ECE 6382

Fall 2023

Homework Set #6

Wronskians

Prob. W1

Consider the following functions:

$$f(x) = \sin(x)$$
$$g(x) = x^3.$$

1) Are the two functions linearly dependent or linearly independent on the interval $x \in (-\infty, \infty)$?

2) Calculate the Wronskian of these two functions. Can the Wronskian be used to conclude anything about whether or not the two functions are linearly independent on the interval? If so, please indicate what the Wronskian tells us about the functions. If not, explain why.

Prob. W2

Repeat Prob. W1 for the following two functions:

$$f(x) = \begin{cases} (1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} 1-(x-3)^2, & 2 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Prob. W3

Repeat Prob. W1 for the following two functions:

$$f(x) = \begin{cases} 1-x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} -(1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sturm-Liouville Theory

Prob. SL1

Consider the matrix [M] defined as follows:

$$[M] = \begin{bmatrix} 1 & 1+i & 1+2i \\ 1-i & 2 & 1+3i \\ 1-2i & 1-3i & 3 \end{bmatrix}.$$

Note that this matrix is Hermetian (the matrix is equal to the conjugate of its transpose) and therefore self-adjoint.

a) Use any software package that you prefer (MATLAB, Mathcad, Maple, Mathematica, etc.) to find the eigenvalues of the matrix [M]. Verify that the eigenvalues are all real. Label your eigenvalues in descending order, so that the first one is the one that is the most positive.

b) Use any software package that you prefer (MATLAB, Mathcad, Maple, Mathematica, etc.) to find the eigenvectors of the matrix M. Take the eigenvectors to be normalized so that the first element is a real number. That is, divide each eigenvector by its first element to get a normalized eigenvector. Then further normalize each eigenvector so that they are all of unit magnitude (any eigenvector dot multiplied with its conjugate is unity).

Also, numerically verify that the inner product between any eigenvector and a different one is zero (there are three such combinations).

Prob. SL2

a) Construct the matrix [E], where the columns of [E] are the eigenvectors of the matrix [M] found in Prob. SL1. Verify numerically (using any software that you wish) that the matrix [E] is a unitary matrix (the conjugate of its transpose is equal to its inverse). That is, verify numerically that

$$[E]\left[E^{t*}\right] = [I],$$

where [I] is the identity matrix.

b) Verify numerically (using any software that you wish) that the following is true:

$$[M] = [E][D]\left[E^{t*}\right],$$

where [E] is a matrix whose column vectors are the eigenvectors, and [D] is a diagonal matrix whose diagonal elements are the eigenvalues.

Green's Functions

Prob. G1

Find the Green's function for the differential equation

$$u''(x)+k^2u(x)=f(x),$$

with

$$u'(a)=u'(b)=0.$$

Use "method 1" (i.e., representing the solution in terms of the solution to the homogenous equation).

Prob. G2

Find the Green's function for the differential equation

$$u''(x)+k^2u(x)=f(x),$$

with

$$u'(a)=u'(b)=0.$$

Use "method 2" (i.e., representing the solution in terms of the eigenfunctions to the following eigenvalue problem:

$$u''(x)+k^2u(x)=\lambda u(x).$$

Note: You can take the last term on the left-hand side across the equal sign and lump it together with the term on the right-hand side, and call this new term $-\lambda'^2u(x)$, where

$$\lambda'^2 \equiv k^2-\lambda.$$

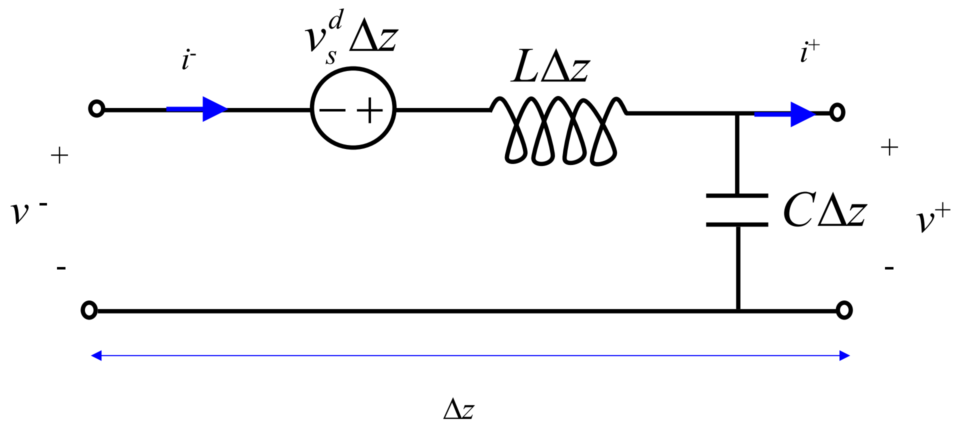
Also, note that it might be easier to find the eigenfunctions if you first choose $a = 0$, and then think about how to modify the eigenfunctions for $a > 0$.

Prob. G3

Consider a transmission line that has a distributed series voltage source on the line (instead of a distributed parallel current source, as was done in the class notes). Please see the figure below. Show that the differential equation for the current at any point on the line is

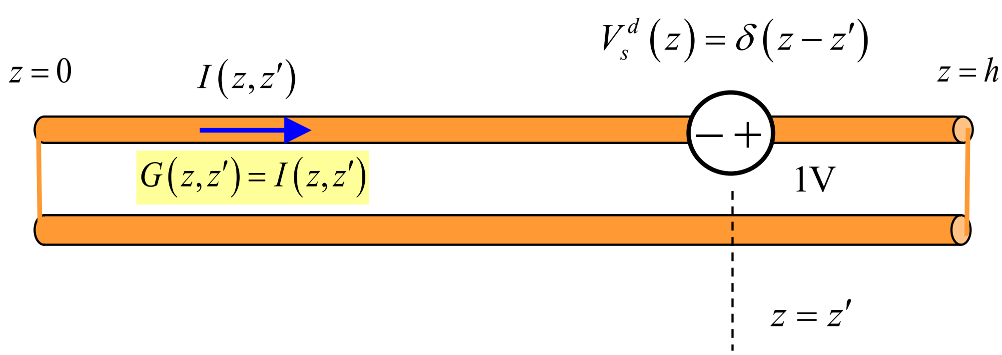
$$\frac{d^2I(z)}{dz^2}+k_z^2I(z)=-j\omega CV_s^d(z),$$

where V_s^d is the series distributed voltage source (V/m) along the line. (Please refer to the Appendix in Notes 19 for the general Telegrapher's equations when distributed sources are included on the transmission line.)



Prob. G4

Consider a transmission line that runs from $z = 0$ to $z = h$. There are short circuits at both ends, as for the problem that was studied in Notes 19. As a continuation of the previous problem, find the Green's function for the current due to a series distributed voltage source on the short-circuited transmission line, using method 1. This Green's function gives the current $I(z,z')$ at z due to a 1V series voltage source at z' (please see the figure below).



Prob. G5 (optional, not to be turned in)

In the class notes a general method for constructing a Green's function (method 1) was used to solve for the Green's function that gives the voltage on a short-circuited transmission line due to a 1A parallel current source. Solve for this same Green's function by starting with the same voltage functions u_1 and u_2, and then applying the Kirchhoff voltage and current laws at $z = z'$, so that the voltage is continuous and the jump in the current corresponds to the 1A source current. Show that you get the same result for the Green's function as obtained in the class notes. Note that away from the current source the current on the transmission line is related to the voltage as

$$\frac{dV}{dz} = -j\omega LI \,.$$