ECE 6382

Fall 2023

Homework Set #5

Homework problems are from *Mathematical Methods for Physicists*, 7^{th} Ed., by Arfken, Weber, and Harris.

Asymptotic Series

Prob. 12.6.1 Obtain the complete asymptotic expansion. (Please see Note 1 below.)
Prob. 12.6.3 Obtain the complete asymptotic expansion. (Please see Note 2 below.)

Prob. A1

Use integration by parts to asymptotically evaluate the following integral:

$$I(\Omega) = \int_0^1 e^{-x} \cos(\Omega x)\, dx.$$

Finding the leading term of the asymptotic expansion is sufficient.

Prob. A2

Consider the function studied in Notes 13,

$$f(z) \equiv e^z E_1(z),$$

Where the exponential integral function is defined as

$$E_1(z) \equiv \int_z^\infty \frac{e^{-t}}{t}\, dt.$$

The asymptotic series is

$$f(z) \sim \frac{1}{z} - \frac{1}{z^2} + \ldots + \frac{(-1)^{n-1}(n-1)!}{z^n} + \ldots$$

As derived in Notes 13, after keeping N terms in the series, an asymptotic estimate for the error is the last term that is neglected, and hence (assuming that z is real here)

$$\text{Error}_N \approx \frac{(-1)^N N!}{x^{N+1}}.$$

(a) If you keep only the leading term of the asymptotic series for $F(z)$, what would you estimate the error in your asymptotic series to be when $x = 10$? What about $x = 100$?

(b) If $x = 10$, how many terms would you estimate would be the optimum number to keep in the asymptotic series to get the best possible approximation of $f(10)$? Make a table of values for the terms in the series to justify your answer.

The Gamma Function

Prob. 13.1.1 (Please see Note 3 below.)

Prob. G1

Using the Sterling series for Ln $\Gamma(z)$ (please see the formulas in Notes 14), calculate an approximate value for 1000! Put your answer in "scientific notation." Your answer should be in the form 1.2345×10^{1234} where you keep 5 significant figures in the mantissa (the number in front of the exponential term), and 10 is raised to an integer power. (The number 1.2345 and the integer 1234 are not the correct values – these are just to show you the correct format.)

(Please see Note 4 below.)

The Stationary Phase Method

Prob. SP1

Determine the leading term of the asymptotic expansion of the following integral using the stationary phase method:

$$I(\Omega) = \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \sin(\theta/4)\, e^{j\Omega\cos(\theta)}\, d\theta\,.$$

The Method of Steepest Descent

Prob. 12.7.2 (Only part (a), the integral with the cosine in it.) (Please see Note 5 below.)

Prob. SDP1

The modified Bessel function of the first kind has an integral definition that is

$$I_0(\Omega) = \frac{1}{2\pi} \int_0^{2\pi} e^{\Omega \sin\theta} \, d\theta \, .$$

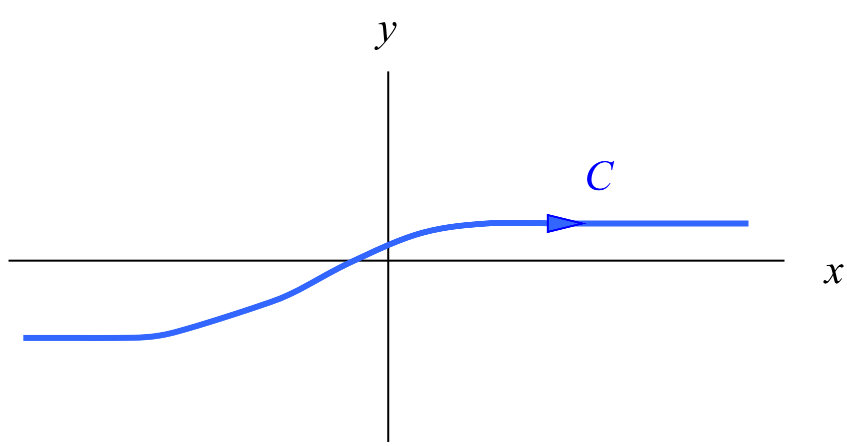
Use the method of steepest descent to asymptotically evaluate the function $I_0(\Omega)$ for large Ω. (That is, find the leading term in the asymptotic expansion of the integral.)

Prob. SDP2

Evaluate the integral

$$I(\Omega) = \int_C \frac{1}{z} e^{j\Omega\left(\frac{z^2}{2} - z\right)} dz$$

For $\Omega >> 1$, where C is the contour extending from ($-\infty - j2$) to ($-\infty + j1$), going above the origin as shown below. Carefully discuss how you are treating any singularities that are present when you deform the integration path to the SDP.



Prob. W1

Determine the first two leading terms of the asymptotic approximation to the following integral, as Ω becomes large, using Watson's Lemma as discussed in Notes 15.

$$I(\Omega) = \int_{-1}^{1} e^{s} \, e^{-\Omega s^2} \, ds$$

This is the form of Watson's Lemma that is used for evaluating integrals of the form

$$I(\Omega) = \int_{-\infty}^{\infty} h(s)\, e^{-\Omega s^2}\, ds,$$

where $h(s)$ is analytic at $s = 0$:

Prob . W2

Determine the complete asymptotic expansion of the following integral using Watson's Lemma as discussed in Notes 16.

$$I(\Omega) = \int_0^{\infty} e^{-t} \sin\left(\pi + \frac{t^2}{\Omega^2}\right) dt.$$

This is the form of Watson's Lemma that is used for evaluating integrals of the form

$$I(\Omega) = \int_0^{\infty} f(s)\, e^{-\Omega s}\, ds,$$

where $f(s)$ may have a branch point at $s = 0$.

(Please see Note 6 below.)

NOTES

Note 1: Try using the substitution $t = u^2$. Also, note that the integral from zero to x can be written as the integral from zero to infinity minus the integral from x to infinity. Note also that $C(\infty) = 1/2$ and $S(\infty) = 1/2$. (This can be found from a math handbook.)

Note 2: Try using the substitution $u = t^2$. Also, take note of the hint given in the book.

Note 3: Try using integration by parts.

Note 4: It might be helpful to use the following property:

$$e^x = 10^y, \text{ where } y = x \log_{10}(e).$$

Note 5: Try extending the path symmetrically from $-s$ to s first, and then try the substitution $z = x/s$. Or, you can try using the hint in the book.

Note 6: Try using a simple substitution to first put the integral into a form that can be analyzed using Watson's Lemma.