ECE 6382

Fall 2023

Homework Set #3

Homework problems are from *Mathematical Methods for Physicists*, 7th Ed., by Arfken, Weber, and Harris.

Branch Cuts

Prob. B1

Consider the function

$$f(z) = z^{1/2} .$$

Assume that we choose the first branch of the function to be such that $f(1) = 1$. (In other words, the angle θ (the argument of z) is zero at the point $z = 1$.) Find the value of $f(z)$ for the first branch of the function at the point $z = -1 - 2i$, if the branch cut is chosen as indicated below.

(a) The branch cut is on the negative real axis (so that $f(z) = \sqrt{z}$).

(b) The branch cut is on the negative imaginary axis.

(c) The branch cut is on the positive imaginary axis.

(d) The branch cut is defined by the curve $y = -x^2$, $x \le 0$.

(e) For part (b), what would the value of the second branch of the function be?

(f) Assume now that the branch cut is on the negative imaginary axis, as in part (b). What would the function be at the point $z = -i$? (Note: This may be a trick question!)

Prob. B2

Consider the function

$$f(z) = \left(z^2 - 1^2\right)^{1/2} .$$

Assume that we take the branch of the function where

$$f(2) = \sqrt{3} .$$

Assume that we use a branch cut that is a straight horizontal line on the x axis that connects the points $z = -1$ to $z = +1$.

Identify and label the regions of the complex plane where $\operatorname{Re} f(z) > 0$.

Prob. B3

Consider the function

$$f(z) = \left(z^2 - a^2\right)^{1/2},$$

where

$$a = a' - ia'',$$

with a' and a'' both positive real numbers. (Note: In the class notes, we considered the case where $a = 1$, a real number.)

The Sommerfeld branch cuts are defined by $\operatorname{Re}(f(z)) = 0$. Show that the branch cuts are described by

$$xy = -a'a'' \text{ and } x^2 - y^2 < a'^2 - a''^2 .$$

Draw a picture of what the Sommerfeld branch cuts look like, assuming that

$$a = 1 - i .$$

Prob. B4

Consider the function

$$f(z) = -i\left(z^2 - 1\right)^{1/2}.$$

Assume that we choose Sommerfeld branch cuts and that on the top sheet of the Riemann surface we have

$$f(2) = -i\sqrt{3} .$$

Find the value of $f(z)$ at each point below, assuming that we start at a point given by $z = 0.5 + 0.5i$ on the top sheet, and then proceed to the next point by moving <u>continuously</u> on the Riemann surface, in either a vertical or a horizontal direction (whichever direction takes you to the next point). Put your answers in rectangular form.

a) $0.5 + 0.5i$
b) $0.5 - 0.5i$
c) $-0.5 - 0.5i$
d) $-0.5 + 0.5i$
e) $0.5 + 0.5i$

Note: On the Riemann surface, what were branch cuts in the complex plane are now really the escalators that connect the two sheets. So we really have Sommerfeld (hyperbolic) escalators.

Prob. B5

Repeat problem B4, assuming that to get to each point (a-e) we stay on the <u>top</u> sheet of the Riemann surface. (Note that the function $f(z)$ no longer stay continuous if you stay on a given sheet of the Riemann surface.)

Prob. B6

Repeat problem B4, assuming that the branch cuts are <u>vertical</u>, in the second and fourth quadrants.

Prob. B7

Draw a cross-sectional sketch (showing the escalators) of what a Riemann surface would look like for the function

$$f(z) = z^{1/3}.$$

Taylor and Laurent Series

Prob. 11.5.1
Prob. 11.5.2
Prob. 11.5.6
Prob. 11.5.7
Prob. 11.5.8

Analytic Continuation

Prob. A1

Using a computational tool such as MATLAB, Mathematica, etc., make a plot of the function $\ln(-1+i-is)$ vs. the (real) parameter s, for $0 < s < 2$. (Assume that the principal branch is chosen for the ln function, so than $\ln(z) = \mathrm{Ln}(z)$). Then make the same type of plot for the "extended" ln function that is the <u>analytic continuation</u> of the usual ln function into the third quadrant. For each plot, show the real and imaginary parts of the function vs. s.

Prob. A2

Using a computational tool such as MATLAB, Mathematica, etc., make a plot of the Y_0 Bessel function $Y_0(-1+i-is)$ vs. the (real) parameter s, for $0 < s < 2$. Then make the same type of plot for the "extended" Y_0 function that is the analytic continuation of the usual Y_0 function into the third quadrant. For each plot, show the real and imaginary parts of the function vs. s.