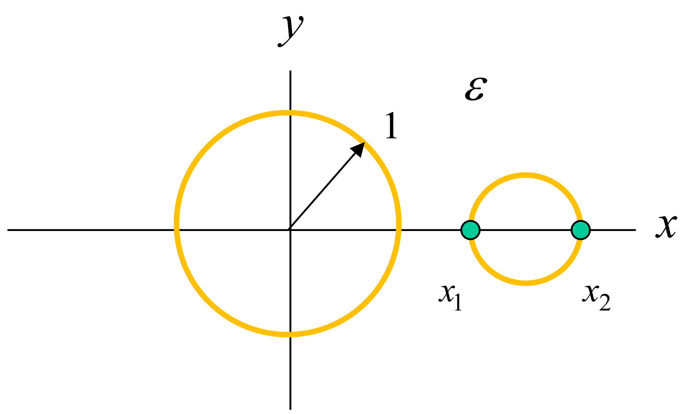
Prob. C4

For the previous problem, show how to generalize your result for the case where the strip now extends from $-w/2$ to $+w/2$ as shown below. The strip is still at V_0 volts and the total charge on the strip (for a one-meter length in the z direction) is still ρ_l. Hint: Consider as a new mapping function $w = f(z)$ that is a simple linear mapping (which is a conformal mapping) that will map from the desired geometry shown below to the one shown above in the previous problem. (It is now convenient to think of the answer to the previous problem $\phi(x,y)$ as being relabeled as $\psi(u,v)$, with (x,y) in the previous problem being relabeled as (u,v).



Prob. C5

Consider the problem of the two circular wires considered in Notes 5, as shown below.



Assume that the right conductor has a total charge of ρ_l [C/m] and the left conductor has a total charge of -ρ_l [C/m]. Also assume that the left conductor is at a potential of zero volts.

Assume we are at a given point (x,y) that is outside of the two conductors. Show how to find the potential and the magnitude of the electric field at this point.

Hint: Please note that the potential inside a coaxial cable type of problem can be found by applying boundary conditions to the solution of the Laplace equation. For a coaxial cable type of problem (no azimuthal angle variation of the potential), the general solution to the Laplace equation (assuming we are in the w plane) is of the form

$$\psi(R) = -A_1 \ln(R) + A_2 \,.$$

Also, recall that you can find the electric field in a coaxial cable type of problem from the potential by using (in the w plane)

$$\underline{E} = -\nabla \Psi \,.$$